A WAY OUT

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ABSTRACT. We present a way out of Russell's paradox for sets in the form of a direct weakening of the usual inconsistent full comprehension axiom scheme, which, with no additional axioms, interprets ZFC. In fact, the resulting axiomatic theory 1) is a subsystem of ZFC + "there exists arbitrarily large subtle cardinals", and 2) is mutually interpretable with ZFC + the scheme of subtlety.

1. NEWCOMP.

Bertrand Russell [Ru1902] showed that the Fregean scheme of full comprehension is inconsistent. Given the intuitive appeal of full comprehension (for sets), this inconsistency is known as Russell's Paradox (for sets). The modern view is to regard full comprehension (for sets) as misguided, and thereby regard Russell's Paradox (for sets) as a refutation of a misguided idea.

We first give an informal presentation of the axiom scheme investigated in this paper. Informally, the full comprehension axiom scheme in the language $L(\in)$ with only the binary relation symbol $\in$ and no equality, is, in the context of set theory,

Every virtual set forms a set.

We use the term "virtual set" to mean a recipe that is meant to be a set, but may be a "fake set" in the sense that it does not form a set. The recipes considered here are of the form $\{x: \varphi\}$, where φ is any formula in $L(\in)$.

Other authors prefer to use the term "virtual class", reflecting the idea that $\{x: \varphi\}$ always forms a class, with the understanding that x ranges over sets. Our terminology reflects the intention to consider only sets, and construct a powerful set existence axiom.

We say that $\{x: \varphi\}$ forms a set if and only if there is a set whose elements are exactly the y such that φ. Here y must not be free in φ (and must be different from x). Thus $\{x: \varphi\}$ forms a set is expressed by

$$(\exists y)(\forall x)(x \in y \leftrightarrow \varphi).$$

Russell showed that

$$\{x: x \notin x\} \text{ forms a set}$$

leads to a contradiction in pure logic.

Our way out of Russell's Paradox is to modify the inconsistent Fregean scheme in this way:

Every virtual set forms a set, or _____.

We refer to what comes after "or" as the "escape clause". The escape clause that we use involves only the extension of the virtual set and not its presentation.

We are now ready to present the comprehension axiom scheme.

NEWCOMP. Every virtual set forms a set, or, outside any given set, has two inequivalent elements, where all elements of the virtual set belonging to the first belong to the second.

To avoid any possible ambiguity, we make the following comments (as well as give a formal presentation in section 2).

1. For Newcomp, we use only the language $L(\in)$, which does not have equality.
2. Here "inequivalent" means "not having the same elements".
3. The escape clause asserts that for any set y, there are two unequal sets z,w in the extension of the virtual set, neither in y, such that every element of z in the extension of the virtual set is also an element of w.

We will show that

a) Newcomp is provable in ZFC + "there exists arbitrarily large subtle cardinals";
b) Newcomp is provable in ZFC + V = L + SSUB, where SSUB is what we call the scheme of subtlety;
c) Newcomp and ZFC + SSUB are mutually interpretable;

d) Newcomp is interpretable in ZFC + "there exists a subtle cardinal", but Newcomp is not provable there, assuming that the latter is consistent;
e) all of the above are provable in the weak fragment of arithmetic EFA (exponential function arithmetic);
f) Newcomp and ZFC + SSUB are equiconsistent, in the sense that their consistencies are provably equivalent in EFA.

As usual, ZFC is formulated with equality; i.e., in the language $L(\in,=)$.

The interpretation of ZFC + SSUB in Newcomp presented here (or coming out of here) takes the following form (when straightforwardly adjusted). Sets in ZFC + SSUB are interpreted to be sets in Newcomp. Membership and equality between the sets in ZFC + SSUB are interpreted as two separate relations between sets in Newcomp defined by two separate formulas with exactly two free variables (no parameters). As normally required of interpretations, the usual connectives and quantifiers are interpreted without change. Every theorem of ZFC + SSUB becomes a theorem of Newcomp when so interpreted.

There is an appropriate sense in which this interpretation is a well founded interpretation. Specifically, it is provable in Newcomp that every set has a minimal element under the above interpretation of the epsilon relation of ZFC + SSUB. We can then draw conclusions such as conservative extension results in the form: any sentence of a certain kind provable in ZFC + SSUB is provable in Newcomp. However, the statement of such results involves various coding apparatus available in Newcomp, and we do not go into this matter here. Suffice it to say that, in an appropriate sense, every arithmetical theorem of ZFC + SSUB is a theorem of Newcomp (and vice versa).

In the interpretation of Newcomp in ZFC + "there exists a subtle cardinal", the sets in Newcomp are interpreted to be some portion of the sets in ZFC + "there exists a subtle cardinal", (an initial segment, possible proper, of the constructible hierarchy), and membership between sets in Newcomp is interpreted as membership between sets in ZFC + "there exists a subtle cardinal". As normally required of interpretations, the usual connectives and quantifiers are interpreted without change. Every theorem of Newcomp becomes a theorem of ZFC + "there exists a subtle cardinal" when so interpreted.

Of course, when interpreting Newcomp in ZFC + “there are arbitrarily large subtle cardinals”, we can use the identity interpretation, since Newcomp is provable there.

The system ZFC + SSUB has logical strength a shade below that of ZFC + SUB, where SUB is “there exists a subtle cardinal”, and substantially stronger than the well studied large cardinal axioms weaker than SUB, such as the existence of Mahlo, weakly compact, or indescribable cardinals. In the well known Chart of Cardinals in [Ka94], p. 471, subtle fits strictly below $\square\ \square\ (\square)^{<\square}_{2}$, and strictly above (in logical strength) “indescribable”, well within the cardinals that are compatible with V = L.

2. SOME FORMALITIES.

We let L(□) be ordinary classical first order predicate calculus with only the binary relation symbol □ (no equality). We assume that x,y,z are distinct variables among the infinitely many variables used in L(□).

Let □ be a formula in L(□). We write z ≡ w for (□u)(u □ z □ u □ w).

NEWCOMP. Let □ be a formula of L(□) in which y,z,w are not free. (□y)(□x)(x □ y □ □) □ (□y)(□z,w)(z,w □ y □ □z ≡ w □ □[x/z] □ □[x/w] □ (□x)((□ □ x □ z) □ x □ w)).

The following definition is used in [Ba75] and [Fr01]. We say that an ordinal □ is subtle if and only if

i) □ is a limit ordinal;
ii) Let C □ □ be closed and unbounded, and for each □ < □ let $A_\square$ □ □ be given. There exists □,□ □ C, □ < □, such that $A_\square$ = $A_\square$ □ □.

It is well known that every subtle ordinal is a subtle cardinal (see [Fr01], p. 3).

We will use the following schematic form of subtlety. SSUB is the following scheme in the language of ZFC with □,=. Let □,□ be formulas, where we view □ as carving out a class on the variable x, and □ as carving out a binary relation on the variables x,y. Parameters are allowed in □,□.

if □ defines a closed and unbounded class of ordinals and □ defines a system $A_□$ □ □, for all ordinals □, then there exists □,□ □ C, □ < □, where $A_□$ = $A_□$ □ □.

This concludes the definitions that are used in the list of agenda items a) - f) at the end of section 1.

In [Fr02] we define □ to be inclusion subtle if and only if
i) □ is a limit ordinal;
ii) Let C □ □ be closed and unbounded, and for each □ < □ let $A_□$ □ □ be given. There exists □,□ □ C, □ < □, such that $A_□$ □ $A_□$.

There is a corresponding scheme SISUB (scheme of inclusion subtlety).

if □ defines a closed and unbounded class of ordinals, and □ defines a system $A_□$ □ □, for all ordinals □, then there exists □,□ □ C, □ < □, such that $A_□$ □ $A_□$.

LEMMA 2.1. An ordinal is subtle if and only if it is inclusion subtle.

Proof: This is Theorem 1.2 of [Fr02]. QED

THEOREM 2.2. SISUB and SSUB are provably equivalent in ZFC.

Proof: This is in clear analogy with Lemma 2.1. The proof is an obvious adaptation of that of Lemma 2.1. QED

In [Fr02] we define □ to be weakly inclusion subtle over □ if and only if
i) □,□ are ordinals;
ii) For each □ < □ let $A_□$ □ □ be given. There exists □ □ □ < □ such that $A_□$ □ $A_□$.

There is a corresponding scheme SWISUB (scheme of weak inclusion subtlety). This is the natural principle used to prove Newcomp in section 3 (we will also use V = L).

if □ defines a system $A_□$ □ □, for all ordinals □, then there exist arbitrarily large ordinals □ < □ such that $A_□$ □ $A_□$.

LEMMA 2.3. The least weakly inclusion subtle ordinal over □ ≥ 2, if it exists, is a subtle cardinal.

Proof: This is Theorem 1.6 of [Fr02]. QED

THEOREM 2.4. SWISUB and SSUB are provably equivalent in ZFC + □SUB.

Proof: This will be an adaptation of the proof of Lemma 2.3. We work in SWISUB + □SUB. Let a closed unbounded class C of ordinals, and $A_{\square}$ □ □, ordinals □, be appropriately given. By Theorem 2.2, it suffices to find □,□ □ C, □ < □, such that $A_{\square}$ □ $A_{\square}$. By Lemma 2.3, there is no weakly inclusion subtle ordinal. So for each □, we have a counterexample $D_{\square}$ □ □, □ < □, to □ is weakly inclusion subtle.

We now proceed exactly as in the proof of Theorem 1.6 in [Fr02]. QED

COROLLARY 2.5. ZFC + SSUB and ZFC + SWISUB are mutually interpretable and equiconsistent (in the sense that the consistency statements are provably equivalent in EFA).

Proof: The interpretation will be V if there is no subtle cardinal, and V(□) if there is a subtle cardinal and □ is the least subtle cardinal. QED

For the interpretation of ZFC + SSUB in Newcomp, we will use another modification of SSUB that is weaker than SWISUB. We call it the scheme of very weak inclusion subtlety, written SVWISUB.

if □ defines a system $A_{\square}$ □ □, for all ordinals □, then there exist 2 □ □ < □ such that $A_{\square}$ □ $A_{\square}$.

THEOREM 2.6. SVWISUB and SSUB are provably equivalent in ZFC + □SUB.

Proof: The same as for Theorem 2.4. In Theorem 1.6 of [Fr02], set □ = 2. QED

COROLLARY 2.7. ZFC + SSUB and ZFC + SVWISUB are mutually interpretable and equiconsistent (in the sense that the consistency statements are provably equivalent in EFA).

Proof: Same as for Corollary 2.5. QED

In section 5, we will interpret SVWISUB in Newcomp + Extensionality.

3. PROOF AND INTERPRETATION OF NEWCOMP.

THEOREM 3.1. Newcomp is provable in ZFC + V = L + SWISUB. In particular, it is provable in ZFC + V = L + SSUB, and interpretable in ZFC + SSUB. Newcomp is interpretable in ZFC + "there exists a subtle cardinal".

Proof: We work in ZFC + V = L + SWISUB. Let S be a proper class given by a formula with set parameters.

Suppose there exists α such that the $x \in S$ for which $S \cap x \subseteq V(\alpha)$ forms a proper class. By replacement, outside of any given set, there exists $x,y \in S$, $x \neq y$, such that $S \cap x = S \cap y$, and we have verified Newcomp for S. So we assume that for all α, $\{x \in S: S \cap x \subseteq V(\alpha)\}$ is a set.

We now construct a one-one surjective function $F:S \to On$ such that

*) for all $x,y \in S$, if $x \in y$ then $F(x) < F(y)$.

For each λ, we define a one-one partial function $F_\lambda:S \to On$ with domain $\{x \in S: S \cap x \subseteq V(\alpha_\lambda)\}$ and range an ordinal, obeying property *) on its domain. Here α_λ will be a strictly increasing ordinal valued function of λ. Moreover, each of the functions F_λ will be strictly extended by the later ones. By the first paragraph in this proof, these domains are sets.

We start the construction with $F_0 = 0$ and $\alpha_0 = 0$. Suppose F_β has been defined for all $0 \leq \beta < \lambda$, according to the previous paragraph, where each function is extended by the later ones. Define If λ is a limit ordinal, then take F_λ to be the union of the F_β, $\beta < \lambda$, and α_λ to be the sup of the α_β, $\beta < \lambda$. Suppose $\lambda = \beta+1$. Define α_λ to be the least ordinal $> \alpha_\beta$ such that $\{x \in S: S \cap x \subseteq V(\alpha_\beta)\} \neq \{x \in S: S \cap x \subseteq V(\alpha_\lambda)\}$. Obviously α_λ exists since the left side is a set and S forms a proper class. Define F_λ to be any one-one extension of F_β, with domain $\{x \in S: S \cap x \subseteq V(\alpha_\lambda)\}$, which is onto an ordinal, and where all new values of F_λ are greater than any values of the old F_β.

Finally, define $F:S \to On$ as the union of the F_λ, over all ordinals λ. Since the α_λ are strictly increasing, every element of S lies in the domain of some F_λ. Since each F_λ is one-one, clearly $F_{\square}$ is one-one. It is also clear that F is onto.

We now verify condition *). Let x,y □ S, x □ y. Let □ be least such that y □ dom($F_{\square}$). Write □ = □+1. Then y □ S, S □ y □ V($\square_{\square+1}$). Note that x □ S, S □ x □ V(□), for some □ < $\square_{\square+1}$. By the definition of $\square_{\square+1}$, we see that x □ S, S □ x □ V($\square_{\square}$). Hence x □ dom($F_{\square}$), y □ dom($F_{\square}$). By the construction of $F_{\square+1}$, we have F(x) < F(y).

It is clear that we cannot make this construction in a definable manner; e.g., S may be all of V. However, V = L provides the needed definable well ordering of V to make this construction go through.

We now define $A_{\square}$ □ □ for all ordinals □. Recall that F is one-one onto. Take $A_{\square}$ = {F(x): x □ S □ F^{-1}□}, where F^{-1}□ is the inverse image of the function F at the point □. Suppose x □ S □ F^{-1}□. By condition *) in the construction of F, we have F(x) < F(F^{-1}□) = □. Thus we see that $A_{\square}$ □ □.

By SWISUB, let □ < □ be arbitrarily large ordinals such that $A_{\square}$ □ $A_{\square}$. Then {F(x): x □ S □ F^{-1}□} □ {F(x): x □ S □ F^{-1}□}. Since F:S □ On is one-one, we have {x: x □ S □ F^{-1}□} □ {x: x □ S □ F^{-1}□}, S □ F^{-1}□ □ S □ F^{-1}□. Also, since F is one-one, F^{-1}□ and F^{-1}□ can be taken to lie outside any given set and are distinct elements of S. We have thus verified the escape clause in Newcomp.

For the second claim, obviously SWISUB is derivable from SSUB, and ZFC + V = L + SSUB is interpretable in ZFC + SSUB by relativizing to L. For the third claim, obviously ZFC + SSUB is interpretable in ZFC + "there exists a subtle cardinal" by using the model (V(□),□) of SSUB, where □ is the least subtle cardinal. QED

THEOREM 3.2. Newcomp is provable in ZFC + "there are arbitrarily large subtle cardinals".

Proof: Assume that the virtual set S does not form a set. As in the proof of Theorem 3.1, we can assume that for all □, {x □ S: S □ x □ V(□)} is a set.

We cannot build all of the one-one partial functions $F_{\square}$:S □ On that were constructed in the proof of Theorem 3.1 with the aid of V = L. But we can define the $\square_{\square}$, for all ordinals □, easily within ZFC. Now let w be the given set in the escape clause of Newcomp, and □ be a subtle cardinal such that w □ V(□). In ZFC, we can well order the set {x □ S: S □ x □ V($\square_\square$)}. We then use this well ordering in order to define the partial functions $F_\square$:S □ On as in the proof of Theorem 3.1, for □ □ □. The final function $F_\square$ is one-one from a subset of S onto an ordinal ≥ □, and obeys x □ y □ $F_\square$(x) < $F_\square$(y).

As in the proof of Theorem 3.1, for □ < □, define $A_\square$ = {F(x): x □ S □ F^{-1}□}. As before, $A_\square$ □ □. Since □ is a subtle cardinal, there are arbitrarily large □ < □ < □ such that $A_\square$ □ $A_\square$. Again, we have S □ F^{-1}□ □ S □ F^{-1}□ for each such choice of □ < □ < □. Note that there are □ choices of such pairs {□,□}, where these unordered pairs are pairwise disjoint. Thus there are □ choices {F^{-1}□,F^{-1}□}, where these unordered pairs are pairwise disjoint. Hence for one of these pairs, both elements lie outside w. QED

THEOREM 3.3. Newcomp is not provable in ZFC + V = L + "there exists a subtle cardinal", assuming the latter is consistent. Newcomp is not provable in ZFC together with any existential sentence in the language of set theory with equality and the power set operation, with bounded quantifiers allowed, assuming the latter is consistent.

Proof: Let M be a model of ZFC + V = L + "there exists a subtle cardinal". Let □ be any subtle cardinal in the sense of M. Let M' be the same as M if M satisfies "there is no strongly inaccessible cardinal > □"; otherwise M' is the restriction of M to the sets of rank less than the first inaccessible cardinal > □ in the sense of M. Then M' satisfies ZFC + V = L.

In M', we construct a definable assignment $A_\square$ □ □, ordinals □, as follows. If □ > □ is a successor ordinal, let $A_\square$ = {0,□-1}. If □ > □ is a limit ordinal of cofinality < □, let $A_\square$ be an unbounded subset of □ of order type cf(□) whose first two elements are 1,cf(□). If □ > □ is the next cardinal after the cardinal □, let $A_\square$ = {2} □ [□,□). For □ □ □, let $A_\square$ = □. Note that the strict sup of every $A_\square$ is □.

We claim that □ < □ < □ and $A_\square$ □ $A_\square$ is impossible. If this holds then either □,□ are both successor ordinals, or □,□ are both nonregular limit ordinals, or □,□ are both successor cardinals. The first and third cases are dispensed with immediately. For the second case, we have cf(□) = cf(□) and $A_\square$,$A_\square$ have order type cf(□) < □. Since $A_\square$

has strict sup □ and $A_\square$ has strict sup □ > □, this is impossible.

We can now convert this construction to a counterexample to Newcomp. The conversion is according to the proof of Theorem 2.5, iii) □ i), in [Fr02], with □ = On and □ = □. For the convenience of the reader, we repeat the relevant part from there with □ = On and □ = □, in the next three paragraphs, working within M′.

We define f:On □ V as follows. f(□) = {f(□): □ □ $A_\square$}. By transfinite induction, each f(□) has rank □. Let S be the range of f. Then S is a transitive set of rank □, where S has exactly one element of each rank < □.

We claim that f(□) □ f(□) □ $A_\square$ □ $A_\square$. To see this, suppose f(□) □ f(□). Then {f(□): □ □ $A_\square$} □ {f(□): □ □ $A_\square$}. Since f is one-one, $A_\square$ □ $A_\square$.

Suppose there exists a 2 element chain {x □≠ y} □ S, rk(x),rk(y) ≥ □. Since x □≠ y, we have rk(x) □ rk(y), and hence □ □ rk(x) < rk(y). Write rk(x) = □ and rk(y) = □. Then x = f(□) and y = f(□). Therefore $A_\square$ □ $A_\square$. Since □ □ □ < □, this contradicts the choice of ($A_\square$).

We now claim that S provides a counterexample to Newcomp in M′. Clearly S is a proper class in M′. However, there are no x,y □ S, x ≠ y, outside of V(□), with S □ x □ y.

For the second claim, we again begin with a model M of ZFC + (□x_1,…,x_k)(□), where □ is a bounded formula in □,=, and the power set operation. Fix a limit cardinal □ in M such that (□x_1,…,x_k □ V(□))(□) holds in M. Choose a well ordering X of V(□) in M, and pass to the submodel L[X] of M. Finally, let M′ is L[X] if there is no strongly inaccessible cardinal above □ in the sense of L[X]; otherwise M′ is the result of chopping off at the first strongly inaccessible cardinal above □ in the sense of L[X]. Then M′ satisfies ZFC + (□x_1,…,x_k)(□). Now repeat the above argument for the first claim to show that M′ does not satisfy Newcomp. QED

4. NEWCOMP + EXT IN NEWCOMP.

In this section, we give an interpretation of Newcomp + Extensionality in Newcomp.

Let us be precise about what these two theories are. Recall that Newcomp is formulated as a scheme in L(□); i.e., without equality. See section 2.

However, we formulate Newcomp + Ext in L(□,=); i.e., with equality. This is convenient for section 5. The axioms of Newcomp + Ext, in addition to the logical axioms for L(□,=), including the equality axioms for L(□,=), are

EXTENSIONALITY. (□z)(z □ x □ z □ y) □ x = y.

NEWCOMP. Let □ be a formula of L(□,=) in which y,z,w are not free. (□y)(□x)(x □ y □ □) □ (□y)(□z,w)(z,w □ y □ □z = w □ □[x/z] □ □[x/w] □ (□x)((□ □ x □ u) □ x □ v)).

We first give the interpretation (and its verification), where a few points remain to be handled formally, as noted. We follow this by a formal treatment of the few remaining points.

In Newcomp, we define x ≡ y if and only if (□z)(z □ x □ z □ y).

Sitting in Newcomp, we call a set x extensional if and only if for any finite sequence x_1 □ x_2 □ ... □ x_k = x, k ≥ 2, for all y, we have x_1 ≡ y □ y □ x_2. This is a place where we need to fill in some details, because we don't have natural numbers and finite sequences readily available in Newcomp.

We interpret the sets for Newcomp + Ext to be the extensional sets. Membership is interpreted as membership. Equality is interpreted as ≡ in Newcomp.

We first check that the interpretations of the equality axioms of Newcomp + Ext are provable in Newcomp.

For the equality axioms, it remains to prove the interpretations of

x = y □ (z □ x □ z □ y)
x = y □ (x □ z □ y □ z).

Let x,y,z be extensional. Assume the interpretation of x = y. This is (□w)(w □ x □ w □ y). Obviously z □ x □ z □ y. We need to check that x □ z □ y □ z. Assume x □ z. We apply the definitional of extensional to x □ z. Since x ≡ y, we have y □ z.

We now check that the interpretation of Ext is provable in Newcomp.

Extensionality reads

(□z)(z □ x □ z □ y) □ x = y.

The first crucial Lemma we need is that every element of an extensional set is extensional. This is informally clear, but we will give a careful proof in Newcomp later.

Let x,y,z be extensional. Assume the interpretation of (□z)(z □ x □ z □ y). Then for all extensional z, z □ x □ z □ y. Now let z be arbitrary. If z □ x then z is extensional, and hence z □ y. If z □ y then z is extensional, and hence z □ x. Thus we conclude that x ≡ y, which is the interpretation of x = y.

We now come to the Newcomp axiom scheme. We want to interpret the universal closure of the Newcomp axiom

1) {x: □} forms a set or, outside any given set, there exists y,z, □y = z, □[x/y], □[x/z], with (□x □ y)(□ □ x □ z)

where y,z are not free in □. Let $v_1,...,v_k$ be a complete list of the free variables in this Newcomp axiom, without repetition.

The interpretation of the universal closure of the above Newcomp axiom is the sentence

2) For all extensional $v_1,...,v_k$, (□ extensional y)(□ extensional x)(x □ y □ □*) or, for any extensional w, there exist extensional y,z □ w, with □y ≡ z, □*[x/y], □*[x,z], and (□ extensional x □ y)(□* □ x □ z)

where □* is the result of relativizing all quantifiers in □ to the extensional sets, and replacing = with ≡.

We can rewrite 2) in Newcomp as

3) For all extensional $v_1,...,v_k$, (□ extensional y)(□ extensional x)(x □ y □ □*) or, for any extensional w, there

exist extensional y,z □ w, with □y ≡ z, □*[x/y], □*[x,z], and (□x □ y)(□* □ x □ z).

We will actually prove the following strengthening of 3) in Newcomp. Let □′ be □* □ x is extensional.

4) For all extensional $v_1,...,v_k$, (□ extensional y)(□x)(x □ y □ □′) or, for any w, there exists y,z □ w, with □y ≡ z, □′[x/y], □′[x,z], and (□x □ y)(□′ □ x □ z).

Let us verify that 4) □ 3) in Newcomp. Clearly (□x)(x □ y □ □′) implies (□ extensional x)(x □ y □ □*) since if x is extensional, then □′ □ □*. Also, any such y,z in 4) are extensional because of the construction of □′. In addition, (□x □ y)(□′ □ x □ z) implies (□x □ y)(□* □ x □ z). To see this, assume (□x □ y)(□′ □ x □ z), and let x □ y and □*. Since y is extensional, x is extensional, and so □′, and hence x □ z.

Note that 4) is almost in the form of the universal closure of a Newcomp axiom. The problem is the displayed existential quantifier. In Newcomp we obviously have

5) For all extensional $v_1,...,v_k$, (□y)(□x)(x □ y □ □′) or, for any w, there exists y,z □ w, with □y ≡ z, □′[x/y], □′[x,z], and (□x □ y)(□′ □ x □ z).

It remains to prove in Newcomp,

6) For all extensional $v_1,...,v_k$, (□y)(□x)(x □ y □ □′) □ (□ extensional y)(□x)(x □ y □ □′).

We first claim that

7) For all extensional $v_1,…,v_k$, (□x,u)(x ≡ u □ (□′ □ □′[x/u]))

where u is not free in □′ and not among $v_1,…,v_k$.

To prove 7), first note that we have

8) For all extensional $v_1,…,v_k$, (□x,u)(x = u □ (□ □ □[x/u]))*

since this is the interpretation of a theorem of logic, and we have secured the interpretation of the equality axioms. Hence we have

9) For all extensional $v_1,\ldots,v_k$, (□x,u)(x ≡ u □ (□* □ □*[x/u])).

For the next paragraph, we will need a second crucial Lemma. If x is extensional and x ≡ y, then y is extensional. This is informally clear, but we will give a careful proof in Newcomp later.

To finish the proof of 7), let $v_1,\ldots,v_k$ be extensional and let x ≡ u. If □′ then x is extensional, and so u is extensional. Also □*, and so by 9), □*[x/u]. Hence □′[x/u]. For the other direction, if □′[x/u] then u is extensional, and so x is extensional. Also □*[x/u], and so □*. Hence □′. This verifies 7).

We now complete the verification of 6) in Newcomp. Let $v_1,\ldots,v_k$ be extensional, and let (□x)(x □ y □ □′). Then y is a set of extensional sets. By 9), y is a set of extensional sets, where any set equivalent to an element of y is an element of y. It is informally clear that y itself is extensional.

Thus we need a third crucial Lemma. If x is a set of extensional sets, and (□y,z)((y □ x □ y ≡ z) □ z □ x), then x is extensional.

This completes the verification of the interpretation of Newcomp + Ext in Newcomp.

We now formally treat extensional sets and the three crucial Lemmas used above. We will work entirely within Newcomp.

LEMMA 4.1. Separation holds. I.e., (□y)(□x)(x □ y □ (x □ a □ □)), where y is not free in □.

Proof: {x: x □ a □ □} does not have two inequivalent elements outside a. So it forms a set. QED

By Lemma 4.1, we can use the notation y ≡ {x: x □ a □ □} for (□x)(x □ y □ (x □ a □ □)}. This determines x up to equivalence.

LEMMA 4.2. For any x,y, there exists z such that (□w)(w □ z □ (w □ x w ≡ y)).

Proof: {w: w □ x   w ≡ y} does not have two inequivalent elements outside x. So it forms a set. QED

By Lemma 4.2, we can use the notation z ≡ x □ {y} for (□w)(w □ z □ (w □ x   w ≡ y)). This determines z up to equivalence. We also use the notation z ≡ {y} for (□w)(w □ z □ w ≡ y), which also determines z up to equivalence.

Recall that, informally in Newcomp, a set x is extensional if and only if for any finite sequence x_1 □ x_2 □ ... □ x_k = x, k ≥ 2, for all y, if x_1 ≡ y then y □ x_2.

To formalize this, we use the notion of an x-set, for any set x.

An epsilon closed subset of a set b is w □ b such that any element of an element of w that lies in b lies in w.

An x-set is a set b such that

i) All sets equivalent to x lie in b;
ii) Every epsilon closed subset of b containing all sets equivalent to x, is equivalent to b.

We say that x is 1-extensional if and only if every set equivalent to an element of x is an element of x.

We say that x is extensional if and only if every element of every x-set is 1-extensional.

We could have used a simpler notion of x-set: x □ b, and every epsilon closed subset of b containing x as an element is equivalent to b. However, the proofs with the present notion are simpler.

LEMMA 4.3. Let b be an x-set. Assume y ≡ {x}. Then any c ≡ b □ {y} is a y-set.

Proof: Clearly every set equivalent to y lies in c. Let z be an epsilon closed subset of c containing all sets equivalent to y. Then all sets equivalent to x lie in z. Let w ≡ z □ b. We claim that w is an epsilon closed subset of b containing all sets equivalent to x. To see this, let v □ u □ w, v □ b. Then v □ u □ z and v □ c, and so v □ z, and hence v □ w. This establishes the claim.

Since b is an x-set, w ≡ b. It remains to show that z contains all elements of c that are not in b. I.e., z contains all sets equivalent to y. This we already have. QED

LEMMA 4.4. Every element of an extensional set is extensional.

Proof: Let x □ y, where y is extensional. Every element of every y-set is 1-extensional. Let u be an element of some x-set b. Let y ≡ {x}. Let c ≡ b □ {y}. By Lemma 4.3, c is a y-set and u □ c. Hence u is 1-extensional. QED

LEMMA 4.5. If x ≡ y then every x-set is a y-set.

Proof: Let b be an x-set. We verify that b is a y-set. Clearly every set equivalent to y lies in b. Suppose z is an epsilon closed subset of b containing all sets equivalent to y as an element. Then z is an epsilon closed subset of b containing all sets equivalent to x as an element, and so z ≡ b. QED

LEMMA 4.6. If x is extensional and x ≡ y, then y is extensional.

Proof: Suppose z is an element of some y-set. By Lemma 4.5, z is an element of some x-set, and so z is 1-extensional. QED

LEMMA 4.7. Let b be an x-set and y be an element of an element of b. Then there is an x-set c such that y □ c.

Proof: Let b,x,y be as given. Let y □ z □ b. Let d ≡ b □ {y}, and c ≡ {z □ d: z □ b   z is an element of an element of b}. Clearly b □ c and y □ c.

Suppose w is an epsilon closed subset of c containing all sets equivalent to x. Let w′ ≡ w □ b. We claim that w′ is an epsilon closed subset of b containing all sets equivalent to x. To see this, let v □ u □ w′, v □ b. Then u,v □ c, and so v □ w′. This establishes the claim. Hence w′ ≡ b.

It remains to verify that c □ w. The elements of c that are not in b are elements of elements of b. Since w is an epsilon closed subset of c, the elements of c not in b lie in w. QED

LEMMA 4.8. Let b ≡ {u}. Then b is a u-set.

Proof: Clearly b contains all sets equivalent to u. Let z be an epsilon closed subset of b containing all sets equivalent to u. Then obviously z ≡ b. QED

LEMMA 4.9. Let b be an x-set, y □ b, □y ≡ x. There exists z □ x such that y is an element of some z-set.

Proof: Let w be the set of all elements of b that lie in some z-set, z □ x, together with the sets equivalent to x. It suffices to show that b □ w. For this it suffices to show that w is an epsilon closed subset of b containing all sets equivalent to x. I.e., w is an epsilon closed subset of b.

Let u □ v □ w, u □ b. First suppose v ≡ x. Let c ≡ {u}. By Lemma 4.8, c is a u-set, u □ x, with u □ c. Hence u □ w.

Now suppose □v ≡ x. Then v lies in some z-set, z □ x. By Lemma 4.7, since u □ v, we see that u lies in some z-set, z □ x. Hence u □ w. QED

LEMMA 4.10. Let x be a 1-extensional set of extensional sets. Then x is extensional.

Proof: Let x be as given, and let y be an element of some x-set b. If y ≡ x then y is 1-extensional. If □y ≡ x then by Lemma 4.9, y is an element of some z-set with z □ x. So y is 1-extensional. QED

THEOREM 4.11. Newcomp + Extensionality is interpretable in Newcomp.

Proof: By the argument given before the Lemmas. The three crucial Lemmas that were cited are Lemmas 4.4, 4.6, and 4.10, respectively.

5. ZFC + V = L + SVWISUB IN NEWCOMP.

In light of Theorem 4.11, we have only to interpret ZFC + V = L + SVWISUB in Newcomp + Extensionality. Throughout this section, we work within Newcomp + Ext.

We will use the term "virtual set" whenever we have a definable property of sets presented as $\{x: \varphi\}$, with parameters allowed.

Until the proof of Lemma 5.39 is complete, we will work entirely within the system Newcomp + Ext.

We will use abstraction notation with braces to indicate virtual sets. We say that these expressions exist to indicate that they form sets.

All lower case letters will represent sets (not virtual sets).

It is convenient to adopt the following terminology. Let A be a virtual set. An expansion in A consists of two elements $x,y \in A$ such that $x \neq y$ and $x \cap A \subseteq y$. We say that an expansion lies outside z if and only if neither component is a member of z. We say that an expansion meets z if and only if at least one component is a member of z.

Obviously, we can reformulate Newcomp in these two ways:

If all expansions in a virtual set meet some given set, then the virtual set forms a set.

If no expansion in a virtual set lies outside some given set, then the virtual set forms a set.

LEMMA 5.1. Any virtual subset of a set forms a set. The empty set exists. The intersection of any virtual set with a set forms a set. For all x,y, $x \cup \{y\}$ exists. For all $k \geq 0$ and $x_1,\ldots,x_k$, $\{x_1,\ldots,x_k\}$ exists.

Proof: The first claim is obvious since all expansions of the virtual subset lie in the set. The second and third claims follow immediately from the first claim. For the fourth claim, note that every expansion in $x \cup \{y\}$ meets x. The fifth claim follows from the second and fourth claims. QED

LEMMA 5.2. Let A be a transitive virtual set. The expansions in A are exactly the $x,y \in A$ with $x \subseteq\neq y$.

Proof: For the forward direction, let $x,y \in A$, $x \neq y$, $x \cap A \subseteq y$. Then $x \subseteq y$ by the transitivity of A. The reverse direction is immediate. QED

We write <y,z> = {{y},{y,z}}.

LEMMA 5.3. x □ {{y}: y □ x} exists. x □ {{y,z}: y,z □ x} exists. x □ {{y,z}: y,z □ x} □ {{{y}}: y □ x} exists. x □ {{y,z}: y,z □ x} □ {<y,z>: y,z □ x} exists.

Proof: Observe that all four virtual sets are transitive, and so we can apply Lemma 5.2. For the first claim, note that no expansions lie outside x. For the second claim, note that no expansions lie outside x □ {{y}: y □ x}. For the third claim, note that no expansions lie outside x □ {{y,z}: y,z □ x}. For the fourth claim, we show that no expansions lie outside x □ {{y,z}: y,z □ x} □ {{{y}}: y □ x}. Let <a,b>,<c,d> be an expansion in x □ {{y,z}: y,z □ x} □ {<y,z>: y,z □ x} outside x □ {{y,z}: y,z □ x} □ {{{y}}: y □ x}. Then {{a},{a,b}} □ {{c},{c,d}}, a ≠ b, c ≠ d. Hence the left and right sides have two elements, and each consist of a set with one element and a set with two elements. So {a} = {c} and {a,b} = {c,d}. Hence a = c and b = d. QED

We write x•x for {<y,z>: y,z □ x}. A binary relation on x is a subset of x•x. We often use R for binary relations, and write R(a,b) for <a,b> □ R.

Note that by Lemma 5.1, all virtual binary relations on x exist (as sets). Using (x•x)(x•x), we can simulate all virtual 4-ary relations on x as sets in the obvious way. We can continue doubling the arity in this way, thus making available all Cartesian powers of x by obvious simulation. This is very powerful, in light of separation (Lemma 5.1), and means that, in an appropriate sense, we have full second order logic over (x,□) at our disposal.

We caution the reader that the existence of x•y is apparently not available in Newcomp + Ext. Even the more fundamental x □ y is apparently not available in Newcomp + Ext.

We say that x is well founded if and only if for all nonempty y □ x, there exists z □ y such that z has no elements in common with y.

We define W(x) if and only if x is transitive and well founded.

We wish to extract an "ordinal" out of the x with W(x). Of course, we can't construct anything like von Neumann ordinals. Instead, we develop the usual rank comparison relation on x, and use its equivalence classes to simulate ordinals.

We define C(R,x,y) if and only if

i) W(x), W(y), and x•y exists;
ii) R □ x•y;
iii) for all z □ x and w □ y, R(z,w) if and only if (□a □ z)(□b □ w)(R(a,b)).

LEMMA 5.4. There is at most one R □ x•y with C(R,x,y).

Proof: Let C(R,x,y), C(R′,x,y), R ≠ R′. Then W(x), W(y), and x•y exists. Let z □ x be epsilon least such that (□w □ y)(R(z,w) □ □R′(z,w)). Then (□a □ z)(□b □ y)(R(a,b) □ R′(a,b)). Hence (□a □ z)(□b □ w)(R(a,b)) □ (□a □ z)(□b □ w)(R′(a,b)). Hence R(z,w) □ R′(z,w). QED

LEMMA 5.5. Let C(R,x,y) and z □ y be transitive. Then C(R □ x•z,x,z).

Proof: By Lemma 5.1, W(z) and x•z exists. Let u □ x and v □ z. Then R(u,v) if and only if (□a □ u)(□b □ v)(R(a,b) □ <a,b> □ x•z). QED

LEMMA 5.6. Let C(R,x,y), C(R′,x,z). Then (□z □ x)(□w □ y □ z)(R(z,w) □ R′(z,w)).

Proof: By Lemma 5.5, C(R □ x•z,x,z), C(R′ □ x•z,x,z). By Lemma 5.4, R □ x•z = R′ □ x•z. QED

LEMMA 5.7. Let W(x), W(y), and x•y exist. Let A be a virtual set of transitive subsets of y s•uch that for all z □ A, there exists R with C(R,x,z). Then there exists R′ with C(R′,x,□A).

Proof: First of all, □A is a transitive subset of y, so W(□A). By Lemma 5.4, for all z □ A, there exists a unique R_z with C(R_z,x,z). Since these R_z are subsets of x•y, we can set R′ to be the union of the R_z, and know that R′ exists. Let a □ x and b □ □A. We verify that R′(a,b) □ (□c □ a)(□d □ b)(R′(a,b)). Let b □ z □ A. The $R_z(a,b)$ □ (□c □ a)(□d □ b)($R_z(a,b)$). By Lemam 5.6, for all a □ x and b □ z, $R_z(a,b)$ □ R′(a,b). QED

LEMMA 5.8. Let W(x), W(y), and x•y exist. There exists a unique R such that C(R,x,y). In particular, if W(x) then there exists a unique R such that C(R,x,x).

Proof: Uniqueness is by Lemma 5.4. For existence, let x,y be as given. We first claim that every z ∈ y is an element of a transitive set w ⊆ y such that for some R, C(R,x,w). Suppose this is false, and let z ∈ y be an epsilon minimal counterexample.

Now every w ∈ z is an element of a transitive set u ⊆ y such that for some R, C(R,x,u). Let z* be the union of all transitive u ⊆ y such that for some R, C(R,x,u). By Lemma 5.7, z ⊆ z* and z* is a transitive subset of y and for some R, C(R,x,z*). Fix such an R. Now z* ∪ {z} is a transitive subset of y. Define R′(a,b) ↔ (R(a,b) ∨ (b = z ∧ (∀c ∈ a)(∃d ∈ b)(R(c,d)))). Then C(R′,x,z* ∪ {z}). This contradicts the choice of z.

We have thus shown that every z ∈ y is an element of a transitive set w ⊆ y such that for some R, C(R,x,w). To complete the proof, apply Lemma 5.7.

The second claim is from the first claim and Lemma 5.3. QED

LEMMA 5.9. Let C(R,x,x). Then R is reflexive, transitive, connected.

Proof: Let C(R,x,x). Suppose R is not reflexive, and let y ∈ x be epsilon minimal such that ¬R(y,y). Then R(x,x) is immediate.

Suppose R is not transitive, and let y,z,w be a counterexample to transitivity, with y chosen to be epsilon minimal. Assume (∀a ∈ y)(∃b ∈ z)(R(a,b)), (∀a ∈ z)(∃b ∈ w)(R(a,b)). Let a ∈ y. Fix b ∈ z, R(a,b). Fix c ∈ w, R(b,c). By the minimality of y, R(a,c).

Suppose R is not connected, and let y,z be a counterexample to connectivity with y chosen to be epsilon minimal. Now ¬(∀a ∈ y)(∃b ∈ z)(R(a,b)), ¬(∀a ∈ z)(∃b ∈ y)(R(a,b)). Let a ∈ y, (∀b ∈ z)(¬R(a,b)). Let b ∈ z, (∀c ∈ y)(¬R(b,c)). Then ¬R(a,b), ¬R(b,a). This contradicts the minimality of y. QED

Let W(x). We write □(x) for the unique R such that C(R,x,x). We write ≡(x) for the relation □(x)(a,b) □ □(x)(b,a). We write <(x) for the relation <(x)(a,b) □ □□(x)(b,a). Analogously for >(x).

The mere appearance of any of the four expressions defined in the previous paragraph will be taken to imply that W(x).

LEMMA 5.10. If W(x), a □ b □ x, then <(x)(a,b). If W(x) and x is nonempty then x has the epsilon minimum element ∅ and □(x) has the unique minimum element ∅. If W(x) and x has more than one element then □(x) without ∅ has the unique minimum element {∅}.

Proof: Let a,b be a counterexample where a is epsilon minimal. We first show that □(x)(a,b). Let c □ a. Then <(x)(c,a). Hence (□c □ a)(□d □ b)(□(x)(c,d)). Therefore □(x)(a,b).

Now suppose □(x)(b,a). Then (□c □ b)(□d □ a)(□(x)(c,d)). Let d □ a, □(x)(a,d). This contradicts the epsilon minimality of a.

Now let W(x) and x be nonempty. Let u be an epilson minimal element of x. Since x is transitive, u = ∅. Obviously, ∅ is □(x) minimum. Also if v □ x is nonempty, then <(x)(∅,x).

Now assume that x has more than one element. Let v be an epsilon minimal element of x\{∅}. Since x is transitive, the only element of v is ∅, and so v = {∅}. Also □({∅},u) for all u □ x\{∅}, since □(∅,w) for all w □ x. Let b □ x, b ≠ ∅,{∅}. Let c □ b, c ≠ ∅. Then □(x)(b,{∅}) is impossible since □□(x)(c,∅). Hence the uniqueness of {∅} as a minimum element of □(x) without ∅ is established. QED

LEMMA 5.11. Assume W(x), a,b □ x. <(x)(a,b) if and only if (□d □ b)(□(x)(a,d)).

Proof: Let a,b □ x be a counterexample to the forward direction, where a is epsilon minimal. We have <(x)(a,b), □(□d □ b)(□(x)(a,d)).

Now □□(x)(b,a). Hence □(□d □ b)(□c □ a)(□(x)(b,a)). Fix d □ b such that (□c □ a)(<(x)(c,d)). We need only show that □(x)(a,d). Let c □ a. Then <(x)(c,d). By the epsilon minimality of a, let □(x)(c,e), e □ d. We have thus shown that (□c □ a)(□e □ d)(□(x)(c,e). This verifies that □(x)(a,d).

For the reverse direction, let d □ b, □(x)(a,d). By Lemma 5.10, <(x)(d,b). Hence <(x)(a,b). QED

LEMMA 5.12. If W(x) then <(x) is well founded. □(x) is connected.

Proof: Let u be a nonempty subset of x. Let y □ x be epsilon minimal such that (□b □ u)(≥(x)(y,b)). Let <(x)(z,y). We claim that z □ u. Suppose z □ u. By Lemma 5.11, let □(x)(z,w), w □ y. This contradicts the epsilon minimality of y.

We have shown that no z □ u has <(x)(z,y). We would be done if y □ u. However, let b □ u, ≥(x)(y,b). We can rule out >(x)(y,b) since by Lemma 5.11, some element of y would dominate b under ≥(x). Hence ≡(x)(y,b). Therefore b □ u is <(x) minimal as required.

Since C(□(x),x,x), by Lemma 5.9, □(x) is connected. QED

This completes our treatment of "ordinals" in Newcomp + Ext. Specifically, we use the □(x), <(x), ≡(x), for x with W(x).

A major difficulty arises because we apparently cannot prove the existence of x □ y, even if W(x) and W(y). Because of this, we don't have flexibility in constructing relations between different sets.

We have discovered a way of comparing □(x) and □(y) for at least the relevant x,y, even though we don't have x □ y. This will be good enough for our purposes.

An x-system is an S □ x•x such that

i) W(x);
ii) (S(a,b) □ ≡(x)(a,c) □ ≡(x)(b,d)) □ S(c,d);
iii) S(a,b) □ <(x)(b,a);
iv) the least strict upper bound of the b's such that S(a,b) is given by a;
v) for all a,b, if <(x)(a,b) and (□y)(S(a,y) □ S(b,y)), then a = ∅ or a = {∅}.

In order to get a clear understanding of x-systems, look at the cross sections {b: S(a,b}}. These cross sections are essentially sets of “ordinals” with “strict sup” a. They have the property that the only proper inclusions between them are the cross sections of ∅,{∅}, which are the “0” and “1” of <(x). In other words, from the point of view of full set theory, an x-system is a counterexample to “rk(x) is weakly inclusion subtle over 2”, in the sense of [Fr02], using the elements of x under the equivalence relation “having the same rank”, where the counterexample also obeys Lemma 2.3 of [Fr02].

In Theorem 2.5 of [Fr02], we use such a counterexample to weak inclusion subtlety over 2 to construct a transitive set where the only proper inclusions among elements have left side ∅ or {∅}. We want to carefully perform this construction here in Newcomp + Ext.

We define D(S,f) if and only if

i) S is an x-system for some necessarily unique x;
ii) f is a univalent set of ordered pairs (function) with domain x;
iii) for all y □ x, f(y) = {f(z): S(y,z)}.

We will need the following technical modification. D(S,f,a) if and only if

i) S is an x-system for some necessarily unique x;
ii) a □ x;
iii) f is a univalent set of ordered pairs (function) with domain {b: □(x)(b,a)};
iv) □(x)(y,a) □ f(y) = {f(z): S(y,z)}.

LEMMA 5.13. Fix D(S,f,a) and D(S,g,b), where S is an x-system. f,g agree on their common domain. If c □ x then the restriction h of f to {d: □(x)(d,c)} has D(S,h,c). f is one-one in the sense that for all y,z □ dom(f), if <(x)(y,z) then f(y) ≠ f(z). For all y,z □ dom(f), f(y) = f(z) □ ≡(y,z). The range of f is a virtual transitive set. For all y,z □ dom(f), if f(y) □ f(z) then S_y □ S_z and □(x)(y,z). All proper inclusions among elements of the range of f have left side ∅ or {∅}.

Proof: Let S be an x-system. Suppose f ≠ g, and let y be <(x) minimal such that f(x) ≠ g(x). But f(y) = {f(z): <(x)(z,y)} = {g(z): <(x)(z,y)} = g(y).

For the second claim, the restriction h is a subset of f, and so exists. D(S,h,c) is immediate by inspection.

For the third claim, let y,z □ dom(f) be such that <(x)(y,z), f(y) = f(z), where y is chosen to be <(x) minimal. By the strict sup condition, let S(z,w), □(y,w). Then f(w) □ f(z), and so f(w) □ f(y). Let S(y,u), f(w) = f(u). Then <(x)(u,y), □(y,w), and so the minimality of y is violated.

The fourth claim follows immediately from the third claim.

The fifth claim is immediate from clause iv).

For the six claim, assume {f(u): S(y,u)} □ {f(u): S(z,u)}. Since f is appropriately one-one, we have {u: S(y,u)} □ {u: S(z,u)}. By the strict sup condition, □(x)(y,z).

For the seventh claim, suppose f(y) □≠ f(z). By the sixth claim, <(x)(y,z) and S_y □≠ S_z. By clause v) in the definition of x-system, we have y = ∅ or y = {∅}. Hence f(y) = ∅ or f(y) = {∅}. QED

Let S be an x-system. We use S* for the virtual set of all sets that occur in the range of some f with (□a)(D(S,f,a)).

LEMMA 5.14. Let S be an x-system. Then S* is transitive and all proper inclusions among elements of S* have left side ∅ or {∅}. x □ S* exists.

Proof: The transitivity of S* follows from the transitivity of the range of each relevant f. Let u,v □ S*, where u,v are in the ranges of f,g, and D(S,f,a), D(S,g,b). By the comparability of a,b under □(x) and lemma 5.13, we have f □ g or g □ f. Hence any proper inclusion among elements of S* is a proper inclusion among some relevant h. So by the last claim of Lemma 5.13, the proper inclusion must have left side ∅ or {∅}.

For the second claim, note that every expansion in x □ S* meets x □ {∅} □ {{∅}}. The latter forms a set by Lemma 5.1. Hence x □ S* exists. QED

According to Lemma 5.3, (x □ S*)•(x □ S*) exists, and in fact we can simulate all Cartesian powers of x □ S*, and hence also simulate full second logic over (x □ S*,□).

LEMMA 5.15. Suppose S is an x-system, b □ x, and for all a with <(x)(a,b), there exists f such that D(S,f,a). Then there exists g such that D(S,g,b).

Proof: Let S,x,b be as given. By Lemma 5.13, for each a with <(x)(a,b), there is a unique f_a such that D(S,f_a,a), and furthermore, these f_a agree on their common domains. Clearly b is either the minimum of <(x), a successor in <(x), or a limit in <(x). If b is the minimum of <(x) then b = ∅ by Lemma 5.10, in which case set g = {<∅,∅>}.

Assume b is the successor of c in <(x). We need to extend the function g_c to be defined on {d: ≡(x)(d,b)} by g(d) = {f(z): S(b,z)}. Note that this extension is a binary relation on the transitive virtual set x □ S* □ {{f(z): S(b,z)}}. By Lemma 5.14, x □ S* exists, the triple union is transitive, and all expansions in the triple union meet x □ S* □ {∅,{∅}}. Hence the triple union exists. Since the desired extension of the function g_c to g_b is a virtual binary relation on the triple union, it exists.

Assume b is a limit in <(x). Let f be the virtual set which is the union of all f_a, <(x)(a,b). This is obviously a binary relation on x □ S*, and therefore it exists. By Lemmas 5.13 and 5.14, this union is itself a function g with domain {a: <(x)(a,b)}, given by g(a) = {g(z): S(b,z)}. We need to extend this function g to be defined on {d: ≡(x)(d,b} by g(d) = {f(z): S(b,z)}. We argue exactly as in the previous paragraph. QED

LEMMA 5.16. Suppose S is an x-system. For all b □ x, there exists f such that D(S,f,b). There exists g such that D(S,g).

Proof: Let b □ x be <(x) minimal such that there is no f with D(S,f,b). Then the hypotheses of Lemma 5.15 hold, and we obtain a contradiction. For the second claim, the case x = ∅ is trivial. If there is a <(x) greatest element b of x then any f with D(S,f,b) has D(S,f). Finally suppose x is nonempty and there is no <(x) greatest element of x. By Lemma 5.14, x □ S* exists. Thus the desired function is a binary relation on x □ S*. Therefore we can take g to be the union of the f such that for some b, D(S,f,b). As in the proof of Lemma 5.15, D(S,g). QED

Let S be an x-system. We define S# to be the unique (set) function such that D(S,S#), according to Lemma 5.16. Note that S#:x □ S*, S# is onto, and x □ S* is a transitive set. In addition, for all b □ x, S#(b) = {S#(z): S(b,z)}, and by Lemma 5.13, S#(b) = S#(c) □ ≡(x)(b,c). We refer to this equivalence as the one-one property of S#.

LEMMA 5.17. Let S be an x-system. W(S*). For all a,b □ x, □(x)(a,b) if and only if □(S*)(S#(a),S#(b)). ≡(S*) is the identity relation on S*. <(S*) is a strict well ordering.

Proof: By Lemma 5.14, S* is transitive. To prove well foundedness, let y be a nonempty subset of S*. Let z = {w □ x: S#(w) □ y}. Let w be a <(x) minimal element of z. We claim that S#(w) is an epsilon minimal element of y. To see this, let u □ S#(w) □ y. Let u = S#(w') □ y, S(w,w'). Then w' □ z, <(x)(w',w), contradicting the <(x) minimality of w. Hence W(S*).

The second claim now makes sense since W(S*). Recall the definition of □(S*) as the unique relation satisfying clauses i) - iii) given just before Lemma 5.4. To establish the second claim, we have only to show that the binary relation R on S* given by R(S#(a),S#(b)) □ □(x)(a,b) obeys conditions i) - iii) with x = y = S*. The only clause of substance is iii).

Let z,w □ S*. We must verify that R(z,w) □ (□a □ z)(□b □ w)(R(a,b)). In other words, let a,b □ x. We must verify that R(S#(a),S#(b)) □ (□u □ S#(a))(□v □ S#(b))(R(u,v)).

The left side is equivalent to □(x)(a,b). The right side is equivalent to (□c|S(a,c))(□d|S(b,d))(□(x)(c,d)). Suppose <(x)(a,b). This is true because S(a,c) □ <(x)(c,a) and the strict sup condition for x-systems. Suppose ≡(x)(a,b). This is true by setting d = c. For the inverse, suppose <(x)(b,a). Choose c such that S(a,c), □(x)(b,c). Any d with S(b,d) will have <(x)(d,b), and therefore not □(x)(c,d).

From the third claim, using Lemma 5.13, we have ≡(S*)(S#(a),S#(b)) if and only if ≡(x)(a,b) if and only if S#(a) = S#(b). By W(S*) and Lemma 5.12, <(S*) is well founded and connected, and so by the third claim, <(S*) is a strict well ordering. QED

The S* are our best approximations to genuine ordinals in Newcomp + Ext. <(S*) is a strict well ordering. Of course, S* is not normally epsilon connected.

LEMMA 5.18. Let S,S′ be two x-systems. For all y,z □ x, S#(y) = S′#(z) □ ≡(x)(y,z). S = S′ □ S* = S′*.

Proof: Let S,S′ be as given. For the first claim, let y,z be a counterexample, with y chosen to be <(x) minimal. We first assume that >(x)(y,z). Now S#(y) = S′#(z). By the strict sup condition on x-systems, let S(y,w), ≥(x)(w,z). Then S#(w) □ S#(y), and so S#(w) □ S′#(z). Let S#(w) = S′#(u), S′(z,u). Clearly □≡(w,u). This contradicts the minimality of z.

Now assume <(x)(y,z). By the strict sup condition on x-systems, let S′(z,b), ≥(x)(b,y). Then S′#(b) □ S′#(z) = S#(y). Let S′#(b) = S#(c), S(y,c). Clearly □≡(b,c), and so we have a violation of the minimality of z. This establishes the first claim.

For the second claim, the forward direction is obvious. Now let S* = S′*. Let y,z be such that S(y,z) □ □S(y,z), where y is <(x) minimal. By symmetry, we can assume S(y,z), □S(y,z).

An obvious transfinite induction (or minimal element) argument, shows that S# and S′# agree below y in the sense of <(x).

We now show that S#(y) ≠ S′#(y). Suppose {S#(z): S(y,z)} = {S′#(z): S′(y,z)}. Then S#(z) □ {S′#(z): S′(y,z)}. Let S#(z) = S′#(w), S′(y,w). Note that □≡(w,z). By the previous paragraph, S′#(w) = S#(w) = S#(z). This contradicts the one-one property of S#.

By the one-one property of S#, S′#, we see that S#(y) is not equaled to any S′#(z), ≡(x)(y,z). By the first claim, S#(y) is not equaled to any S′#(z), □≡(x)(y,z). Hence S#(y) is an element of S* that is not an element of S′*. QED

Let S be an x-system and S′ be a y-system. We say that S,S′ are equivalent if and only if S* = S′*.

We say that R is an isomorphism relation from □(x) onto □(y) if and only if the following holds. We do not require that R exists; i.e., R may only be a virtual relation. Since we have ordered pairs, R can always be viewed as a virtual set.

i) the domain of R is x and the range of R is y;
ii) if R(a,b) and R(c,d) then □(x)(a,c) □ □(y)(b,d).

We say that □(x) and □(y) are isomorphic if and only if there is an isomorphism relation from □(x) onto □(y).

LEMMA 5.19. Let x,y be given. Suppose some x-system is equivalent to some y-system. Then □(x) and □(y) are isomorphic. If □(x) and □(y) are isomorphic, then every x-system is equivalent to some y-system and vice versa.

Proof: Before we begin, we make some comments about the formalization of Lemma 5.19. The problem arises because the definition of isomorphic □(x) involves virtual relations. For the second claim, this just means that we are making infinitely many assertions. For the first claim, we mean that the virtual isomorphism can be given uniformly in x,y, and the x-system and the y-system; i.e., by a single specific virtual relation with parameters, as will be clear in the construction below.

Let S be an x-system, S′ be a y-system, S* = S′*. We define the virtual relation R from x to y as follows. Let a □ x and b □ y. Take R(a,b) □ S#(a) = S′#(b). We claim that R is an isomorphism relation from □(x) onto □(y).

To see that the domain of R is x, let a □ x. Then S#(a) □ S* = S′*. Let b □ y, S′#(b) = S#(a). Then R(a,b). Analogously, the range of R is y.

Now let R(a,b), R(c,d). By Lemma 5.17, □(x)(a,c) □ □(S*)(S#(a),S#(c)), and □(y)(b,d) □ □(S′*)(S′#(b),S′#(d)). Hence □(y)(b,d) □ □(S*)(S#(a),S#(c)). So □(x)(a,c) □ □(y)(b,d).

Now let R be a virtual isomorphism relation from □(x) onto □(y). We now show that every x-system is equivalent to some y-system. Let S be an x-system. Define S′ to be the binary relation on y given by S′(z,w) if and only if there exists a,b □ x such that R(a,z), R(b,w), and S(a,b). Note that S′ exists since it is a virtual subset of the set y•y.

We have to verify that S* = S′*. I.e., that S# and S′# have the same ranges. It suffices to show that R(a,b) □ S#(a) =

S’#(b). Let a,b □ x be a counterexample with a chosen to be <(x) minimal. We can find such a,b by using separation on x, despite the fact that R is only virtual. Observe that S#(a) = {S#(c): S(a,c)}, S’#(b) = {S’#(d): S’(b,d)}. We have R(a,b), S#(a) ≠ S’#(b).

We first show that S#(a) □ S’#(b). We will then show that S’#(b) □ S#(a), and so S#(a) = S’#(b). This contradicts the choice of a,b.

Let S(a,c). We show that S#(c) □ {S’#(d): S’(b,d)} = S’#(b). Since R(a,b), we let d □ y be such that S’(b,d) and R(c,d). By the minimality of a, we have S#(c) = S’#(d). Hence S#(c) □ {S’#(d): S’(b,d)} = S’#(b). This establishes S#(a) □ S’#(b).

To establish S’#(b) □ S#(a), let S’(b,d). We show that S’#(d) □ {S#(c): S(a,c)}. Since R(a,b), we let c □ x be such that S(a,c) and R(c,d). By the minimality of a, we have S’#(d) = S#(c). Hence S’#(d) □ {S#(c): S(a,c)} = S#(a). This establishes S’#(b) □ S#(a). QED

We refer to the conclusion of the second claim of Lemma 5.19 as “x,y have equivalent systems”.

Let W(x). The initial segments of x are the subsets of x closed under □(x). The proper initial segments are written $x_{<a}$, a □ x. The initial segments of an x-system S also has the obvious meaning, and the proper initial segments are written $S_{<a}$, a □ x. We also use the notation $x_{\square a}$, $S_{\square a}$, with the obvious meaning. And we also use the notation $S\#_{\square a}$, $S\#_{<a}$, again with the obvious meaning.

LEMMA 5.20. Suppose that there exists an x-system. It is not the case that x and one of its proper initial segments have equivalent systems.

Proof: Suppose x and $x_{<a}$ have equivalent systems. By Lemma 5.19, □(x) and □($x_{<a}$) are isomorphic. This violates the well foundedness of x, using separation on x, even if the isomorphism is virtual. QED

LEMMA 5.21. Let x,y have systems, and assume that for every proper initial segment of x there is a proper initial segment of y with equivalent systems, and vice versa. Then x,y have equivalent systems.

Proof: Let x,y be as given. Suppose $x_{<a}$ and $y_{<b}$ have equivalent systems. Since x,y have systems, so do $x_{<a}$ and $y_{<b}$. By Lemma 5.20, $y_{<b}$ is the unique initial segment of y such that $x_{<a}$ and $y_{<b}$ have equivalent systems.

Define the virtual relation R(a,b) if and only if $x_{<a}$ and $y_{<b}$ have equivalent systems. We claim that R is a virtual isomorphism from □(x) onto □(y). By hypothesis, the domain of R is x and the range of R is y. Suppose R(a,b) and R(c,d). Then $x_{<a}$ and $y_{<b}$ have equivalent systems. Also $x_{<c}$ and $y_{<d}$ have equivalent systems. By Lemma 5.19, we obtain a virtual isomorphism from $x_{<a}$ onto $y_{<b}$, and a virtual isomorphism from $x_{<c}$ onto $y_{<d}$. Suppose □(x)(a,c), and <(x)(d,b). Then we get a virtual isomorphism from $x_{<a}$ onto an initial segment of $y_{<d}$, and hence onto a proper initial segment of $y_{<b}$. But this gives a virtual isomorphism from $y_{<b}$ onto a proper initial segment of $y_{<b}$, which is a virtual subset of the set y•y. Hence using separation, we have an actual isomorphism from $y_{<b}$ onto a proper initial segment of $y_{<b}$, which contradicts the well foundedness of <(y). The other direction is symmetric.

We have thus verified that R is a virtual isomorphism from □(x) onto □(y). By Lemma 5.19, x,y have equivalent systems. QED

LEMMA 5.22. Let x,y have systems. Either x and some proper initial segment of y have equivalent systems, or y and some proper initial segment of x have equivalent systems, or x,y have equivalent systems. The cases are mutually exclusive, and the choice of proper initial segments is unique.

Proof: Let x,y have systems. Let B = {a □ x: (□b □ y)($x_{<a}$ and $y_{<b}$ have equivalent systems)}. We claim that (□(x)(c,a) □ a □ B) □ c □ B. To see this, assume $x_{<a}$ and $y_{<b}$ have equivalent systems. By Lemma 5.19, we obtain a virtual isomorphism from $x_{<a}$ onto $y_{<b}$. This induces a virtual isomorphism from $x_{<c}$ onto some $y_{<d}$. By Lemma 5.19, we see that $x_{<c}$ and $y_{<d}$ have equivalent systems.

Note that for a □ B, the associated b □ y is unique up to ≡(y). We let C be the set of all these associated b □ y that are used in B, closing up under ≡(y). As in the previous paragraph, C is also closed downward under □(y).

We the well foundedness of <(x), either B = x or B is a proper initial segment of x. Also either C = y or C is a proper initial segment of y.

No matter which of these four cases holds, we can apply Lemma 5.21. We obtain that B,C have equivalent systems. This establishes the first claim, provided we can rule out the fourth case where B is a proper initial segment of x and C is a proper initial segment of y. Let B = $x_{<a}$, and C = $y_{<b}$. Then a □ B, and so <(x)(a,a), which is impossible.

For the second claim, if more than one case holds, or if the choice of proper initial segments is not unique, then we obtain that x and a proper initial segment of x have equivalent systems, or y and a proper initial segment of y have equivalent systems. This violates Lemma 5.20. QED

Suppose x,y have systems. By Lemma 5.22, either x and some unique proper initial segment of y have equivalent systems, or y and some unique proper initial segment of x have equivalent systems, or x,y have equivalent systems, and these three cases are mutually exclusive. By Lemma 5.21, we obtain a (rather simple) virtual isomorphism relation corresponding to these three cases. The relevant virtual isomorphism relations are defined uniformly in x,y. We call it the virtual comparison relation for x,y.

In light of the previous paragraph, we make the following definition, assuming x,y have systems. COMP(x,y)(a,b) if and only if the virtual comparison relation for x,y holds at a,b.

We now construct an x such that □(x) has "order type □". The idea is to construct {∅,{∅},{{∅}},...}.

We say that x is simple if and only if

i) W(x);
ii) every nonempty element of x is the singleton of an element of x.

We say that x is very simple if and only if x is simple and x has an epsilon maximal element.

LEMMA 5.23. Let x be simple and y be an epsilon maximal element of x. x is the least transitive set containing y as an element. The singleton of any element other than y is an element. No simple set has more than one epsilon maximal element.

Proof: Let x,y be as given. Let z be a transitive set with y □ z. We claim that x □ z. First observe that x □ z is nonempty, transitive and well founded, and so ∅ □ x □ z. Now let b be an epsilon minimal element of x not in z. Then b ≠ ∅. Write b = {c}, c □ x. Since z is transitive, c □ z. This contradicts the minimality of b.

For the second claim, let z be the set of all elements of x whose singleton lies in x, together with the element y. To see that z is transitive, let a □ b □ z. Then b is a nonempty element of x, and so b = {a}. Therefore a □ z. By the first claim, x □ z.

The third claim follows immediately from the second. QED

LEMMA 5.24. The very simple sets are comparable under inclusion.

Proof: Let x,y be very simple. Assume x is not a subset of y. Let b be an epsilon least element of x that is not in y. If b = ∅ then y is empty, in which case y □ x. So we assume b ≠ ∅. Let b = {c}, c □ x. Then c □ y. By Lemma 5.23, if c is not the epsilon maximal element of y, then b = {c} □ y. Therefore c is the epsilon maximal element of y. By Lemma 5.23, y is the least transitive set containing c as an element. But x is a transitive set containing c as an element. Hence y □ x. QED

LEMMA 5.25. In a very simple set, every proper inclusion between elements of x has left side ∅. If x is a very simple set with epsilon maximal element y, then x □ {{y}} is very simple.

Proof: Let a,b □ x, a □≠ b. Then b ≠ ∅, and so write b = {c}, c □ x. Then a = ∅.

Let x be very simple with epsilon maximal element y. Clearly x □ {{y}} is transitive and well founded. Let b □ x □ {{y}}, b ≠ ∅. If b □ x then b = {c} for some c □ x □ {{y}}. If b = {y} then just note that y □ x □ {{y}}.

It remains to show that {y} is an epsilon maximal element of x □ {{y}}. Now {y} □ {y} is impossible since then {y} = y, and so y □ y, violating the well foundedness of x. It remains to show that no element of x includes {y} as an element. Let {y} □ b □ x. Then {y} □ x, violating that y is the epsilon maximal element of x. QED

LEMMA 5.26. There is a unique simple set with no epsilon maximal element. It is the union of all very simple sets. Every proper inclusion among elements of this set has left side ∅.

Proof: Let A be the virtual union of all very simple sets. Let a □≠ b, a,b □ A. Let a □ x, b □ y, where x,y are very simple. By Lemma 5.24, we can assume a,b □ x. By Lemma 5.24, a = ∅. Therefore all expansions in A meet {∅}. Hence A exists.

To see that A is simple, we first check that W(A). Clearly A is transitive since it is the union of transitive sets. To see that A is well founded, let z be a nonempty subset of A. Let x be very simple, where x,z have an element in common. Let b be an epsilon minimal element of x □ z. We claim that b is an epsilon minimal element of z. This follows from the transitivity of x.

To finish the argument that A is simple, let b be a nonempty element of A. Then b □ x for some very simple x. Hence b = {c} for some c □ x. Hence b = {c} for some c □ A.

We now show that A has no epsilon maximal element. Let b □ x, where x is very simple, with epsilon maximal element y. By Lemma 5.23, b = y or {b} □ x. If {b} □ x then {b} □ A and b is not an epsilon maximal element of A. Now assume b = y. By Lemma 5.25, x □ {{y}} is very simple. Hence again {b} □ A, and so b is not an epsilon maximal element of A.

We now prove that A is the unique simple set with no epsilon maximal element. It suffices to prove that for any two simple sets y,z with no epsilon maximal elements, we have y □ z. Suppose this is false, and let b be an epsilon minimal element of y that is not in z. Then b ≠ ∅, since z is nonempty and transitive. Let b = {c}, c □ y. Then c □ z. Since c is not epsilon maximal in z, let c □ d □ z. Then d = {e} for some e □ z. Hence c = e and d = {c} = b □ z.

By the first paragraph, every proper inclusion among elements of A has left side ∅. QED

LEMMA 5.27. The unique simple set with no epsilon maximal element is the least set containing ∅ as an element, and closed under the singleton operation.

Proof: This unique set is A = the union of all very simple sets, according to Lemma 5.26. Obviously ∅ A, because {∅} is very simple. Also by Lemmas 5.23 and 5.25, A is closed under the singleton operation.

Now let y be any transitive set containing ∅ as an element, and closed under the singleton operation. Suppose A is not a subset of y. Let x be an epsilon minimal element of A that is not in y. Clearly x ≠ ∅ since ∅ y. Since A is simple, let x = {b}, b A. Then b y, and so x = {b} y. QED

We use the notation ^ for the set in Lemma 5.27, which is the union A of all very simple sets.

Note that ^ with ∅ as 0 and the singleton operation as successor forms a successor structure with second order induction in the following standard sense. The successor of any element is not 0. Any two elements with the same successor are equal. Every set that includes 0 as an element, and is closed under successor, contains the whole structure as a subset.

This means that we have full second order arithmetic at our disposal for use in the rest of the section.

LEMMA 5.28. W(^). (^) is a well ordering with no greatest element and no limit point.

Proof: W(^) since ^ is simple. Hence (^) is well founded.

To obtain that (^) is a well ordering, it suffices to show that ≡(^)(x,y) x = y. Let x,y be a counterexample, where x is epsilon minimal. If x = ∅ then from (^)(y,∅) we obtain y = ∅. So x ≠ ∅. Let x = {b}. We argue similarly that y ≠ ∅, and so let y = {c}. Since x ≠ y, we have b ≠ c. Since ≡(^)(x,y), we have (^)(x,y), (^)(x,y). By the definition of (^), we have (^)(b,c), (^)(c,b), and so ≡(^)(b,c). This contradicts the minimality of x.

That □(□^) has no greatest element follows from the fact that □^ has no epsilon maximal element, the first claim, and Lemma 5.10. QED

□^ certainly gives us full second order arithmetic, but how are we going to use it to interact with the binary relations on any x with W(x)? This is answered by the following.

LEMMA 5.29. If W(x) then x □ □^ exists. In fact, W(x □ □^).

Proof: By Lemma 5.26, every proper inclusion between elements of □^ has left side ∅. Therefore all expansions in x □ □^ meet x □ {∅}. Hence x □ □^ exists. The second claim follows easily. QED

This means that what is essentially the Cartesian powers of x □ □^, and therefore separation on them, are available.

LEMMA 5.30. Suppose there is an x-system. Then there is an x □ {x}-system.

Proof: Let S be an x-system. Obviously W(x □ {x}) since transitivity and well foundedness are preserved. In Lemma 2.3 of [Fr02], a construction is given that in our contexts amounts to modifying S to another x-system with the property that every cross section contains an element at the bottom level (i.e., 0), or an element at an odd level (i.e., odd as in odd ordinal level). These levels refer to levels in □(x). Bringing this construction into this context requires ordinal arithmetic, which is fully available here. An x □ {x}-system is formed by augmenting the x-system with a top level cross section which is the set of all elements of x that lie at a nonzero even level in □(x). QED

Note that □(□^) has no greatest element. We want to extend Lemma 5.36 by constructing x □ {x,{x},{{x}},...}, with this same property.

LEMMA 5.31. Assume that there is an x-system. There exists a set z such that x □ z and z is closed under singletons. There is a least such z. There is an x □ z-system. □(x □ z) has no greatest element. □(x) is a proper initial segment of □(x □ z).

Proof: We use □^ for the construction. By induction, for each n □ □^, there is a unique function f_n with domain {i:

□(□^)(i,n)} such that f(∅) = x and each f({i}) = {f(i)}. Using W(x), each f_n is one-one. The union of the ranges of the f_n is a virtual set B containing x as an element and closed under singletons. Note that every expansion within the virtual set x □ B meets x □ {x}. Hence x □ B exists. Hence B exists. Clearly B is contained in any set z such that x □ z and z is closed under singletons. Therefore B serves as the z.

We now verify W(x □ z). By induction, every element of B = z is either x or the singleton of an element of z. Hence x □ z is transitive. For well foundedness, if the nonempty subset of x □ z meets x then take an epsilon minimal element in x. Show that this is epsilon minimal by induction in the construction of z. Otherwise, use induction in the construction of z.

To see that there is an x □ z-system, start with the x □ {x}-system provided by Lemma 5.30. Extend it to the rest of x □ z by taking the cross section at each b □ z\{x} to consist of just the unique element of b. Obviously □(x □ z) has no greatest element since x □ z has no epsilon maximal element. Also, □(x) is the proper initial segment of □(x □ z) determined by the point x. QED

We now wish to build the constructible hierarchy on every x with W(x). We run into trouble if we wish to compare the constructible hierarchy on x with the constructible hierarchy on y, where W(x), W(y). There is where we will need that x,y have systems; i.e., there is an x-system and there is a y-system.

Rather than reinvent the wheel, we will use a fairly strong version of a standard sentence □ in L(□) used in standard treatments of the constructible hierarchy in set theory with the property that the well founded models of □ are exactly the structures which, when factored by the equivalence relation of extensional equality, are isomorphic to some (L(□),□), where □ is a limit ordinal. This is normally done in L(□,=) in connection with the verification of the generalized continuum hypothesis in L, but here we stay within L(□).

We wish to be more explicit about □ for two reasons. One is for the sake of expositional completeness, and the other is because we are going to use □ formally, especially in Lemma 5.38.

We take □ to be the conjunction of the following sentences in L(□). We do not make much effort to be economical.

i) extensionality, pairing, union, every set has a transitive closure, every nonempty set has an epsilon least element, every set has a cumulative rank function, every set whose cumulative rank function is onto a finite ordinal has a power set, there is no greatest ordinal;
ii) if there is a limit ordinal then the satisfaction relation of every set under epsilon exists;
iii) if there is a limit ordinal then for all ordinals □, there is a function on □ which follows the usual definition by transfinite recursion of the constructible hierarchy;
iv) if there is a limit ordinal then every set lies somewhere in the constructible hierarchy defined in iii);
v) $□_0$ separation.

Here an ordinal is an epsilon connected transitive set. A finite ordinal is an ordinal which is not a limit ordinal and no element is a limit ordinal. A cumulative rank function on x is an ordinal valued function f with domain x such that each f(y) is the strict sup of the f(z), z □ y. It is well known how to finitely axiomatize $□_0$ separation in the presence of extensionality, pairing, and union. In ii), the satisfaction relation is defined using finite sequences from the set for the assignments, where a finite sequence is taken to be a function from an element of the least limit ordinal into the set.

LEMMA 5.32. Let W(x), x ≠ ∅, and □(x) have no greatest element. There exists A □ x•x and binary relation R □ A•A such that (A,R) satisfies □, R is well founded, and the relative level in the internal constructible hierarchy of an ordered pair in A is given by its first coordinate's position in □(x), and every element of x is the first coordinate of an element of A. I.e., for all (a,b),(c,d) □ A, (A,R) satisfies "(a,b) occurs at an earlier stage in the constructible hierarchy (the L(□)'s) than (c,d)" if and only if <(x)(a,c). Furthermore, equivalence (having the same elements) in the sense of (A,R) is the same as the equivalence relation E on A given by (a,b) E (c,d) □ (≡(x)(a,c) □ ≡(b,d)).

Proof: By Lemma 5.29, x □ □^ exists. So we can simulate any of its Cartesian powers and use separation on them. We can treat □(x) as a well ordering, provided that for each b □

x, we carry along all c with ≡(x)(b,c). We can build a coded version of the constructible hierarchy up through □(x) (in fact well beyond that point) as a binary relation on an appropriate subset of x•x. There is no difficulty assembling this construction so as to be a binary relation on an appropriate subset of x•x where the first coordinate is used for the level of the constructible hierarchy. We use the □^ in x □ □^ in order to code formulas and set up the needed satisfaction relations for the successor steps in the constructible hierarchy. One delicate point is that we need to use a function from the finite sequences from any given infinite initial segment of □(x), mod ≡(x), into that same initial segment, mod ≡(x). Since we don't properly have the set of all such finite sequences in this environment, we use a suitable mapping from any given infinite initial segment of □(x), cross □^, into that same initial segment, which serves as an appropriate finite sequence mechanism. Because we have what amounts to full second order logic on (x □ □^,□) at our disposal, we can explicitly provide such a finite sequence mechanism with the help of some relevant ordinal arithmetic. QED

The construction of (A,R) in Lemma 5.32 is done uniformly in x. So for x ≠ ∅, W(x), □(x) with no greatest element, we write L[x] for the (A,R), A □ x•x, R □ A•A, given by the proof of Lemma 5.32. We write dom(L[x]) for the A. The idea of the notation is that this is the code for the initial segment of the constructible hierarchy along □(x).

We use the obvious notions of initial segment and proper initial segment of L[x], where we only cut off at limit stages. We also use the obvious notions of isomorphism relations between initial segments of L[x] and initial segments of L[y].

It is convenient to define W'(x) if and only if x ≠ ∅, □(x) has no greatest element, and there is an x-system. Note that W'(□^). In particular, R □ □^•□^ given by R(n,m) □ n = {m} is an □^-system.

Recall the definition of the virtual relation COMP(x,y) after the proof of Lemma 5.21.

LEMMA 5.33. Let W'(x), W'(y). There is a virtual isomorphism relation from L[x] onto a proper initial segment of L[y], or a virtual isomorphism relation from L[y] onto a proper initial segment of L[x], or a virtual isomorphism relation from L[x] onto L[y]. The choice of cases and proper initial segments, as well as the comparison of levels, is by COMP(x,y). Furthermore, at most one of the three possibilities can apply, the proper initial segment is unique, and the isomorphism is unique.

Proof: We suppose that COMP(x,y) is an isomorphism relation from □(x) onto □(y). The other cases are handled analogously.

COMP(x,y) obviously provides the right way of matching levels, but does not give us the desired virtual isomorphism relation. For this, we have to use the systems. Let S be an x-system, S' a y-system, where S,S' are equivalent. I.e., S* = S'*. Since L[x] respects the equivalence relation ≡(x), we can forward image L[x] via S# to get A* □ S*•S* and L*[x] □ A*•A*. Similarly, we forward image L[y] via S'# to get B* □ S*•S* and L*[y] □ B*•B*. Both of these forward images also satisfy □, are well founded, and the level in the internal constructible hierarchy of an ordered pair is given by its first coordinate's position in □(S*). It is not clear that L*[x] = L*[y]. However, S* □ □^, its Cartesian powers, and full separation are available, and so we can prove that L*[x] and L*[y] are isomorphic, by an actual, not merely virtual, isomorphism. This yields the desired virtual isomorphism relation from L[x] onto L[y] by composition. The comparison of levels by COMP(x,y) is preserved.

The final claim is by obvious transfinite induction (or minimal element) arguments. Firstly, observe that any virtual isomorphism must preserve levels in the constructible hierarchy, since the levels are well founded (here we use only separation on x and y separately). The uniqueness of the isomorphism constitutes infinitely many statements. These are again proved by obvious transfinite induction (or minimal element) arguments. We do these arguments separately on x and on y, and do not need x □ y. QED

For x,y with W'(x), W'(y), we let LCOMP(x,y) be the unique virtual comparison isomorphism relation between L[x] and L[y] given by the proof of Lemma 5.33. Note that LCOMP(x,y) is given uniformly in x,y.

We are now prepared to construct the full constructible universe. The points will be pairs (x,b), where W'(x) and b

□ dom(L[x]). The epsilon relation between points (x,b) and (y,c) holds if and only if LCOMP(x,y)(b) is satisfied in L[y] to be an element of c. We caution the reader that LCOMP(x,y) is not a function, but rather an isomorphism relation. However, it is functional when we factor out by the equivalence relations ≡(x) and ≡(y).

We use L for the virtual set of these points, and $□_L$ for this virtual epsilon relation on L. Thus our notation for the full constructible universe in this context is $(L,□_L)$. Both coordinates are virtual.

LEMMA 5.34. $(L,□_L)$ is well founded in the sense that every nonempty subset of L has an $□_L$ minimal element.

Proof: Let z □ L be nonempty. Let (y,c) □ z,L. By W'(y), let (x,b) be chosen such that LCOMP(x,y)(b) exists and is of minimum level in the constructible hierarchy of L[y].

We claim that (x,b) is an $□_L$ minimal element of z. To see this, let (w,d) $□_L$ (x,b), (w,d) □ z,L. Then LCOMP(w,x)(d) is satisfied in L[x] to be an element of b, and therefore is of lower level than b in the constructible hierarchy of L[x]. Therefore LCOMP(w,y)(d) is of lower level than c in the constructible hierarchy of L[y]. This contradicts the minimality of (x,b). QED

Let □' be the following variant of □: any set is an element of a well founded transitive set satisfying □.

LEMMA 5.35. □' logically implies □.

Proof: Assume □'. Then any list of sets of standard integer length all lie in some well founded transitive set satisfying □.

To derive extensionality, let x,y have the same elements, and let z be given. Let x,y,z □ w, where w is a transitive set satisfying □. Then x,y have the same elements in the sense of □, and hence x □ z □ y □ z.

To derive foundation, let x be a nonempty set, and x □ y, y a transitive set satisfying □. Then x has an epsilon minimal element in the sense of y. So x has an epsilon minimal element.

We leave pairing and union to the reader.

Let x be given, and let x □ y, where y is a well founded transitive set satisfying □. Then according to y, x has a cumulative rank function. Hence it is easily seen that this cumulative rank function in the sense of y is a cumulative rank function.

To establish □$_0$ separation, put all the parameters in some well founded transitive set, and apply □$_0$-separation inside there.

Let x be a well founded transitive set satisfying □. Then the cumulative rank function for x exists. It is easy to see that its range is a limit ordinal. From this we obtain the ordinal □ as the least limit ordinal. We can apply induction to □, as long as the predicate forms a subset of □. In particular, we have □$_0$ induction.

We claim that every x is an element of some transitive set y satisfying □ where □ □ y. To see this, let x □ y □ z, where y,z are transitive sets satisfying □. Then the cumulative rank function for y exists in z, and hence □ □ z.

To derive that x has a transitive closure, let x □ y, where y is a transitive set satisfying □. Then x has a transitive closure u, in the sense of y. Clearly x □ u and u is transitive. Now the characterization of the transitive closure as the set of terms in backwards epsilon chains is provable from □. Hence this characterization holds in y. To verify that u is the actual transitive closure, we use □$_0$ induction.

Suppose that the cumulative rank function of x is onto a finite ordinal. We claim that all subsets of x lie in y. To see this, let z □ x. We show that the intersection of z with the first i elements of x lies in y by □$_0$ induction on i, using the cumulative rank function of x.

We now show that the satisfaction relation for any (x,□) exists. Write x □ y □ z, where y,z are transitive sets satisfying □. Then z satisfies that there is a limit ordinal. So z satisfies that (x,□) has a satisfaction relation, and therefore (x,□) has a satisfaction relation.

Let □ be an ordinal, □ □ x, x a transitive set satisfying □. The constructible hierarchy as a function defined by transfinite recursion on exists in the sense of x. Therefore the constructible hierarchy as a function defined by transfinite recursion on exists.

Let y be a transitive set satisfying , with   y. The internal constructible hierarchy in y exhausts y. As in the previous paragraph, the internal constructible hierarchy in y is an initial segment of the external constructible hierarchy. Since every x lies in some transitive set satisfying which also has the element , it follows that every set appears in the constructible hierarchy. QED

LEMMA 5.36. (L,$_L$) is a well founded model of ′ and hence of .

Proof: By Lemmas 5.34 and 5.35, we have only to verify that ′ holds in (L,$_L$). Let W′(x). By Lemma 5.31, let W′(y), where (x) is a proper initial segment of (y). Look at the constructible hierarchy internal to L[y]. Internally, one of the points will be an L() where, externally, the is the length of (x). Let b be such a point. Then in (L,$_L$), (y,b) will be satisfied to be a well founded transitive set satisfying . Also, every (x,a) L will have (x,a) $_L$ (y,b). QED

Recall the scheme SVWISUB, which we have only discussed in the context of ZFC. We wish to discuss it in the context of . It is formulated identically.

Let # be the following scheme, which is a weakening of SVWISUB.

#. If defines a system A , for all ordinals , where the strict sup of each A is , then there exist 2   < such that A A.

LEMMA 5.37. SVWISUB is provable in + #.

Proof: By a straightforward adaptation of Theorem 2.3 of [Fr02]. This requires only the development of some simple ordinal arithmetic in . QED

LEMMA 5.38. ZFC + V = L + SVWISUB is provable in + # + SUB.

Proof: By Lemma 5.37, we already have SVWISUB. Also V = L follows immediately from . We have only to obtain ZFC.

Note that Lemma 2.3 can be proved just from $\square$, and so there is no weakly inclusion subtle ordinal over 2.

By an obvious use of SVWISUB, we obtain the existence of a limit ordinal. Let $\square$ be the least limit ordinal. Then it is easily verified that $\square$ obeys the axiom of infinity.

Next we verify replacement on $\square$. This is a strong analog of "On is uncountable" for eventual use in the adaptation of the proof of Theorem 1.6 of [Fr02].

Suppose $\square_0 < \square_1 < \ldots$ is unbounded, and definable. Pick a counterexample to the weak inclusion subtlety over 2 of $\square_0$. Next pick a counterexample to the weak inclusion subtlety over 2 of $\square_1$, and place the part of the assignment to ordinals $\geq \square_0$ on top. Continue in this way.

There is no problem making the construction in Lemma 2.3 of [Fr02] in this context for $\square = \square_i$ and $\square = 2$, uniformly in i. In fact, we can adjust the "2x+1" so that in the sets assigned to the limit ordinals for $\square = \square_i$, the least integer present is always 2i+1. This guarantees that we have given a counterexample to SVWISUB. Hence we have replacement on $\square$.

Now we verify the analog of "$\square$ is a cardinal". Specifically, that there is no definable map from On one-one into an ordinal. This is a straightforward adaptation of Lemmas 1.4 and 1.5 of [Fr02].

We now wish to prove the analog of "$\square$ is subtle" by adapting the proof of Lemma 1.6 of [Fr02]. We first need the analog of Theorem 1.2 of [Fr02], which is "On is inclusion subtle implies On is subtle". This is no problem. Here these notions are, as always, formulated on On through definable assignments. We already have "On is an uncountable cardinal" by the previous paragraph and $\square$ replacement. The adaptation of the proof of Lemma 1.6 of [Fr02] is clear and shows that "On is subtle".

It is now easy to verify replacement. Suppose replacement fails on $\square$. Then we obtain a definable mapping partially from $\square$ into ordinals which is unbounded. We can adjust this map so that the range, C, is closed and unbounded. We can then give an easy counterexample to the subtlety of On by assigning singletons to sufficiently large elements of C.

We now verify separation. Separation can be proved from □ together with replacement. It suffices to fix an ordinal □ and find a limit ordinal □ > □ such that L(□) is an elementary substructure of L with respect to n quantifier prenex formulas, for any standard integer n. It is clear what we mean by the first k stages L(□),L($\square_1$),L($\square_2$),..., of the obvious Skolem hull construction starting with L(□+1), where □ = $\square_0$ < $\square_1$ ..., and each $\square_{i+1}$ is picked minimally greater than $\square_i$, and k ≥ 0. Note that we apparently cannot prove that this exists for all k ≥ 0. However, if it exists for k then it exists for k+1, and the construction for k+1 extends the construction for k.

By replacement, we obtain □ such that the above finite sequences that exist all end with an ordinal < □. Now we can use foundation to conclude that there are such sequences of every finite length, which can be put together into an infinite sequence with limit □ □ □. Then L(□) is the desired partial elementary substructure of L. This suffices to establish separation since the standard integer n is arbitrary.

Finally, we verify power set. Fix L(□), □ ≥ □, and suppose that new subsets of L(□) appear arbitrarily high up in the constructible hierarchy. Since there is a bijection from L(□) onto □, we see that new subsets of □ appear arbitrarily high up in the constructible hierarchy. Let B be the unbounded class of all ordinals □ > □ such that L(□+1)\L(□) has an element from L(□). By □ replacement, B has a closed unbounded class C of limits, none of which lie in B.

We define $A_\square$ □ □, □ □ C, as the first subset of □, in the constructible hierarchy, lying in L(□+1)\L(□), where □ is the least element of B greater than □. Since the $A_\square$ are all different subsets of □, we have a counterexample to the subtlety of On. QED

LEMMA 5.39. ZFC + V = L + SVWISUB is interpretable in □ + #.

Proof: This follows easily from Lemma 5.38. QED

We now wish to verify that # holds in (L,$\square_L$). By Lemma 5.36, □ holds in (L,$\square_L$). This will provide an interpretation of □ + # in Newcomp + Ext.

In ordinary set theoretic terms, the hypothesis for # is

H1) an assignment $A_\alpha \subseteq \alpha$, for all ordinals α, where the strict sup of every A_α is α.

The conclusion is

C1) the existence of $2 \leq \alpha < \beta$ such that $A_\alpha \subseteq A_\beta$.

Suppose the hypothesis H1) holds in $(L,\in_L)$. This gives us a virtual assignment $A_\alpha \subseteq \alpha$, for all "ordinals" α, where we have to be careful about what notion of ordinal is being used here. These will be the "ordinals" as given by pairs (x,d), where W'(x) and d ∈ x. These are "measured" according to the position of d in □(x). I.e., we are factoring by the equivalence relation ≡(x). Of course, the same "ordinal" is also "measured" by pairs (y,e), W'(y), provided the position of e in □(y) is the same as the position of d in □(x). These positions are compared by COMP(x,y).

A "set of ordinals" is therefore given by pairs (x,u), where W'(x) and u ⊆ x. Here we require that u is closed under the equivalence relation ≡(x). There is the obvious relationship between another such pair (y,v), representing the same "set of ordinals".

Thus H1) is represented by

H2) a virtual function F that maps the pairs (x,d), W'(x), d ∈ x, to sets $F(x,d) \subseteq x_{<d}$, where F(x,d) is closed under ≡(x) and the strict sup of F(x,d) is d in the sense of □(x). The function F will produce the same "set of ordinals" at other (x',d') that represent the same "ordinal".

Assume that C1) fails in $(L,\in_L)$. I.e.,

H3) all "inclusions" among the A_α have α = "0" or "1", all in the sense of $(L,\in_L)$.

We need to appropriately interpret this statement using Lemma 5.10. The interpretation is that

H4) if F(x,d) ⊆ F(x,e) then d = ∅ or d = {∅}.

To complete the contradiction, we now use this data to build a giant transitive set, with a system, that goes through the roof of L.

This situation is a more exotic form of the construction used in Lemma 5.16 that starts with an x-system S and defines S# and S*. Also see the paragraph after Lemma 5.16.

It is convenient to rearrange F so as to assign an x-system to each x with W'(x). Thus we have the following, which heavily uses separation.

H5) Let G be the virtual function such that for each x with W'(x), G(x) is the x-system R ⊆ x•x, where R(a,b) ↔ b ∈ F(x,a). We have the following crucial coherence property. Let y be such that W'(y). Then either the x-system G(x) is isomorphic to a proper initial segment of the y-system G(y), or the y-system G(y) is isomorphic to a proper initial segment of the x-system G(x), or the x-system G(x) is isomorphic to the y-system G(y), where, in any case, the virtual isomorphism relation is COMP(x,y).

We emphasize that, even though F,G are virtual, when localized to x with W'(x), they become actual in view of separation (i.e., every virtual subset of a set is a set).

We now use the construction arising out of Lemma 5.16. We consider the various G(x)# and their ranges G(x)*. Again, these cohere using the COMP(x,y). In particular, the G(x)* cohere in the following strong sense. For any two G(x)*,G(y)*, one is an actual initial segment of the other. This is formulated using ∪(G(x)*), ∪(G(y)*). So when we take ∪'s, the relevant isomorphism relations are identity functions. In light of H4) or Lemma 5.14, we see that any proper inclusion among these G(x)* has left side ∅ or {∅}.

To verify that # holds in $(L,\in_L)$, it suffices to derive a contradiction from the hypothesis H5).

LEMMA 5.40. Assume the hypothesis H5). Let E be the virtual union of the G(x)*. Then E is a transitive virtual set. Any proper inclusion among elements of E has left side ∅ or {∅}. E is well founded. E is nonempty. W'(E). For all x with W'(x), COMP(x,E) maps all of x.

Proof: The transitivity of E is immediate since it is the union of transitive sets. Any proper inclusion among elements of E is a proper inclusion among elements of some G(x)* by coherence. Hence by Lemma 5.14, it must have left side ∅ or {∅}. Hence all expansions in E meet {∅,{∅}}. Therefore E is a set. Now E is well founded because it is the union of well founded transitive sets. E is nonempty because W’(□^) and G(□^)* is nonempty. □(E) is a direct limit of the □(G(x)*), each one of which has no maximum element. Hence □(E) has no maximum element. W’(E) has now been established. For the final claim, if W’(x) then G(x)* is an initial segment of E. Hence COMP(x,E). QED

LEMMA 5.41. The last two claims of Lemma 5.40, W’(E), and for all x with W’(x), COMP(x,E) maps all of x, are jointly impossible.

Proof: By Lemma 5.31, we can find x with W’(x), where COMP(x,E) maps a proper initial segment of x onto E. This violates trichotomy. QED

LEMMA 5.42. H5) is false. # holds in (L,$\square_L$).

Proof: The first claim is by Lemma 5.41. The second claim is by the first claim and the paragraph just before Lemma 5.40. QED

THEOREM 5.43. ZFC + V = L + SVWISUB is interpretable in Newcomp + Ext. ZFC + V = L + SSUB is interpretable in Newcomp.

Proof: By Lemma 5.39, ZFC + V = L + SVWISUB is interpretable in □ + #. By Lemmas 5.36 and 5.42, □ + # holds in (L,$\square_L$), and this has been shown in Newcomp + Ext. Hence □ + # is interpretable in Newcomp + Ext. This establishes the first claim. For the second claim without V = L, use the first claim together with Theorem 4.11 and Corollary 2.7. By standard relativization to L, we can add V = L. QED

THEOREM 5.44. Newcomp, Newcomp + Ext, ZFC + SSUB are mutually interpretable.

Proof: ZFC + SSUB is interpretable in Newcomp by Theorem 5.43. Newcomp + Ext is interpretable in Newcomp by Theorem 4.11. Newcomp is interpretable in ZFC + SSUB by Theorem 3.1. QED

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