APPLICATIONS OF LARGE CARDINALS TO GRAPH THEORY

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INTRODUCTION

In [Fr97] we presented the first examples of statements in discrete and finite mathematics with a clear combinatorial meaning, which are proved using large cardinals, and shown to require them. The large cardinals in question are the subtle cardinals of finite order.

Since then we have been engaged in the development of such results of greater relevance to mathematical practice. In January, 1997 we presented some new results of this kind involving what we call "jump free" classes of finite functions. This Jump Free Theorem is treated in section 2.

Gill Williamson had the remarkable insight that the Jump Free Theorem could be applied to give information concerning various natural distance functions in subgraphs of a given graph. Williamson proposed several kinds of assertions in this vein, and proved that some of them do indeed follow from the Jump Free Theorem. We showed that some others also follow from the Jump Free Theorem, and also simplified and streamlined the applications. Williamson has informed us that these applications belong to a general class of problems of interest to a wide community of graph theorists, combinatorialists, and computer scientists.

We also were able to prove that the more elementary of these applications of the Jump Free Theorem could be directly proved without the Jump Free Theorem using classical Ramsey theory. This established that these particular applications can be proved well within the usual axioms for mathematics.

Next, Sam Buss and Williamson collaborated to extend our results to obtain more of the applications of the Jump Free Theorem just using classical Ramsey theory. Then we proved a general theorem on decreasing classes of functions, just using classical Ramsey theory, that covers this Buss/Williamson collaboration as special cases.

It is still not yet clear whether the remaining, more sophisticated applications of the Jump Free Theorem can be proved within the usual axioms for mathematics. These more sophisticated applications can be stated with more and more structure of the kind considered standard in graph theory and computer science, and form a virtually open ended series of applications. They bear a clear technical resemblance to the Jump Free Theorem, and we consider a proof of the independence of at least some natural version of these applications to be within reach of our technology.

More recently, Williamson has proposed an inductive model which leads to more general applications of the Jump Free Theorem. We have sharpened his original application along these lines, and also extended it in a multivariate way. We have succeeded in showing that one of the multivariate forms is indeed independent of the usual axiom for mathematics, as formalized by ZFC. In fact, it requires use of the same large cardinals that the Jump Free Theorem requires. This development is discussed in section 4.

1. DECREASING CLASSES OF FUNCTIONS

We use N for the set of all nonnegative integers. For $x \in Nk$ let min(x) be the minimum coordinate of x, and let max(x) be the maximum coordinate of x. For $A \subseteq Nk$, let fld(A) be the set of all coordinates of elements of A.

A function f is said to be reflexive in Nk if and only if

i) $dom(f) \subseteq Nk$;
ii) $rng(f) \subseteq fld(A)$.

Let T(k) be the set of all reflexive functions in Nk with finite domain.

Let f,g be any functions that are into N. We write $f \geq g$ if and only if for all $x \in dom(f)$, $f(x) \geq g(x)$. Thus in particular, this implies that $dom(f) \subseteq dom(g)$.

Let X be a set. A regressive value of f on X is an $n \geq 0$ such that there exists $x \in X$ with $f(x) = n < min(x)$.

Let $S \subseteq T(k)$. We say that S is full if and only if for all finite $A \subseteq Nk$, some element of S has domain A.

We say that S is decreasing if and only if for all $f,g \in S$, if $dom(f) \subseteq dom(g)$ then $f \geq g$.

Let $f,g \in T(k)$. We say that h is an order isomorphism from f onto g if and only if

i) h is the order preserving bijection from fld(dom(f)) onto fld(dom(g));
ii) if $f(x1,...,xk) = y$ then $g(h(x1),...,h(xk)) = h(y)$.

For $A,B \subseteq Nk$ we say that h is an order isomorphism from A onto B if and only if

i) h is the order preserving bijection from fld(A) onto fld(B);
ii) $(x1,...,xk) \in A$ if and only if $(h(x1),...,h(xk)) \in B$.

We say that f,g are order isomorphic if and only if there is an order isomorphism from f onto g. We say that A,B are order isomorphic if and only if there is an order isomorphism from A onto B.

We say that $S \subseteq T(k)$ is order invariant if and only if every element of T(k) that is order isomorphic to an element of S is an element of S.

LEMMA 1.1. Let $k,p \geq 1$ and $S \subseteq T(k)$ be full, decreasing, and order invariant. Then some $f \in T(k)$ has at most kk regressive values on some $Ek \subseteq S$, $|E| = p$. In fact, there exists an infinite $E \subseteq N$ such that the following holds. Let $n \in E$, $f \in S$ have domain {0,1,...,n-1}^k, and ot be an order type of k-tuples. Then either

i) for all x ∈ Ek of order type ot with max(x) < n, f(x) ≥ min(x); or

ii) for all x,y ∈ Ek of order type ot with max(x) < n, f(x) = f(y) < min(E).

Proof: Firstly observe that for every finite A ⊆ Nk there is a unique element V[A] of S with domain A. Secondly observe that if A,B are two finite order isomorphic subsets of Nk then the functions V[A] and V[B] are order isomorphic via the same unique isomorphism.

Let x,y ∈ Nk. We define x <* y if and only if

i) x,y have the same order type;
ii) max(x) < min(y);
iii) for all 1 ≤ i,j ≤ k, |xi - xj| ≤ |yi - yj|.

[It turns out that we can weaken ii) to say that min(x) < min(y) without affecting the argument; the choice is a matter of style.]

CLAIM 1. Let x <* y, x ∈ Ak, and V[Ak](x) < min(x). Then there exists B ⊆ N, |A| = |B|, such that the unique order isomorphism from Ak onto Bk sends x to y, and fixes V[Ak](x).

Proof: The spreading out condition, x <* y, assures that the elements of A can be appropriately moved to the elements of B in such a way that the position of x in Ak is the same as the position of y in Bk. The elements of A that are < min(x) can be left untouched. This completes the proof of the claim.

Let k,S be as given. Define F:Nk → N as follows. Let x in Nk. Set F(x) to be the least possible V[Bk](x) for B ⊆ N.

CLAIM 2. If x ≤* y and f(x) < min(x) then F(x) ≥ F(y).

Proof: Let A be such that V[Ak](x) = F(x) < min(x). Choose B according to claim 1. Then F(x) = V[Ak](x) = V[Bk](y) >= F(y) by the order invariance of V and the definition of F.

We are now ready to complete the proof of Lemma 1.1. By Ramsey's theorem applied to 2k-tuples, fix D be an infinite subset of N such that for each order type ot of k-tuples, either

i) for all x <* y ∈ Dk of order type ot, F(x) < min(x) implies F(x) = F(y); or

ii) for all x <* y $\in$ Dk of order type ot, not(F(x) < min(x) implies F(x) = F(y)).

Let ot be given. We claim that ii) cannot hold. Suppose ii) holds. Let x1 <* x2 <* ... be from Dk and have order type ot. Then the values of F at these x's are distinct and < min(x1). According to claim 2, the values of F at these x's must be decreasing. This is the required contradiction.

Hence clause i) holds. By applying Ramsey's theorem again within D, we obtain an infinite D' $\subseteq$ D such that for each order type ot, either

iii) for all x in D'k of order type ot, F(x) < min(x); or

iv) for all x in D'k of order type ot, F(x) $\geq$ min(x).

If clause iii) holds for D' then clause i) holds for D and so for D'. Hence for each order type ot, either

v) for all x,y in D'k of order type ot, F(x) = F(y) < min(x); or

vi) for all x in D'k of order type ot, F(x) $\geq$ min(x).

Since in v), we can take x so that min(x) = min(D’), we can replace min(x) by min(D’).

Let x in D'k. By the definition of F, there is a sufficiently large m > max(x) such that V[A](x) = F(x) as long as A includes {0,...,m}^k. Hence we can further prune D' to obtain an infinite E $\subseteq$ D' which satisfies the properties demanded of the Lemma. □

LEMMA 2. Let k $\geq$ 1 and S $\subseteq$ T(k) be full and decreasing. Then there exists a full decreasing and order invariant S* $\subseteq$ T(k) such that every element of S* is order isomorphic to an element of S.

Proof: We first construct an infinite sequence A1 $\supseteq$ A2 $\supseteq$ A3 ... of subsets of N, each including the next, using Ramsey's theorem, such that the following holds. Let i $\geq$ 1 and f,g $\in$ V, where dom(f),dom(g) are order isomorphic subsets of Ai whose fields have cardinality $\leq$ i. Then f,g are order isomorphic. This is accomplished by coloring the unordered i-tuples u from Ai-1 according to a complete description of the order types of all functions in S whose domain is included in u^k in terms of the exact ensemble of positions in u occupied by the elements of their domain. There are only finitely many colors.

We now take S* to be the set of all elements f of T(k) such that there exists i ≥ 1 and g ∈ S where f,g are order isomorphic and fld(dom(g)) ⊆ Ai. It is clear that S* is order invariant, and that every element of S* is order isomorphic to an element of S. It is also clear that S* is full.

It remains to show that S* is decreasing. Let A ⊆ B ⊆ N^k be finite sets. Let dom(f) = A, dom(g) = B, f,g ∈ S*, |B| = i. Let B' be order isomorphic to B, fld(B') ⊆ Ai. Let h be the order isomorphism from B onto B'. Let A' ⊆ B' be the image of h on A. Let f',g' be the unique elements of S with domains A',B'. Then g is order isomorphic to g'. Now since f is order isomorphic to some f' ∈ S with fld(dom(f')) ⊆ Ai, we see that f is order isomorphic to f'. The order isormorphism in both cases must be h (or its restriction to A). Now since V is decreasing, f'(h(x)) ≥ g'(h(x)). Now f'(h(x)) = h(f(x)) and g'(h(x)) = h(g(x)). Hence h(f(x)) ≥ h(g(x)). So f(x) ≥ g(x) as required. □

DECREASING CLASS THEOREM. Let k,p ≥ 1 and S ⊆ T(k) be full and decreasing. Then some f ∈ T(k) has at most kk regressive values on some Ek ⊆ dom(f), |E| = p. In fact, there exists E ⊆ A ⊆ N, |E| = p, and f ∈ S, dom(f) = Ak, such that the following holds. Let ot be an order type of k-tuples. Then either

i) for all x ∈ Ek of order type ot, f(x) ≥ min(x); or
ii) for all x,y ∈ Ek of order type ot, f(x) = f(y) < min(E).

Proof: By Lemma 2, let S* be a full, decreasing, and order invariant subset of T(k) such that every element of S* is order isomorphic to an element of V. Now apply Lemma 1 to S*. We obtain an infinite E ⊆ N. Cut E down to the first p elements. Choose A to be the initial segment up to but not including the next element of E. Now take an order isomorphic copy of V*[Ak] in S. □

The conclusion of the Decreasing Class Theorem involving order types is a recurring theme throughout the paper. We make the following important definition.

Let k ≥ 1 and f ∈ T(k). We say that f is regressively regular over E if and only if the following holds. Let ot be an order type of k-tuples. Then either

i) for all $x \in E^k$ of order type ot, $f(x) \geq \min(x)$; or

ii) for all $x,y \in E^k$ of order type ot, $f(x) = f(y) < \min(E)$.

Implicit in this definition is the condition $E^k \subseteq \mathrm{dom}(f)$.

2. JUMP FREE CLASSES OF FUNCTIONS

Let $S \subseteq T(k)$. We say that S is jump free if and only if the following holds: Let $f,g \in T(k)$ and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Suppose that for all $y \in \mathrm{dom}(f)$, if $\max(y) < \max(x)$ then $f(y) = g(y)$. Then $f(x) \geq g(x)$. [This says that there is no jump from f to g at x. There would be a jump from f to g at x if $f(x) < g(x)$.]

Here is the Jump Free Theorem:

JUMP FREE THEOREM. Let $k \geq 1$ and $S \subseteq T(k)$ be full and jump free. Then some $f \in T(k)$ has at most k^k regressive values on some $E^k \subseteq \mathrm{dom}(f)$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some E of cardinality p.

The jump free theorem is closely related to the following Proposition A from [Fr97].

For $x \in N^k$, let $|x| = \max(x)$.

A function assignment for N^k is a mapping U which assigns to each finite subset A of N^k, a unique function

$$U(A):A \rightarrow A.$$

Let U be a function assignment for N^k. We say that U is #-decreasing if and only if for all finite $A \subseteq N^k$ and $x \in N^k$,

either $U(A) \subseteq U(A \cup \{x\})$, or there exists y such that $|y| > |x|$ such that $|U(A)(y)| > |U(A \cup \{x\})(y)|$.

PROPOSITION A [Fr97]. Let $k,p > 0$ and U be a #-decreasing function assignment for N^k. Then some U(A) has $\leq k^k$ regressive values on some $E^k \subseteq A$, $|E| = p$.

In [Fr97] it is shown that it is necessary and sufficient to use subtle cardinals of every finite order in order to prove Proposition A.

There seem to be difficulties in deriving the Jump Free Theorem from Proposition A. However, there is a closely related stronger version of Proposition A which does immediately imply the Jump Free Theorem, and it has virtually the same proof as Proposition A. Thus the same large cardinals are used.

To formulate Proposition A′, we say that U is a function assignment′ for Nk if and only if for all finite A ⊆ Nk, U(A):A → fld(A)k. Obviously fld(A)k includes A but may include more than A. So every function assignment for Nk is a function assignment′ for Nk.

PROPOSITION A′. Let k,p > 0 and U be a #-decreasing function assignment′ for N^k. Then some U(A) has ≤ k^k regressive values on some E^k ⊆ A, |E| = p.

Proof: Same as for Proposition A as given in [Fr97]. The proof is conducted in ZFC + (∀n)(there exists an n-subtle cardinal). □

We now prove the Jump Free Theorem using A′.

LEMMA 2.1. Let U be a function assignment′ for Nk. The following is a necessary and sufficient condition for U to be #-decreasing. Suppose that A,B ⊆ Nk are finite, x ∈ A ∩ B, and for all y ∈ A with |y| < |x|, we have U(A)(y) = U(B)(y). Then U(A)(x) = U(B)(x) or |U(A)(x)| > |U(B)(x)|.

Proof: The condition on U is called <1,<2-*-decreasing in section 3 of [Fr97], where <1 = <2 is the ordering on Nk given by x <1 y ↔ |x| < |y|. The result is from Theorem 3.10 in [Fr97]. The proof works for function assignment′. □

LEMMA 2.2. Let S ⊆ T(k) be full and jump free. Then for all finite A ⊆ Nk there is a unique f ∈ S with domain A.

Proof: Suppose f,g ∈ S with domain A. We prove by induction on max(x) that for all x ∈ A, f(x) = g(x). Suppose x ∈ A and for all y ∈ A, if |y| < |x| then f(y) = g(y). We now use jump free. Since there is no jump from f to g at x, we see that f(x) ≥ g(x). On the other hand, since there is no jump from g to f at x, we see that g(x) ≥ f(x). So f(x) = g(x). □

Let S ⊆ T(k) be full and jump free, and A ⊆ Nk be finite. In light of Lemma 2.1, we define S*(A) to be the unique f ∈ S with domain A.

LEMMA 2.3. Proposition A' implies the Jump Free Theorem.

Proof: Assume Proposition A. Let S be a full and jump free subset of T(k). We now define a function assignment' U for Nk associated with S. Let A ⊆ Nk be finite and x ∈ Nk. We define U(A)(x) = (S*(A)(x),…,S*(A)(x)). By Proposition A', let Ek ⊆ A ⊆ Nk, A finite, |E| = p, where U(A) has at most kk regressive values on Ek. Then obviously S*(A) also has at most kk regressive values on Ek.

For the remainder of the Jump Free Theorem, we choose p' >> p and assume |E| = p'. We then apply the classical Ramsey theorem to E obtain the appropriate E' ⊆ E, |E| = p with the required properties. The coloring of an unordered k-tuple X from E is by the table of regressive values of S*(A) on Xk. □

We would like to derive Proposition A from the Jump Free Theorem. There seem to be some difficulties in doing this. However, we can derive Lemma 5.2 of [Fr97]. In [Fr97], Lemma 5.2 was shown to be independent of ZFC, and in fact require subtle cardinals of finite order to prove. We now present Lemma 5.2 [Fr97].

A function system U for N^k is a mapping U from finite A ⊆ N^k into functions U(A):A → fld(A).

Let FPF(N^k) be the set of all finite partial functions from N^k into N. For f ∈ FPF(N^k) we let fld(f) = fld(graph(f)).

Let DFNL(N^k) be the set of all H:FPF(N^k)xN^k → N such that for all f ⊆ g from FPF(N^k) and x ∈ N^k,

i) H(f,x) ≥ H(g,x);
ii) H(f,x) ∈ fld(f) ∪ $\{x_1,...,x_k\}$.

Here DFNL stands for "decreasing functional."

Each H ∈ DFNL(N^k) generates a function system U for N^k by the following inductive process.

Let $A \subseteq N^k$ be finite. We define RCN(A,H) as the unique $F:A \to fld(A)$ such that for all $x \in A$, $F(x) = H(F|\{y \in A: |y| < |x|\},x)$. Note that for each fixed $H \in DFNL(N^k)$, RCN(A,H) defines a function system. We call such a function system an inductive function system for N^k. Here RCN stands for "recursion."

For $x,y \in N^k$, we write $x \subseteq y$ if and only if every coordinate of x is a coordinate of y.

We say that $A \subseteq N^k$ is closed if and only if for all $x \subseteq y$ with $y \in A$, we have $x \in A$.

In [Fr97] we used the following definition of regressively regular: Let f be a nonempty partial function from $N^k \to N^r$ and $E \subseteq N$. We say that f is regressively regular over E if and only if the following holds.

i) $E^k \subseteq dom(f)$;

ii) Let $x,y \in E^k$ be of the same order type. If $|f(x)| < min(x)$ then $|f(y)| < min(y)$ and $f(x) = f(y)$.

Note that this agrees with the definition given at the end of the previous section for r = 1. To see this, suppose f is regressively regular over E in the sense here. Suppose $x,y \in Ek$ are of the same order type and $f(x) < min(x)$. Then only case ii) at the end of the previous section can hold, in which case $f(x) = f(y) < min(E) \leq min(y)$. On the other hand suppose f is regressively regular over E in the sense of [Fr97] just above. Let ot be an order type. Suppose there exists $x \in Ek$ of order type ot with $f(x) < min(x)$. Let $y \in Ek$ be of order type ot with $min(y) = min(E)$. Then $f(x) = f(y) < min(y) = min(E)$.

PROPOSITION B (Lemma 5.2 [Fr97]). Let $k,p > 0$ and U be an inductive function system for N^k. Then there exists closed A such that U(A) is regressively regular over some E of cardinality p.

Proof of B from Jump Free Theorem: Let U be an inductive function system for Nk. Let $H \in DFNL(Nk)$, where for all finite $A \subseteq Nk$, U(A) = RCN(A,H), where $H \in DFNL(Nk)$.

First of all, we observe that the set of all U(A) forms a full and jump free subset of T(k). To see this, let $A,B \subseteq Nk$

be finite, $x \in A \cap B$, and assume that for all $y \in A$, if $|y| < |x|$ then $U(A)(y) = U(B)(y)$. Then $U(A)(x) = H(U(A)|\{y \in A: |y| < |x|\},x)$, and $U(B)(x) = H(U(B)|\{y \in A: |y| < |x|\},x)$. Now by hypothesis, $U(A)|\{y \in A: |y| < |x|\} \subseteq U(B)|\{y \in A: |y| < |x|\}$, and so $U(A)(x) \geq U(B)(x)$.

Having seen that the set of all U(A) forms a full and jump free subset of T(k), we can then apply the Jump Free Theorem to obtain everything that is required except the closedness of A. So we have to be more sophisticated about this, as in the proof of Lemma 5.2 [Fr97].

For finite $A \subseteq N^k$, write $A' = \{x \in A$: for all $y \subseteq x$, $y \in A\}$. We modify the RCN construction as follows.

Let $H \in DFNL(N^k)$. Define MRCN(A,H) as the unique $F:A \to fld(A)$ such that for all $x \in A$, $F(x) = H(F|\{y \in A' :|y| < |x|\},x)$. Clearly MRCN(A,H) defines a function system for N^k.

Note that every value of MRCN(A,H) is a coordinate of an element of A' or a coordinate of x. We let V(A) = MFCN(A,H).

We again observe that the set of all V(A) forms a full and jump free subset of T(k). To see this, let $A,B \subseteq Nk$ be finite, $x \in A \cap B$, and assume that for all $y \in A$, if $|y| < |x|$ then $V(A)(y) = V(B)(y)$. Then $V(A)(x) = H(V(A)|\{y \in A': |y| < |x|\},x)$, and $V(B)(x) = H(V(B)|\{y \in B': |y| < |x|\},x)$. Now by hypothesis, $V(A)|\{y \in A': |y| < |x|\} \subseteq V(B)|\{y \in B': |y| < |x|\}$, and so $V(A)(x) \geq V(B)(x)$.

Hence the set S of all V(A) is a full and jump free subset of T(k). In fact, we have $S^*(A) = V(A)$.

Let $A \subseteq Nk$ be finite. Note that V(A) is the unique $F:A \to fld(A)$ such that for all $x \in A$, $F(x) = H(F|\{y \in A': |y| < |x|\},x)$. And U(A') is the unique $G:A' \to fld(A')$ such that for all $x \in A'$, $G(x) = H(G|\{y \in A': |y| < |x|\},x)$. Hence $U(A') \subseteq V(A)$.

Now by the Jump Free Theorem applied to S, fix E,A such that $Ek \subseteq A \subseteq Nk$, $|E| = p$, A finite, such that the following holds. Let ot be an order type of k-tuples. Then either

i) for all $x \in Ek$ of order type ot, $V(A)(x) \geq \min(x)$; or

ii) for all $x,y \in E_k$ of order type ot, $V(A)(x) = V(A)(y) < \min(x)$.

Now obviously $E_k \subseteq A'$. Also clearly $V(A)$ is regressively regular over E. Hence $U(A') \subseteq V(A)$ is also regressively regular over E. ∎

We summarize the results of this section.

THEOREM 2.4. The Jump Free Theorem can be proved using subtle cardinals of every finite order, but not with subtle cardinals of any fixed finite order. I.e., it can be proved in ZFC + $(\forall n)$(there exists an n-subtle cardinal), but not in ZFC + {there exists an an n-subtle cardinal}$_n$.

3. DISTANCE FUNCTIONS IN GRAPHS

It is natural for our purposes to consider both undirected graphs and directed graphs.

Let $k \geq 1$. Here a digraph G in N^k is a pair (V,E), where V = V(G) is a subset of N^k and E = E(G) is a set of ordered pairs of distinct elements from V. V = V(G) is the set of all vertices, and E = E(G) is the set of all edges. Thus we are considering directed graphs with no multiple edges, and no loops.

We let $DGI(N^k)$ be the set of all directed graphs in N^k. We let $DGO(N^k)$ be the set of all directed graphs on N^k; i.e., where the vertex set is all of N^k. A directed graph is said to be finite if it has at most finitely many vertices.

The elements of $DGO(N^k)$ have a clear geometric meaning. The elements of $DGO(N^k)$ can be viewed as a set of directed line segments connecting various pairs of distinct elements of N^k.

Let G be in $DGI(N^k)$. A path in G is a sequence $x_1,\ldots,x_n$ from V(G), $n \geq 1$, such that for all $1 \leq i < n$, (x_i,x_{i+1}) is an edge joining x_i to x_{i+1}. The length of this path is n-1, which is the number of edges (with repetition).

Let $A \subseteq V(G)$. We write G|A for the subgraph of G induced by A. Thus if $G \in DGI(N^k)$ then G|A = (A,E') where $\{x,y\} \in E'$ if and only if $(x,y) \in E(G)$ and $x,y \in A$. The digraphs of the form G|A are called the vertex induced subgraphs of G. G|A is the subgraph induced by A.

For 1 ≤ i ≤ k, the i-th coordinate plane in [0,∞)k is the set of all elements of [0,∞)k consisting of all vectors whose i-th coordinate is 0. The distance of x ∈ Nk to the i-th coordinate plane is just the i-th coordinate of x, written xi.

We can view any graph in DGI(Nk) as a geometric object which is enclosed in the k coordinate planes in [0,∞)k.

Let G be in DGI(Nk) or UGI(Nk). We will be defining several distance functions d[G]:V(G) → N. The simplest such distance functions are the coordinate functions di[G](x) = xi. Also dmin[G](x) = min(x) = min(x1,…,xk) plays a significant role. These have obvious geometric interpretations in terms of distances to specific coordinate planes, and the shortest distance to some coordinate plane. Also note that the values of these distance functions depend only on the vertex x and do not depend on the graph G.

We will, however, be considering more sophisticated distance functions d[G] which give more subtle ways of measuring a "distance" from a vertex x in G to a (or some) coordinate plane. For these more sophisticated distance functions, d[G](x) depends on G and not only on x; i.e., for x ∈ V(G) ∩ V(G’), we in general do not have d[G](x) = d[G’](x).

The first of these more sophisticated distance functions is given by d*min[G](x) = min{min(y): there is a path in G from x to y}. This is well defined since there is always a path from x to x of length 0. This distance function can be defined as the minimum distance to some coordinate plane of any vertex that can be accessed from x by a path in G.

We can view d*min[G](x) as the length of the shortest way of getting to a coordinate plane from x, where traveling within G is not counted.

A cube in Nk is a k fold Cartesian product A1 x … x Ak, where A1,…,Ak are nonempty subsets of N of the same cardinality. The length of a cube is the cardinality of its factors. A cube may be finite or infinite.

We will often use "min" to indicate the usual min function, min:Nk → N.

THEOREM 3.1. Let k,p >= 1 and G ∈ DGO(Nk). There exists finite A ⊆ Nk such that d*min[G|A] = min on some cube ⊆ A of length p, or is constant on some cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d*min[G|A] is regressively regular over some E of cardinality p; and A can be taken to be a Cartesian power.

We can interpret Theorem 3.1 as saying that either there is a cube of length p where the shortest distance to a coordinate plan from its elements involves no traveling within G, or there is a cube of length p where the shortest distance to a coordinate plane is the same from all of its elements.

Proof of Theorem 3.1: Let k,p ≥ 1 and G ∈ DGO(Nk). Let S ⊆ T(k) be the set of all functions d*min[G|A] such that A ⊆ Nk is finite. Then obviously S is full and decreasing. Now apply the Decreasing Class Theorem to obtain Ek ⊆ A ⊆ Nk, |E| = kp, such that the following holds. Let ot be an order type of k-tuples. Then either

i) for all x ∈ Ek of order type ot, d*min[G|A](x) = min(x); or
ii) for all x,y ∈ Ek of order type ot, d*min[G|A](x) = d*min[G|A](y) < min(E).

This establishes the second claim. For the first claim, consider the cube C = E1 x … x Ek, where E1,…,Ek is the partition of E into k consecutive sets of cardinality p. Then C ⊆ A is a cube of length p all of whose elements are of the same order type. Then C is as required. □

We now add some additional structure to Theorem 3.1. Let (Nk)* be the set of all finite sequences from Nk (including the empty sequence). Nk is naturally embedded in (Nk)* as (Nk)1.

Let G ∈ DGI(Nk) and Nk ⊆ X ⊆ (Nk)*. We now define the distance function d*min[G;X] as follows. Let x ∈ V(G). Then d*min[G;S](x) = min{min(y): there is a path P from x to y in G, where P ∈ X}.

Note that even if, say, x,y,z is a path in G that lies in X, we don't know that x,y is a path in G that lies in X. Obviously x,y is a path in G, but it may not lie in X.

THEOREM 3.2. Let k,p ≥ 1, G ∈ DGO(Nk), and Nk ⊆ X ⊆ (Nk)*. There exists finite A ⊆ Nk such that d*min[G|A;X] = min on some cube ⊆ A of length p, or is constant on some cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d*min[G|A;X] is regressively regular over some E of cardinality p; in fact, A can be taken to be a Cartesian power.

Proof: Same as for Theorem 3.1. □

Actually, under the presence of such additional structure, there is no need to consider the graph. Thus we give a streamlined version of Theorem 3.2 that is easily seen to be equivalent. For any A ⊆ Nk, we let A* be the set of all nonempty finite sequences from A.

Let Nk ⊆ X ⊆ (Nk)* and x ∈ Nk. We define d*min[X]:Nk → N by d*min[X](x) = min{min(y): some element of X starts with x and ends with y}.

THEOREM 3.3. Let k,p ≥ 1 and Nk ⊆ X ⊆ (Nk)*. There exists finite A ⊆ Nk such that d*min[X ∩ A*] = min on some cube ⊆ A of length p, or is constant on some cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d*min[X ∩ A*] is regressively regular over some E of cardinality p; in fact, A can be taken to be a Cartesian power.

Proof: Same as for Theorem 3.1. It also follows from Theorem 3.2 by setting G to be the complete graph on Nk. We can derive Theorems 3.1 and 3.2 from Theorem 3.3 by using X′ = the set of all elements of X which are paths in G. □

Further applications of the Decreasing Class Theorem can be given with additional structure involved. Williamson has many suggestions along these lines.

We now introduce a more delicate kind of distance function d#min[G], G ∈ DGI[Nk], due to Williamson. A terminal vertex in G is a vertex out of which there are no edges.

A terminal path in G ∈ DGI(Nk) is a path x1,…,xn such that xn is a terminal vertex.

Let x be a vertex in G. We define d#min[G](x) = min{min(y): there is a terminal path in G from x to y}.

THEOREM? Let k,p >= 1 and G ∈ DGO(Nk). There exists finite A ⊆ Nk such that d#min[G|A] ≥ min on some cube ⊆ A of length p, or constant on some cube ⊆ A of length p.

Refutation: To be filled in later.

In order to turn this Proposition into a Theorem, we say that G in DGO(Nk) is downward if and only if for all edges (x,y), |x| > |y|. (Recall that |x| = max(x)).

The following Theorem was originally a conjecture of Williamson. We prove it using the Jump Free Theorem, and hence using large cardinals. We do not know if it can be proved in ZFC.

THEOREM 3.4. Let k,p ≥ 1 and G in DGO(Nk) be downward. There exists finite A ⊆ Nk such that d#min[G|A] ≥ min on some cube ⊆ A of length p, or constant on some ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[G|A] is regressively regular over some E of cardinality p.

Proof: We use the Jump Free Theorem. Fix a downward G in DGO(Nk). For each A ⊆ Nk we can consider the function min#[G|A]:A → fld(A). Unfortunately, the set of all these functions is not necessarily jump free.

However, we instead consider the function d#min[G|A]':A → fld(A) defined as follows: d#min[G|A]'(x) = d#min[G|A](x) if x is not a terminal vertex; max(x) otherwise. Now let S be the set of all functions d#min[G|A]'. We now prove that S ⊆ T(k) is full and jump free. We have only to verify jump free.

CLAIM 1. Let G ∈ DGI(Nk) and x ∈ V(G). Then d#min[G](x) = min({d#min[G](y): (x,y) ∈ E(G)} if x is not terminal in G; min(x) otherwise.

Proof: Let G,x be as given, and assume x is not terminal in G. If (x,y) is an edge in G then d#min[G](y) ≥ d#min[G](x). Now let x = x1,...,xk be a terminal path in G, where d#min[G](x) = min(xk), and k >= 2. Then d#min[G](x2) ≤ min(xk) = d#min[G](x).

CLAIM 2. S is jump free.

Proof: Fix finite A,B ⊆ Nk and x ∈ A ∩ B. Assume that for all y ∈ A, if max(y) < max(x) then d#min[G|A]'(y) = d#min[G|B]'(y). We wish to prove that d#min[G|A]'(x) ≥ d#min[G|B]'(x).

case 1. x is a terminal vertex in G|A. Then d#min[G|A]' = max(x). Since G is downward, clearly d#min[G|B]' ≤ max(x) = d#min[G|A]'(x) as required.

case 2. x is not a terminal vertex in G|A. Then x is not a terminal vertex in G|B. Hence d#min[G|A]'(x) = d#min[G|A](x) and d#min[G|B]'(x) = d#min[G|B]'(x). By Claim 1 it suffices to prove that

min{d#min[G|A](y): (x,y) in E(G|A)} ≥ min{d#min[G|B](y): (x,y) ∈ E(G|B)}.

Now let (x,y) be an edge in G|A. Since G is downward, |x| > |y|, and so d#min[G|A]'(y) = d#min[G|B]'(y). We claim that d#min[G|A](y) = d#min[G|B](y). This clearly suffices to establish the desired inequality.

Suppose y is not terminal in G|A. Then d#min[G|A]'(y) = d#min[G|B]'(y) = d#min[G|A](y) = d#min[G|B](y).

Suppose y is terminal in G|A. Then d#min[G|A]'(y) = |y| = d#min[G|B]'(y). Then y must be terminal in G|B since otherwise, d#min[G|B]'(y) < |y| by the downwardness of G. Hence d#min[G|A](y) = d#min[G|B](y) = min(y).

We are now prepared to complete the proof of Theorem 3.4. Since S is jump free, we can use the Jump Free Theorem to obtain an f ∈ S and Ek ⊆ dom(f), |E| = p, such that on each order type in Ek, f is either ≥ min or constant and < min(E) on Ek. I.e., d#min[G|A]' is either ≥ min or constant and < min(E) on any given order type in Ek.

Now let x ∈ A and suppose d#min[G|A]'(x) ≥ min(x). If x is terminal in G|A then d#min[G|A](x) = min(x). If x is not terminal in G|A then d#min[G|A](x) = d#min[G|A]'(x) ≥ min(x). Hence in any case d#min[G|A](x) ≥ min(x).

On the other hand, suppose d#min[G|A]'(x) < min(x). Then x is not terminal in G|A. Hence d#min[G|A]'(x) = d#min[G|A](x). □

We can generalization Theorem 3.4 in the spirit of Theorem 3.3. We say that X ⊆ (Nk)* is admissible if and only if

i) Nk ⊆ X ⊆ (Nk)* without the empty sequence;
ii) if (x1,…,xk) ∈ X then |x1| > … > |xk|;
iii) for all (x1,…,xn),(y1,…,ym) ∈ X, if xn = y1 then (x1,…,xn,y2,…,ym) ∈ X;
iv) for all (x1,…,xn),(x1,…,xn,y1,…,ym) ∈ X, we have (xn,y1,…,ym) ∈ X.

Note that the set of all paths in any G ∈ DGO(Nk) is admissible.

We say that x ∈ X is terminal in X if and only if x has no proper extension in X.

We say that a sequence is from x to y if and only if it begins with x and ends with y.

Define d#[X](x) = min{min(y): there exists a sequence from x to y which is a maximal element of X}. More generally, let A ⊆ Nk. We define d#[X|A]:A → fld(A) by d#[X|A] = min{min(y): there exists a sequence from x to y which is a maximal element of X ∩ A*}.

THEOREM 3.5. Let k,p ≥ 1 and X ⊆ (Nk)* be admissible. There exists finite A ⊆ Nk such that d#min[X|A] ≥ min(x) on a cube ⊆ A of length p, or is constant on a cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[X|A] is regressively regular over some E of cardinality p.

Proof: Let X ⊆ (Nk)* be admissible.

CLAIM 1. Let x ∈ V(G) and A ⊆ Nk be finite. Then d#min[X|A](x) = min{d#min[X|A](y): |y| < |x| and there is an element of X ∩ A* from x to y} if x is not terminal in X ∩ A*; min(x) otherwise.

Proof: Suppose that x is terminal in X ∩ A*. Then d#min[X|A](x) = min(x), and the rhs is also min(x). Now suppose that x is not terminal in X ∩ A*.

Let P be an element of X of length ≥ 2. Extend P to a maximal element of X, and let y be the last term in P. Then |y| < |x| and d#min[X|A](y) ≤ min(y). This establishes that every term in the min that defines the lhs bounds a term that appears in the min that defines the rhs.

On the other hand, let P ∈ X, P from x to y, |y| < |x|. Let P′ be a maximal element of X ∩ A* which starts with y and ends in some z. Then min(z) = d#min[X|A](y). By admissibility, PP′ is a maximal element of X ∩ A*. Hence every term in the min that defines the rhs is a term in the min that defines the lhs. This establishes claim 1.

In order to get a useful jump free set, we define d#min[X|A]':A → fld(A) by d#min[X|A]′(x) = min{min(y): there exists a sequence from x to y which is a maximal element of X

∩ A*} if x is not terminal in X ∩ A*; |x| otherwise. Let S = {d#min[X|A]’: A ⊆ Nk is finite}.

CLAIM 2. S is a full jump free subset of T(k).

Proof: We have only to verify jump free. Let A,B ⊆ Nk be finite and x ∈ A ∩ B. Suppose that for all y ∈ A, if |y| < |x| then d#min[X|A]'(y) = d#min[X|B]'(y). We must verify that d#min[X|A]'(x) ≥ d#min[X|B]'(x).

case 1. x is terminal in X ∩ A*. Then d#min[X|A]'(x) = |x| ≥ d#min[G|B,X]'(x).

case 2. x is not X-terminal in X ∩ A*. Then x is not terminal in X ∩ B*. Now d#min[X|A]'(x) = d#min[X|A](x) = min{d#min[X|A](y): |y| < |x| and there is an element of X ∩ A* from x to y} ≥ min{d#min[X|B](y): |y| < |x| and there is an element of X ∩ B* from x to y} = d#min[X|B](x).

We now complete the proof of Theorem 3.5. Since S is jump free, we can use the Jump Free Theorem to obtain an f ∈ S and Ek ⊆ dom(f), |E| = p, which is either ≥ min or constant and < min(E) on any given order type in Ek.

Now let x ∈ A and suppose min#[X|A]'(x) ≥ min(x). If x is terminal in X ∩ A* then min#[X|A](x) = min(x). If x is not terminal in X ∩ A* then min#[X|A](x) = min#[X|A]'(x) ≥ min(x). Hence in any case min#[X|A](x) ≥ min(x).

On the other hand, suppose min#[X|A]'(x) < min(x). Then x is not terminal in X ∩ A*. Hence min#[X|A]'(x) = min#[X|A](x). ∎

Williamson has also defined the following variants of d#min. Let G ∈ DGI(Nk), x ∈ V(G), A ⊆ Nk, and X ⊆ (Nk)* be admissible. Define d#min*[G](x) = min{min(y): there is a terminal path in G from x to y} if this min is ≤ min(x); min(x) otherwise. More generally, define d#min*[X](x) = min{min(y): there exists a sequence from x to y that is a maximal element of X} if this min is ≤ min(x); min(x) otherwise. Also define d#min*[X|A](x) = min{min(y): there exists a sequence from x to y that is a maximal element of X ∩ A*} if this min is ≤ min(x); min(x) otherwise.

Note that if d#min[G|A](x) ≤ min(x) then d#min[G|A](x) = d#*min[G|A](x). And more generally, if d#min[X|A](x) ≤ min(x) then d#min[X|A](x) = d#*min[X|A](x). So we have the following immediate Corollary of Theorem 3.5:

THEOREM 3.6. Let k,p >= 1 and G ∈ DGO(Nk) be downward. There exists finite A ⊆ Nk such that d#*min[G|A] = min(x) on a cube ⊆ A of length p, or is constant on a cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#*min[G|A] is regressively regular over some E of cardinality p.

And the following additional Corollary:

THEOREM 3.7. Let k,p >= 1, G ∈ DGO(Nk) be downward, and X ⊆ (Nk)* be admissible. There exists finite A ⊆ Nk such that d#*min[X|A] = min(x) on a cube ⊆ A of length p, or is constant on a cube ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#*min[X|A] is regressively regular over some E of cardinality p.

4. WILLIAMSON'S INDUCTIVE MODEL

Let G ∈ DGI(Nk) be downward. Williamson builds on the direct inductive definition of d#min[G]. For all x ∈ V(G), define d#min[G](x) = min{d#min[G](y): (x,y) ∈ E(G)} if y is not terminal in G; min(x) otherwise.

In other words, every x ∈ V(G) determines its "distance" out of the graph to a coordinate plane by surveying such "distances" from each of the vertices x points to, and taking the minimum value.

More generally, maybe x does not survey these "distances" from all of the vertices x points to; rather just from some of the vertices x points to. Williamson thinks of the vertices of G as civil servants who determine their "distances" by surveying information from their immediate subordinates (i.e., vertices they point to in the graph) as to their distances. But the civil servants are skeptical and do not rely on all such information from their immediate subordinates.

Williamson implements this idea that only some of the "distances" from immediate subordinates are considered by starting with a binary relation R between edges in G (or even just pairs from Nk) and elements of N. Thus R ⊆ (Nk•Nk)•N. Civil servant x considers the “distance” n of an immediate subordinate y if and only if R((x,y),n). Civil servant x then takes the min only over these “distances” when determining x’s “distance.” If there are no such “distances” to consider, then x’s “distance” is considered to be min(x) by default.

Williamson then makes the following formal definition.

Let G ∈ DGI(Nk) be downward and let R ⊆ (Nk•Nk)•N. For x ∈ V(G) define d#min[G,R](x) = min{d#min[G,R](y): R((x,y),d#min[G,R](y))} if this min is nonempty; min(x) otherwise.

This leads naturally to the following possible application of the Jump Free Theorem:

THEOREM 4.1. Let k,p ≥ 1, G in DGO(Nk) be downward, and R ⊆ (Nk•Nk)•N. There exists finite A ⊆ Nk such that d#min[G|A,R] ≥ min on some cube ⊆ A of length p, or constant on some ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[G|A,R] is regressively regular over some E of cardinality p.

Williamson noticed a difficulty in verifying that the appropriate set of functions is jump free. So he adds the following very natural hypothesis which makes everything work straightforwardly:

THEOREM 4.2. Let k,p ≥ 1, G in DGO(Nk) be downward, R ⊆ (Nk•Nk)•N, and assume that R((x,y),n) implies n ≤ min(x). There exists finite A ⊆ Nk such that d#min[G|A,R] = min on some cube ⊆ A of length p, or constant on some ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[G|A,R] is regressively regular over some E of cardinality p.

Proof: Let S be the set of all functions d#min[G|A,R], where A ⊆ Nk is finite. It is easy to verify that each d#min[G|A,R] is reflexive by induction on max(x), x ∈ A. To verify jump free, let A,B ⊆ Nk be finite, x ∈ A ∩ B, and assume that for all y ∈ A, if max(y) < max(x) then d#min[G|A,R](y) = d#min[G|B,R](y). Now by the hypothesis on R, we see that d#min[G|B,R](x) ≤ min(x). So we can assume that d#min[G|A,R](x) < min(x). Hence the min that defines d#min[G|A](x) is a nonempty min and is clearly over a subset of the min that defines d#min[G|B](x). Hence d#min[G|A,R](x) ≥ d#min[G|B,R](x).

Now that we have proved that S is a jump free subset of T(k), we obtain the conclusion of the lemma by applying the Jump Free Theorem. □

We have been able to prove Theorem 4.1 even though the associated set of functions is not necessarily jump free. This is analogous to the situation with regard to Theorem 3.3.

Proof of Theorem 4.1: The proof is analogous to the proof of Theorem 3.3. We use the Jump Free Theorem. Fix a downward G in DGO(Nk), and R ⊆ (Nk•Nk)•N. For each finite A ⊆ Nk we consider the function d#min[G|A,R]′:A → fld(A) defined as follows: d#min[G|A,R]′(x) = d#min[G|A,R](x) if the min defining d#min[G|A,R](x) is nonempty; max(x) otherwise. Note that we have defined d#min[G|A,R]′ in a noninductive manner from d#min[G|A,R], where the latter has been defined inductively.

Now let S be the set of all functions d#min[G|A,R]′. We now prove that S ⊆ T(k) is full and jump free. We have only to verify jump free. Note that since G is downward, an easy argument by induction on max(x) shows that for all x ∈ A, d#min[G|A,R]′(x) ≤ max(x).

Fix finite A,B ⊆ Nk and x ∈ A ∩ B. Assume that for all y ∈ A, if max(y) < max(x) then d#min[G|A,R]′(y) = d#min[G|B,R]′(y). We wish to prove that d#min[G|A,R]′(x) ≥ d#min[G|B,R]′(x).

case 1. The min defining d#min[G|A,R](x) is empty. Then d#min[G|A,R](x) = max(x). And so we have d#min[G|B,R]′(x) ≤ max(x) = d#min[G|A,R]′(x) as required.

case 2. The min defining d#min[G|A,R](x) is nonempty. Then d#min[G|A]′(x) = d#min[G|A,R](x) ≥ d#min[G|B,R](x) = d#min[G|B,R]′(x). This is because the min defining d#min[G|B,R](x) must be nonempty and moreover contain the min defining d#min[G|A,R](x) as a subset.

Since S is jump free, we can use the Jump Free Theorem to obtain an f ∈ S and Ek ⊆ dom(f), |E| = p, such that on each order type in Ek, f is either ≥ min or constant and < min(E) on Ek. I.e., d#min[G|A,R]′ is either ≥ min or constant and < min(x) on any given order type in Ek.

Now let x ∈ A and suppose d#min[G|A,R]′(x) ≥ min(x). If the min defining d#min[G|A,R](x) is empty then d#min[G|A,R](x) = min(x). If this min is nonempty then d#min[G|A,R](x) = d#min[G|A,R]′(x). So in any case d#min[G|A,R](x) ≥ min(x).

On the other hand, suppose d#min[G|A,R]′(x) < min(x). Then the min defining d#min[G|A,R](x) is nonempty, in which case d#min[G|A,R](x) = d#min[G|A,R](x). □

We are now going to give a multivariate form of Williamson’s model, in which each civil servant x takes into account the “distances” of one or more subordinates when determining x’s “distance.” We are particularly interested in giving a streamlined version of this multivariate model, rather than the most general form, since we are going to show that it is independent of the usual axioms for mathematics (ZFC). Accordingly, we first revisit Williamson’s original model and give a streamlined version of it.

In particular, we remove any mention of the graph in Williamson’s original model, show that this stremlined form is outright equivalent to Williamson’s formulation.

Let R ⊆ Nk•Nk•N and A ⊆ Nk. We define d#min(A,R):A → N by induction as follows. For x ∈ A, d#min(A,R)(x) = min{d#min(A,R)(y): y ∈ A & max(y) < max(x) & R(x,y,n)} if this min is nonempty; min(x) otherwise.

THEOREM 4.3. Let k,p ≥ 1 and R ⊆ Nk•Nk•N. There exists finite A ⊆ Nk such that d#min[A,R] ≥ min on some cube ⊆ A of length p, or constant on some ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[A,R](x) is regressively regular over some E of cardinality p.

Proof: We actually show that this is equivalent to Theorem 4.1. First assume Theorem 4.1. Let k,p,R be as given. Let G be the complete graph on Nk. Let R′ ⊆ (Nk•Nk)•Nk be given by R′((x,y),n) → R(x,y,n). Then 4.3 for k,p,R follows from 4.1 for k,p,R′,G.

Now assume Theorem 4.3. Let k,p,R,G be given for 4.1. Let R′⊆ Nk•Nk•N be given by R′(x,y,n) → ((x,y) ∈ E(G) & R((x,y),n)). Then 4.1 for k,p,R,G follows from 4.3 for k,p,R′. □

We now present the multivariate form of Williamson’s model. We say that F:B → C is a partial function if and only if F is a function whose domain is included in B and range is included in C.

We say that F:Nk•(Nk•N)r → N is a partial selection function if and only if for all defined F(x,y1,n1,y2,n2,…,yr,nr), there exists 1 ≤ i ≤ r such that F(x,y1,n1,y2,n2,…,yr,nr) = ni.

For A ⊆ Nk, define d#min[A,F] by induction as follows. For x ∈ A, d#min[A,F] = min{F(x,y1,n1,y2,n2,…,yr,nr): y1,…,yr ∈ A & max(y1),…,max(yr) < max(x) & d#min[A,F](y1) = n1 & … & d#min[A,F](yr) = nr} if this min is nonempty; min(x) otherwise. Obviously the min is taken only over defined F(x,y1,n1,y2,n2,…,yr,nr).

The significance of F(x,y1,n1,y2,n2,…,yr,nr) = ni is that the committee consisting of x (ex officio) and his subordinates y1,…,yr, after considering the respective “distances” n1,…,nr, of y1,…,yr, have come to the collective opinion of the “distance” ni.

THEOREM 4.4. Let k,p,r ≥ 1 and F:Nk•(Nk•N)r → N be a partial selection function. There exists finite A ⊆ Nk such that d#min[A,F] ≥ min on some cube ⊆ A of length p, or constant on some ⊆ A of length p. In fact, there exists finite A ⊆ Nk such that d#min[A,R] is regressively regular over some E of cardinality p.

Proof: The proof is completely analogous to the proof of Theorem 4.1. We use the Jump Free Theorem. Fix k,p,r,F as given. For each finite A ⊆ Nk we consider the function d#min[A,F]′:A → fld(A) defined as follows: d#min[A,F]′(x) = d#min[A,F](x) if the min defining d#min[A,F](x) is nonempty; max(x) otherwise.

Now let S be the set of all functions d#min[A,R]′. We now prove that S ⊆ T(k) is full and jump free. We have only to verify jump free.

Fix finite A,B ⊆ Nk and x ∈ A ∩ B. Assume that for all y ∈ A, if max(y) < max(x) then d#min[A,F]′(y) = d#min[B,F]′(y). We wish to prove that d#min[A,F]′(x) ≥ d#min[G,F]′(x).

Note that by an easy induction on max(x), we see that d#min[A,F]′(x) ≤ max(x).

case 1. The min defining d#min[A,F](x) is empty. Then d#min[A,F](x) = max(x). And so we have d#min[G,F]′(x) ≤ max(x) = d#min[A,F)(x) as required.

case 2. The min defining d#min[A,F](x) is nonempty. Then d#min[A,F]′(x) = d#min[A,F](x) ≥ d#min[G,F](x) = d#min[B,R]′(x). This is because the min defining d#min[B,F](x) must be nonempty and moreover contain the min defining d#min[A,R](x) as a subset.

We can now use the Jump Free Theorem to obtain an f ∈ S and Ek ⊆ dom(f), |E| = p, such that on each order type in Ek, f is either ≥ min or constant and < min(E) on Ek. I.e., d#min[A,F]′ is either ≥ min or constant and < min(x) on any given order type in Ek.

Now let x ∈ A and suppose d#min[A,F]′(x) ≥ min(x). If the min defining d#min[A,F](x) is empty then d#min[A,F](x) = min(x). If this min is nonempty then d#min[A,F](x) = d#min[A,F]′(x). So in any case d#min[A,F](x) ≥ min(x).

On the other hand, suppose d#min[A,F]′(x) < min(x). Then the min defining d#min[G|A,F](x) is nonempty, in which case d#min[G|A,F](x) = d#min[G|A,F](x). ∎

We are now going to show that Theorem 4.4 requires large cardinals to prove. In fact, the same large cardinals are required to prove Theorem 4.4 that are required to prove the Jump Free Theorem.

We assume Theorem 4.4, and derive Lemma 5.3 of [Fr97], which was shown to have the required metamathematical properties. We now present a self contained statement of Lemma 5.3.

For tuples x of natural numbers, we use |x| for max(x).

We write FPF(Nk) for the set of all finite partial functions from Nk into N; i.e., the domain is a finite subset of Nk and the range is a subset of N. For A ⊆ Nk we write fld(A) for the set of all elements of N that are a coordinate of some element of A.

Let QF be the set of all propositional combinations of atomic formulas of the form x < y, where x and y are variables representing elements of N. We assume that elements of QF are in disjunctive normal form.

Let f ∈ FPF(N^k). If y ∈ N^{kr} then we write f(y) for $(f(y_1,...,y_k),...,f(y_{kr-k+1},...,y_{kr}))$.

It is important to adhere to the convention that f(y) is defined if and only if each of the t components are defined. I.e., f(y) is defined if and only if $y \in dom(f)^r$.

Let BEF(q,r,k) be the set of all bounded existential formulas of the following form:

$$B(x) = (\exists y \in dom(F)^r)(|y| < |x| \ \& \ D(x,y,F(y))),$$

where x abbreviates the list of variables $x_1,...,x_q$, y abbreviates the list of variables $x_{q+1},...,x_{q+kr}$, and D is in QF. Here F is viewed as a function symbol representing a finite partial function from $N^k \to N$.

If we specify an actual $f \in FPF(N^k)$ and $x \in N^q$, then it is clear what we mean by asserting that B(x) is true in f.

Let $B \in BEF(k+1,r,k)$ and A be a finite subset of N^k. We define Df(B;A) as the unique f:A → fld(A) such that for all x $\in$ A, f(x) = min{j $\in$ fld(A): j = |x| or B(x,j) is true in f}.

Finally, here is the Lemma from [Fr97] that we are going to derive from Theorem 4.4:

LEMMA 5.3 [Fr97]. Let k,p > 0 and $B \in BEF(k+1,r,k)$. Then there exists finite closed $A \subseteq N^k$ such that Df(B;A) is regressively regular over some E of cardinality p.

The closedness of A ⊆ Nk was defined in section 2.

We first get into the trenches and expand the definition of Df(B;A) in more primitive notation. Let A ⊆ Nk be finite. We consider the unique f:A → fld(A) such that for all x ∈ A,

I) f(x) = min{j ∈ fld(A): j = |x| or (∃y1,…,yr ∈ A)(|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j))},

where D is quantifier free. We write this f as I(D,A).

This definition is obviously a correct definition of Df{B;A).

In this terminology, the recursion involved in Theorem 4.4 is of the following form. Let A ⊆ Nk be finite. We consider the unique f:A → fld(A) such that for all x ∈ A,

II) f(x) = min{j: (∃y1,…,yr ∈ A)(|y1|,…,|yr| < |x| & R(x,y1,…,yr,f(y1),…,f(yr),j) & j = some f(yi))} if this min is nonempty; min(x) otherwise,

where R ⊆ (Nk)r+1•Nr+1 is fixed in advance of any choice of finite A ⊆ Nk. We write this f as II(R,A).

We spell out the relevant Propositions based on I) and II).

PROPOSITION A. Let k,p,r ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables (see the definition of QF above). There exists finite closed A ⊆ Nk such that I(D,A) is regressively regular over some E of cardinality p.

PROPOSITION B. Let k,p,r ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. There exists finite A ⊆ Nk such that II(D,A) is regressively regular over some E of cardinality p.

LEMMA 4.5. Proposition A implies Lemma 5.3 [Fr97]. Theorem 4.4 implies Proposition B.

Proof: Obvious. □

The remainder of this section is devoted to the derivation of Proposition A from Proposition B. By Lemma 4.5., this means that Theorem 4.4 implies Lemma 5.3 [Fr97]. Hence by [Fr97], Theorem 4.4 implies the consistency of certain large cardinal axioms, and therefore is independent of the usual axioms for mathematics.

PROPOSITION C. Let k,p,r ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. There exists finite closed A ⊆ Nk such that II(D,A) is regressively regular over some E of cardinality p.

We aim to derive Proposition C from Proposition B. Let k ≥ 1 and A ⊆ Nk. We define A′ = {x ∈ A: (∀y ⊆ x)(y ∈ A)}.

Let D ⊆ (Nk)r+1•Nr+1. We inductively define f:A → fld(A) with the intention that f|A′ = II(D,A′).

For x ∈ A set f(x) = min{j: (∃y1,…,yr ∈ A′)(|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j) & j = some f(yi))} if this min is nonempty; min(x) otherwise.

We write this f as f*.

LEMMA 4.6. Let k ≥ 1, D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables, and A ⊆ Nk. Then f*|A′ = II(D,A′). Also there exists t ≥ 1 and D′ ⊆ (Nk)t•Nt given by a quantifier free formula in (k+1)(t) variables such that f* = II(D′,A).

Proof: The first claim is by straightforward induction. The second claim is seen by replacing each existential quantification over A′ with a multiple existential quantification over A. □

LEMMA 4.7. Proposition B implies Proposition C.

Proof: Let k,p,r,D be for Proposition C. We apply Proposition B for k,p,t,D′, where t,D′ are given by Lemma 4.6. We obtain f* = II(D′,A) which is regressively regular over E, |E| = p.

Note that Ek ⊆ A′. And obviously f*|A′ = II(D,A′) is regressively regular over Ek. □

If x is a tuple from N then we write card(x) for the number of distinct terms in x.

Let A ⊆ Nk be finite. We consider the unique f:A → fld(A) such that for all x ∈ A,

III) f(x) = min{j ∈ fld(A): j < |x| & (∃y1,…,yr ∈ A) (|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j))} if this min is nonempty and card(x) ≥ 2; min(x) otherwise,

where D is quantifier free. We write this f as III(D,A).

PROPOSITION D. Let k,p,r ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. There exists finite closed A ⊆ Nk such that III(D,A) is regressively regular over some E of cardinality p.

LEMMA 4.8. Proposition C implies Proposition D.

Proof: Let k,p,r,D be as given by Proposition D. We apply Proposition C to k,p,r+1,D′, where D′ is given by D′(x,y1,…,yr+1,n1,…,nr+1,j) ↔ (card(x) ≥ 2 & D(x,y1,…,yr,n1,…,nr,j)).

From Proposition C, we obtain finite closed A ⊆ Nk and f:A → fld(A) such that for all x ∈ A, f(x) = min{j: (∃y1,…,yr+1 ∈ A)(|y1|,…,|yr+1| < |x| & D’(x,y1,…,yr+1,f(y1),…,f(yr+1),j) & j = some f(yi))} if this min is nonempty; min(x) otherwise.

I.e., for all x ∈ A,

f(x) = min{j: (∃y1,…,yr+1 ∈ A)(|y1|,…,|yr+1| < |x| & (card(x) ≥ 2 & D(x,y1,…,yr,f(y1),…,f(yr),j)) & j = some f(yi))} if this min is nonempty; min(x) otherwise.

Hence for all x ∈ A,

f(x) = min{j: (∃y1,…,yr+1 ∈ A)(|y1|,…,|yr+1| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j)) & j = some f(yi))} if this min is nonempty and card(x) ≥ 2; min(x) otherwise.

In particular, if x ∈ A and card(x) = 1, then f(x) = min(x).

Note that in the above min, since A is closed, yr+1 can be set to be any (m,…,m) where m < |x|, in which case the f(yr+1) is m. And note that by induction on |x|, x ∈ A, we see that f obeys the inequality f(x) ≤ |x|. Thus the possible f(yr+1) in the above min are all elements of fld(A) that are < |x|. Thus for all x ∈ A,

f(x) = min{j ∈ fld(A): j < |x| & (∃y1,…,yr+1 ∈ A)(|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j))} if this min is nonempty and card(x) ≥ 2; min(x) otherwise.

Note that this is exactly III(D,A), and so f is as desired for Proposition D. □

Let A ⊆ Nk. We now consider the unique f:A → fld(A) such that for all x ∈ A,

IV) f(x) = min{j ∈ fld(A): j < |x| & (∃y1,…,yr ∈ A) (|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j))} if this min is nonempty and card(x) ≥ 3; |x| otherwise,

where D is quantifier free. We write this f as IV(D,A).

PROPOSITION E. Let k,p,r ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. There exists finite closed A ⊆ Nk such that IV(D,A) is regressively regular over some E of cardinality p.

We now aim to show that Proposition D implies Proposition E.

Let k ≥ 1. For all x ∈ Nk+1, let x- be the result of chopping off the first term.

Let A ⊆ Nk+1 be finite. We define A# to be the set of all x ∈ Nk such that (min(x),x) ∈ A and (|x|,x) ∈ A.

Let f:A → fld(A). We define f#:A# → fld(A) as follows. Let x ∈ A#. If card(x) ≤ 2 or f(|x|,x)) = min(x), then set f#(x) = |x|. Otherwise set f#(x) = f(min(x),x).

For any B ⊆ Nt and x ∈ Nt, let B<x = {y ∈ B: |y| < |x|}. If f:B → fld(B), let f<x = f|B<x. Let B≤x = B<x ∪ {x} and f≤x = f|B≤x.

Now let D ⊆ (Nk)r+1•Nr+1. For finite A ⊆ Nk+1, we inductively define an f:A → fld(A) with the intention that f# = IV(D,A#).

Let x ∈ A, and suppose f<x has been defined.

case 1. x = (min(x-),x-) and card(x-) ≥ 3. Define f(x) = min{j ∈ fld(A#): j < |x| & (∃y1,…,yr ∈ A#)(|y1|,…,|yr| < |x| & D(x-,y1,…,yr,(f<x)#(y1),…,(f<x)#(yr),j))} if this min is nonempty; min(x) otherwise.

case 2. x = (|x-|,x-) and card(x-) ≥ 3. Define f(x) = the least term of x above min(x) if the min in case 1 is nonempty; min(x) otherwise.

case 3. otherwise. Define f(x) = min(x).

We write this f as ^(D,A).

LEMMA 4.9. Let k ≥ 1, D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables, and A ⊆ Nk+1 be finite and closed. Then ^(D,A)# = IV(D,A#).

Proof: We prove by induction on x ∈ A that ^(D,A≤x)# ⊆ IV(D,A#). Assume that this is true for all y ∈ A with |y| < |x|. Then ^(D,A<x)# ⊆ IV(D,A#).

First suppose that x- ∉ A#. Then (A≤x)# = (A<x)#, and so ^(D,A≤x)# = ^(D,A<x)# ⊆ IV(D,A#).

Next suppose that card(x-) ≤ 2, x ∈ A#. Then ^(D,A≤x)#(x) = |x| = IV(D,A#), and so ^(D,A ≤x) ⊆ IV(D,A#).

For the remainder of the proof, assume that x- ∈ A# and card(x-) ≥ 3.

If x = (min(x-),x-) then

^(D,A≤x)(x) =
min{j ∈ fld(A#): j < |x| & (∃y1,…,yr ∈ A#)(|y1|,…,|yr| < |x| & D(x-,y1,…,yr,(f<x)#(y1),…,(f<x)#(yr),j))} if this min is nonempty; min(x) otherwise.

And if x = (max(x-),x-) then

^(D,A≤x)(x) = the least term of x above min(x) if the min above is nonempty; min(x) otherwise.

Now by the definition of #, if the min above is nonempty then ^(D,A≤x)#(x-) is the min above. And by case 2, if the min above is empty then ^(D,A≤x)#(x-) = |x-|. This is in exact accordance with the definition of IV(D,A#). Hence ^(D,A≤x)# ⊆ IV(D,A#). □

LEMMA 4.10. Let k ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. Then there exists D′ ⊆ (Nk+1)r+1•(Nk+1)r+1 given by a quantifier free formula in (k+1)(r+1) variables such that the following holds. For finite A ⊆ Nk+1, ^(D,A) = III(D′,A).

Proof: By inspection of the definition of ^(D,A). This definition is of the right form. □

LEMMA 4.11. Proposition D implies Proposition E.

Proof: Let k,r,p,D be as given for Proposition V. We apply Proposition IV for k+1,r,p,D′, where D′ is given by Lemma 4.9.

Thus we obtain finite closed A ⊆ Nk+1 and Ek+1 ⊆ A, |E| = p′, such that III(D′,A) is regressively regular over E. Then obviously Ek ⊆ A# and A# ⊆ Nk is closed. We have to verify that ^(D,A)# = IV(D,A#) is regressively regular over E.

We have that for all order types ot of k+1-tuples, either

i) for all x ∈ Ek+1 of order type ot, ^(D,A)(x) ≥ min(x); or
ii) for all x,y ∈ Ek of order type ot, ^(D,A)(x) = ^(D,A)(y) < min(E).

Now let ot′ be an order type of k-tuples. Suppose that for some y ∈ Ek of order type ot′, ^(D,A)#(y) < min(y). By the definition of ^(D,A)#, we see that card(y) ≥ 3 and the min defining ^(D,A)((min(y),y)) is nonempty. Hence ^(D,A)#(y) = ^(D,A)((min(y),y)) < min(y). Therefore for all z ∈ Ek of order type ot′, ^(D,A)((min(z),z)) = ^(D,A)((min(y),y)) < min(E), and hence ^(D,A)#(z) = ^(D,A)((min(z),z)) = ^(D,A)#(y). This verifies that ^(D,A)# = IV(D,A#) is regressively regular over E. □

Note that we can put IV) in a somewhat more convenient form that more closely resembles our target I):

For A ⊆ Nk, consider the unique f:A → fld(A) such that for all x ∈ A,

IV) f(x) = min{j ∈ fld(A): j = |x| or (∃y1,…,yr ∈ A) (|y1|,…,|yr| < |x| & D(x,y1,…,yr,f(y1),…,f(yr),j))} if card(x) ≥ 3; |x| otherwise,

where D is quantifier free.

We now want to prove Proposition A from Proposition E. Let k ≥ 1 and A ⊆ Nk+2. For x ∈ Nk+2, let x-- be the result of chopping off the first two terms from x. Define Sk+2 = {x ∈ Nk+2: x1 < x2 < |x--|}. Define A′ = {x--: x ∈ A ∩ Sk+2}.

Let f:A → fld(A). We say that f is good if and only if for all x,y ∈ A ∩ Sk+2, if x-- = y-- then f(x) = f(y). If f is good then we define f′:A′ → fld(A) by f′(x--) = f(x), where x ∈ A ∩ Sk+2.

Now let D ⊆ (Nk)r+1•Nr+1. For finite A ⊆ Nk+1, we inductively define an f:A → fld(A) with the intention that f′ = I(D,A′). Let x ∈ A.

case 1. x ∈ Sk+2. Set f(x) = min{j ∈ fld(A): j = |x| or (∃y1,…,yr ∈ A ∩ Sk+2)(|y1|,…,|yr| < |x| & D(x--,y1--,…,yr--,f(y1),…,f(yr),j))}.

case 2. x1 > x2 > |x--|. Set f(x) = min{j ∈ fld(A): j = |x| or (∃y ∈ A ∩ Sk+2)(x-- = y-- & j = f(y))}.

case 3. otherwise. Set f(x) = |x|.

We write this f as α(D,A).

LEMMA 4.12. Let k ≥ 1, D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables, and A ⊆ Nk+1 be finite and closed. Then α(D,A) is good and α(D,A)′ = I(D,A′).

Proof: Straightforward induction, which is left to the reader. □

LEMMA 4.13. Let k ≥ 1 and D ⊆ (Nk)r+1•Nr+1 be given by a quantifier free formula in (k+1)(r+1) variables. Then there exists D′ ⊆ (Nk+2)r+1•(Nk+2)r+1 given by a quantifier free formula in (k+3)(r+1) variables such that the following holds. For finite A ⊆ Nk+2, α(D,A) = IV(D′,A).

Proof: By inspection of the definition of α(D,A). This definition is of the appropriate form. □

LEMMA 4.14. Proposition E implies Proposition A.

Proof: Let k,r,p,D be for Proposition A. We apply Proposition E for k+2,r,p+4,D′, where D′ is as given by Lemma 4.13. By Proposition E we obtain finite A ⊆ Nk+2 and E ⊆ N of cardinality p such that α(D,A) is regressively regular over E. Let E′ = {E3,…,Ep+2}. Let x,y ∈ E′k be of the same order type, where I(D,A′)(x) < min(x). Recall from Lemma 4.12 that α(D,A)′ = I(D,A′).

We have x,y ∈ A′ and (E1,E2,x),(E1,E2,y) ∈ A ∩ Sk+2. Hence α(D,A)′(x) = α(D,A)((E1,E2,x)) and α(D,A)′(y) = α(D,A)((E1,E2,y)). Also α(D,A)((Ep+4,Ep+3,x)) = α(D,A)′(x) and α(D,A)((Ep+4,Ep+3,y)) = α(D,A)′(y). Hence by the regressive regularity of α(D,A) over E, we have α(D,A)′(x) =

α(D,A)′(y) < min(E). Hence α(D,A)′ = I(D,A′) is regressively regular over E′. ∎

We have now proved the following.

THEOREM 4.15. Proposition B implies Proposition A. Theorem 4.4 implies Lemma 5.3 [Fr97]. Theorem 4.4 can be proved using subtle cardinals of every finite order, but not with subtle cardinals of any fixed finite order. I.e., it can be proved in ZFC + (∀n)(there exists an n-subtle cardinal), but not in ZFC + {there exists an an n-subtle cardinal}n.

Proof: The first two claims are from the preceding. Theorem 4.4 can be proved using subtle cardinals of every finite order since it was proved here from the Jump Free Theorem; and see Theorem 2.4. The metamathematical claims concerning Lemma 5.3 [Fr97] are from [Fr97]. ∎