THE AXIOMATIZATION OF SET THEORY BY EXTENSIONALITY, SEPARATION, AND REDUCIBILITY

preliminary report
by

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INTRODUCTION

All axiomatizations in sections 1,2,4-8 are in the language L(∈,W) with just ∈ and the constant symbol W standing for a Subworld. Think of W as yesterday's world, and think of the quantifiers in the theory as ranging over today's world. The philosophy is that since the universe cannot be completed, every time we reflect on the universe and what we have reflected on previously, we obtain a larger universe.

We lead off with a very simple reaxiomatization of ZF without Foundation using just three principles: Extensionality, Subworld separation, and Reducibility. We then use a modified form of Reducibility, and obtain an outright derivation of ZF\Foundation plus the existence of a large cardinal somewhere between indescribables and subtles, appropriately formulated in ZF without choice. The formulations are the same as the usual formulations if the axiom of choice is added. Also, ZF with these large cardinals corresponds to ZFC with these same large cardinals in one of several senses including equiconsistency.

In sections 1-2, the axiomatizations prove that W is transitive. However, to get the bigger large cardinals in sections 4-8, we weaken subworld separation in a natural say to allow for W to be not transitive, and strengthen Reducibility in various ways.

From the point of view of the technical development of current large cardinal theory, the idea behind W in sections 4-8 is that it is the range of an elementary embedding from a rank into a rank, and what we are axiomatizing in clear and simple terms is that image.

In sections 9-10 we write down axioms that are much more directly motivated by the technical development of current large cardinal theory, by instead directly axiomatizing an elementary embedding via a unary function symbol. There is no subworld W. Instead we use epsilon, the unary function symbol representing the embedding, and equality. Equality is much more convenient here because of the function symbol, and we don't use equality as a primitive in the earlier sections.

Although this is comparatively brutal, it leads to very elegant axiomatizations for some very large cardinals.

It is already obvious that a lot more can be done both with the axiomatization of W and the direct axiomatization of elementary embeddings, in order to get all kinds of other cardinals. But I think that this initial paper is adequate for laying out the program.

We want to clarify the relationships between these axiomatizations and the usual systems of set theory. Let S be one of our axiomatizations from sections 1,2,4-8, and let T be a standard system of set theory. The language of S is always L(∈,W), where W is a constant symbol. The language of T is taken to be L(∈,=).

The strongest relationship between S and T is that the theorems of S in L(∈) - i.e., that don't mention W - are identical to the theorems of T in L(∈). Once we have such a result, we can use the classical results about standard formal systems of set theory in order to draw additional conclusions about related standard formal systems. This is exactly the situation that prevails in section 1.

In the other sections, we do not have appropriately elegant results of this kind. Instead, we prove somewhat weaker relationships between S and standard systems T of set theory.

In many cases, we can outright derive a standard system T of set theory (without Foundation and without the axiom of Choice) within the system S. When this is not possible, at least we are able to prove the existence of a standard model of T within S.

A standard model for the language L(∈) or L(∈,=) is defined to be a system (x,∈), where i) x is a nonempty transitive set; ii) every subset of every element of x is an element of x; and ii) every nonempty subset of x has an ∈-minimal element. A standard model of a theory in L(∈) or L(∈) or L(∈,=) is a standard model x such that the theory holds in the structure (x,∈). This are the same as the ranks (V(α),∈).

In the other direction, we often prove the existence of a standard model of the theory T in L(∈,W) within the standard set theory S. Here a standard model has the same meaning, except that W is allowed to be interpreted as any subset of the domain; i.e., W does not have to be a rank. However, in sections 1 and 2, we can also get W to be a rank. This cannot be done in sections 4-8.

Standard models have become such a standard vehicle in the comparison of set theories, that we feel justified in considering all of the axiomatizations in this paper as reaxiomatizations of set theory.

A vital feature of the standard set theories associated with the axiomatizations presented here is that they are missing the axiom of choice. This is an essential feature. For instance, ZF does not prove the existence of a standard model of each theorem of ZFC; in fact, ZF does not prove the existence of a standard model of Zermelo set theory with the axiom of choice.

Thus in this paper, we relate our axiomatizations to extensions of ZF by large cardinal axioms. In each case, we have chosen an appropriate version of the large cardinal axiom so that if ZF is replaced by ZFC then the resulting system is equivalent to a system which is familiar in the set theory literature. But one would like to know the relationship between the system with ZF and the system with ZFC. This relationship cannot be gauged by considering standard models.

The normal way of gauging this relationship is through $\in$ models. An epsilon model is the same as a standard model except the subset condition is removed.

In the case of standard systems of set theory relevant to sections 1 and 2, it is well known that by using the constructible sets construction, one obtains a close correspondence between the systems with and without choice in terms of $\in$ models and $\in$ interpretations.

In sections 4-8, an appropriate close correspondence is known through results of Hugh Woodin, where he forces the axiom of choice to hold in generic extensions of models without the axiom of choice, and preserves various large cardinal axioms. Actually, the large cardinal axioms get weakened somewhat in the process, but the correspondences are quite strong. In the case of a single measurable cardinal, such an exact correspondence has been well known through early work of Jack Silver.

Of course, in sections 8 and 10, the axiom of choice cannot be added because of Kunen's inconsistency result from the 1960's.

1. ZF\Foundation

Let $L(\in,W)$ be the classical first order predicate calculus based on the binary relation symbol $\in$ and the constant symbol W, with the usual axioms and rules of inference. Let K(W) be the theory in $L(\in,W)$ given by the following axioms.

1. Extensionality (EXT). $(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)$.
2. Subworld Separation (SS). $x \in W \rightarrow (\exists y \in W)(\forall z)(z \in y \leftrightarrow (z \in x \ \&\ \varphi))$, where φ is a formula in $L(\in,W)$ in which y is not free.
3. Reducibility (RED). $(x_1,\ldots,x_n \in W \ \&\ \varphi) \rightarrow (\exists y \in W)(\varphi)$, where $n \geq 0$ and φ is a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n,y$.

In 2, x,y,z are the variables x_1,x_2,x_3, respectively. In 3, $x_1,\ldots,x_n,y$ are the variables $x_1,\ldots,x_{n+1}$, respectively.

It is understood that the underlying logic has universal generalization as a rule (or a derived rule), and so we are at liberty to state axioms with free variables. Thus we can equivalently formulate RED as follows:

$$(\forall x_1,\ldots,x_n \in W)((\exists y)(\varphi) \to (\exists y \in W)(\varphi)),$$

where $n \geq 0$ and φ is a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n,y$.

For any formula φ in $L(\in)$, let φ^W be the result of relativizing all quantifiers to W.

Define $x = y$ if and only if $(\forall z)(z \in x \leftrightarrow z \in y)$.

LEMMA 1.1. Let φ be a formula in $L(\in,W)$ which does not mention x2. The following is provable in K(W). $x = x$. $x = y \to y = x$. $(x = y \ \&\ y = z) \to x = z$. $x = y \to (\varphi \leftrightarrow \varphi[x/y])$.

Proof: These are a version of the axioms for equality. Use EXT to derive these. □

LEMMA 1.2. Let φ be a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n$, $n \geq 0$. The following is provable in K(W). $x_1,\ldots,x_n \in W \to (\varphi \leftrightarrow \varphi^W)$.

Proof: By induction on φ. It suffices to assume this is provable for φ and to verify this is provable for $(\exists y)(\varphi)$. Let $x_1,\ldots,x_n$ be such that all free variables of $(\exists y)(\varphi)$ are among $x_1,\ldots,x_n$. Let φ' be an alphabetic variant of φ (i.e., change in bound variables) in which xn+1 does not occur, and consider $\psi = (\exists z)(\varphi'[y/z])$. Then by RED and the induction hypothesis, $\psi \leftrightarrow (\exists z \in W)(\varphi'[y/z]) \leftrightarrow (\exists z \in W)(\varphi'[y/z]^W) \leftrightarrow (\exists z \in W)(\varphi'^W[y/z]) \leftrightarrow (\exists y \in W)(\varphi^W) \leftrightarrow (\exists y)(\varphi)^W$ as required. □

LEMMA 1.3. The following is provable in K(W). $(\exists x)(x \in W)$. $x \in y \in W \to x \in W$.

Proof: For the first claim, apply RED in the case $n = 0$ to the statement $(\exists x)(x \in x \to x \in x)$.

For the second claim, let $x \in y \in W$. By SS, $(\exists z \in W)(\forall w)(w \in z \leftrightarrow (w \in y \ \&\ w \in W))$. Fix $z \in W$ with this property. Suppose $(\exists w)(w \in y \ \&\ \neg w \in z)$. Then by RED, $(\exists w \in W)(w \in y \ \&\ \neg w \in$

z), which is a contradiction. Hence $(\forall w)(w \in y \leftrightarrow w \in z)$. Therefore $y = z$, and so $x \in z$. Hence $x \in W$. □

LEMMA 1.4. The following is provable in K(W). $(x \in W \ \& \ y \subseteq x) \rightarrow y \in W$.

Proof: Let $x \in W$ and $y \subseteq x$. By SS, $(\exists z \in W)(\forall w)(w \in z \leftrightarrow (w \in x \ \& \ w \in y))$. Let $z \in W$ have this property. Then $z = y$. Hence $y \in W$. □

We now show that each of the axioms of ZF(rfn)\F is provable in K(W).

Before doing this, we can simplify the axioms of ZF(rfn)\F by a well known trick as follows.

1. Extensionality. $(\forall x)(x \in y \leftrightarrow x \in z) \rightarrow (\forall x)(y \in x \leftrightarrow z \in x)$.
2. Pairing. $(\exists x)(y,z \in x)$.
3. Union. $(\exists x)(\forall y \in w)(\forall z \in y)(z \in x)$.
4. Separation. $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \ \& \ \varphi))$, where φ is a formula in $L(\in)$ in which x is not free.
5. Power set. $(\exists x)(\forall y)(y \subseteq z \rightarrow y \in x)$.
6. Reflection. $(\exists \text{ transitive } x)(y_1,\ldots,y_n \in x \ \& \ (\forall z_1,\ldots,z_m \in x)((\exists w)(\varphi) \rightarrow (\exists w \in x)(\varphi)))$, where $m,n \geq 1$ and φ is a formula in $L(\in)$ whose free variables are among $y_1,\ldots,y_n,z_1,\ldots,z_m,w$.
7. Infinity. $(\exists x)(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x))$.

LEMMA 1.5. Pairing is provable in K(W).

Proof: Obviously $(\forall y,z \in W)(\exists x)(y,z \in x)$. Hence by RED, $(\forall y,z)(\exists x)(y,z \in x)$. □

LEMMA 1.6. Union is provable in K(W).

Proof: By Lemma 1.3, $(\forall w \in W)(\exists x)(\forall y \in w)(\forall z \in y)(z \in x)$, because we can set $x = W$. Hence by taking contrapositives and applying RED, we obtain $(\forall w)(\exists x)(\forall y \in w)(\forall z \in y)(z \in x)$. □

LEMMA 1.7. Separation is provable in K(W).

Proof: Let φ be a formula in $L(\in)$ whose free variables are among $y_1,\ldots,y_n$, $n \geq 1$. By SS, we have $(\forall y_1,\ldots,y_n \in W)(\exists x)(\forall y)(y$

$\in x \leftrightarrow (y \in z \ \& \ \varphi)$). By taking contrapositives and applying reducibility, we obtain $(\forall y_1,\ldots,y_n)(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \ \& \ \varphi))$. ∎

LEMMA 1.8. Power set is provable in K(W).

Proof: By Lemma 1.4, $(\forall z \in W)(\exists x)(\forall y)(y \subseteq z \rightarrow y \in x)$, because we can set x = W. Hence by taking contrapositives and applying RED, we obtain $(\forall z)(\exists x)(\forall y)(y \subseteq z \rightarrow y \in x)$. ∎

LEMMA 1.9. Reflection is provable in K(W).

Proof: Let φ be a formula in $L(\in)$ whose free variables are among $y_1,\ldots,y_n,z_1,\ldots,z_m,w$, where $m,n \geq 1$. By RED, it suffices to prove Reflection for $y_1,\ldots,y_n \in W$. We claim that x = W works. To see this, by Lemma 1.3, W is a nonempty transitive set with $y_1,\ldots,y_n \in W$. Let $z_1,\ldots,z_m \in W$. By RED, $(\exists w)(\varphi) \rightarrow (\exists w \in x)(\varphi)$. ∎

LEMMA 1.10. Infinity is provable in K(W).

Proof: By Separation (Lemma 1.7) and Lemma 1.3, $\emptyset$ exists, and $\emptyset \in W$. Now suppose $y \in W$. By Lemma 1.3, clearly $(\exists z)(\forall w)((w \in y \vee w = y) \rightarrow w \in z)$, by setting z = W. Hence by RED, $(\exists z \in W)(\forall w)((w \in y \vee w = y) \rightarrow w \in z)$. Fix $z \in W$ with this property. Then by SS, $y \cup \{y\} \in W$. We have proved that $\emptyset \in W \ \& \ (\forall y \in W)(y \cup \{y\} \in W)$. ∎

LEMMA 1.11. Every theorem of ZF(rfn)\F is a theorem of K(W).

Proof: We have verified the derivability of all of the axioms of ZF(rfn)\F in K(W). ∎

LEMMA 1.12. Let $\varphi_1,\ldots,\varphi_k$ be formulas in $L(\in)$ whose free variables are among $x_1,\ldots,x_n,y$, $n \geq 1$. The following is provable in ZF(rfn)\F. There exists Y such that

i) Y is a nonempty transitive set;

ii) for all $x_1,\ldots,x_n \in Y$, if there exists y such that φ, then there exists $y \in Y$ such that φ.

Proof: Let $\psi(x_1,\ldots,x_n,y,z,w)$ be the conjunction of

$(y = z \rightarrow w = \omega)$ & $((y = 1$ & $z = \omega) \rightarrow \varphi_1(x_1,\ldots,x_n,y,z,w))$ & … & $((y = k$ & $z = \omega) \rightarrow \varphi_k(x_1,\ldots,x_n,y,z,w))$. Then apply reflection to ψ. □

LEMMA 1.13. The existential quantification by W of the conjunction of the universal closures of any finite number of axioms of K(W) is provable in ZF(rfn)\F.

Proof: From Lemma 1.11. □

We say that a transitive set x is well founded if and only if every nonempty subset of x has an $\in$-minimal element. We say that a set x is hereditarily well founded if and only if x is an element of some well founded transitive set. We write HW for the universe of hereditarily well founded sets.

It is well known how one can prove within ZF\F that all axioms of ZF hold in HW. From this fact, we can read off a direct relationship between K(W) and all of ZF.

THEOREM 1.14. a) The relativization of every theorem of ZF to HW is a theorem of K(W);

b) Every theorem of K(W) in the language $L(\in)$ is a theorem of ZF;

c) K(W) and ZF prove the same sentences relativized to HW in the language $L(\in)$;

d) A formula in $L(\in)$ is provable in K(W) if and only if it is provable in ZF(rfn)\F;

e) The theorems of K(W) are exactly the formulas in $L(\in,W)$ such that the existential quantification by W of its universal closure is a theorem of ZF(rfn)\F.

2. Indescribable and subtle cardinals

Let $K_1(W)$ be the following theory in the language $L(\in,W)$.

1. EXT.
2. SS.
3. RED′. $(x \subseteq W$ & $\varphi) \rightarrow (\exists y,z \in W)(y \subseteq x$ & $\varphi[x/y])$, where φ is a formula in $L(\in)$ whose free variables are among x,z, and y does not occur in φ.

Here $x,y,z_1,\ldots,z_n$ are the variables $x_1,\ldots,x_{n+2}$, respectively. $\varphi[x/y]$ is the result of replacing all free occurrences of x in φ by y.

The indescribable level of the large cardinal hierarchy, which is normally studied in the context of ZFC, makes perfectly good sense in ZF, and hence in ZF without Foundation. It is natural in this context to make the minor modification of considering indescribable ordinals rather than indescribable cardinals, although one proves that indescribable ordinals are cardinals in an appropriate sense within ZF.

We will show that K1(W) proves ZF(rfn)\F + “there exists an extremely indescribable ordinal”, as well as the existence of a standard model of ZF + “there exists an extremely indescribable ordinal,” and the existence of an $\in$ model of ZFC + “there exists an extremely indescribable cardinal.”

On the other hand, ZFC + “there exists a subtle cardinal” proves the existence of a standard model of $K_1(W)$.

First we review indescribable and subtle cardinals in ZFC. Let $n \geq 1$ and κ be a cardinal. We say that κ is n-th order indescribable if and only if for all $R \subseteq V(\kappa)$ and first order sentence φ, if $(V(\kappa+n),\in,R)$ satisfies φ, then there is an $\alpha < \kappa$ such that $(V(\alpha+n),\in,R \cap V(\alpha))$ satisfies φ.

We say that κ is totally indescribable if and only if κ is n-th order indescribable for all $n \geq 2$. This definition agrees with the one given in [Ka94], p. 59.

It is natural to go a bit further, and so we say, in ZFC, that κ is extremely indescribable if and only if for all $R \subseteq V(\kappa)$ and first order sentence φ, if $(V(\kappa+\kappa),\in,R)$ satisfies φ, then there is an $\alpha < \kappa$ such that $(V(\alpha+\alpha),\in,R \cap V(\alpha))$ satisfies φ.

Let κ be a cardinal. We say that $f:\kappa \to S(\kappa)$ is regressive if and only if for all $\alpha < \lambda$, $f(\alpha) \subseteq \alpha$. (Here S is the power set operation).

In ZFC, we say that κ is a subtle cardinal if and only if

i) κ is a cardinal;

ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:\kappa \to S(\kappa)$, there exists $\alpha < \beta$, $\alpha,\beta \in C \cap \kappa$, such that $f(\alpha) = f(\beta) \cap \beta$.

The definition of extreme indescribability in ZF is identical to the definition given above in ZFC except that it is natural to simply drop the requirement that the ordinal be a cardinal. Thus we speak of extremely indescribable ordinals. Without Foundation, ordinals are defined to be $\in$-connected transitive sets such that every nonempty subset has an $\in$-least element.

In ZF(rep)\F, we say that $V(\alpha)$ is strongly inaccessible if and only if the range of every function from an element of $V(\alpha)$ into $V(\alpha)$ is an element of $V(\alpha)$. Here ZF(rep) is formulated with Replacement.

LEMMA 2.1. It is provable in ZF(rep)\F that every strongly inaccessible rank satisfies ZF, and that if α is an extremely indescribable ordinal then $V(\alpha)$ is strongly inaccessible. There is an $\in$-interpretation of ZFC + “there exists an extremely indescribable cardinal” in ZF(rep)\F + “there exists an extremely indescribable ordinal” using the constructible sets in the usual way.

Proof: The first claim is standard, and no Choice is involved. For the second claim, first note that the theory of the cumulative hierarchy - i.e., the $V(\gamma)$’s - is formalizable in ZF(rep)\F. For the second claim, we can formalize the theory of the constructible hierarchy of sets within ZF(rep)\F. We claim within ZF(rep)\F that if α is an extremely indescribable ordinal then the following holds:

> for all $R \subseteq V(\alpha)$, $R \in L$, and first order sentence φ, if $(V(\alpha+\alpha) \cap L,\in,R)$ satisfies φ, then there is a $\beta < \alpha$ such that $(V(\alpha+\alpha),\in,R \cap V(\beta))$ satisfies φ.

Here L indicates the proper class of constructible sets. From the theory of constructible sets within ZF(rep)\F, we can replace $R \in L$ by strengthening the sentence φ, so that this becomes a special case of the extreme indescribability of $V(\alpha)$. And of course the relativization to L of every axiom of ZFC is a theorem of ZF(rep)\F. This argument already establishes the final claim from elementary absoluteness considerations and from the first claim. □

LEMMA 2.2. The following is provable in ZFC. Let κ be a subtle cardinal. Then κ is strongly inaccessible. And let $f:\kappa \to V(\kappa)$ be such that $f(\alpha) \subseteq V(\alpha)$, and let $C \subseteq \kappa$ be closed and unbounded. Then there exists $\alpha < \beta$ from C such that $f(\alpha) = f(\beta) \cap V(\alpha)$.

Proof: For the first claim, see [Ba73]. From the strong inaccessibility of κ, one can construct a one-one map h from $V(\kappa)$ into $S(\kappa)$ such that $\{\alpha < \kappa$: h maps $V(\alpha)$ onto the bounded elements of $S(\alpha)$, and maps $V(\alpha+1)$ onto $S(\alpha)\}$ is closed and unbounded. The second claim is easy to verify using this map. □

With this background material on large cardinals, we are now prepared to discuss $K_1(W)$.

LEMMA 2.3. Let φ be a formula in $L(\in,W)$ which does not mention x2. The following is provable in K(W). $x = x$. $x = y \rightarrow y = x$. $(x = y \ \& \ y = z) \rightarrow x = z$. $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$.

Proof: These are a version of the axioms for equality. Use EXT to derive these. □

LEMMA 2.4. Let $n \geq 0$. The following is provable in $K_1(W)$. W is nonempty. $\emptyset$ exists. $(\forall x_1,\ldots,x_n)(\{x_1,\ldots,x_n\}$ exists). $(\forall x_1,\ldots,x_n \in W)(\{x_1,\ldots,x_n\} \in W)$.

Proof: To see that W is nonempty, apply RED′ with x = W and φ arbitrary. We now prove the third claim by induction on $n \geq 1$.

For the basis case, let n = 1. We first claim that $(\forall x)(\exists y)(x \in y)$. To see this, let x be a counterexample. By RED′, we find a counterexample $x \in W$. This is a contradiction. Next we claim that $(\forall x)(\{x\}$ exists). Suppose this is false, and let y be such that $(\exists x \in y)(\{x\}$ does not exist). By RED′, there is such a $y \in W$. But this violates SS.

Now suppose $(\forall x_1,\ldots,x_n)(\{x_1,\ldots,x_n\}$ exists). We wish to prove that $(\forall x_1,\ldots,x_{n+1})(\{x_1,\ldots,x_{n+1}\}$ exists). We first prove that $(\forall x_1,\ldots,x_{n+1})(\exists y)(x_1,\ldots,x_{n+1} \in y)$. We can assume that the x's are distinct.

By RED′, let $x_1 \in W$ be such that $\neg(\forall x_2,\ldots,x_{n+1})(\exists y)(x_1,\ldots,x_{n+1} \in y)$. Now use $\{x_1\} \subseteq W$ to obtain $x_2 \in W$ such that

$\neg(\forall x_3,\ldots,x_{n+1})(\exists y)(x_1,\ldots,x_{n+1} \in y)$. Continue in this way, finally using $\{x_1,\ldots,x_n\} \subseteq W$ to obtain $x_{n+1} \in W$ such that $\neg(\exists y)(x_1,\ldots,x_{n+1} \in y)$. But we can set $y = W$, thus obtaining a contadiction.

Thus we have proved $(\forall x_1,\ldots,x_{n+1})(\exists y)(x_1,\ldots,x_{n+1} \in y)$. Now suppose y is such that $(\exists x_1,\ldots,x_{n+1} \in y)(\{x_1,\ldots,x_{n+1}\}$ does not exist). By RED′, let $y \in W$ have this property. This contradicts SS.

For the fourth claim, let $x_1,\ldots,x_n \in W$. Then $\{x_1,\ldots,x_n\} \subseteq W$. Apply RED′ to "$\{x_1,\ldots,x_n\}$ has exactly m elements and $y = \{x_1,\ldots,x_n\}$," where {x1,…,xn} actually has exactly m elements. $\square$

LEMMA 2.5. $K_1(W)$ extends $K(W)$. $K_1(W)$ proves ZF(rfn)\F.

Proof: We have only to prove RED in $K_1(W)$. Let $x_1,\ldots,x_n \in W$. and suppose $\varphi(x_1,\ldots,x_n,y)$. By adjusting φ, we can, for the purposes at hand, assume that $x_1,\ldots,x_n$ are distinct. Then we can apply an appropriate instance of RED′ with $x = \{\{x_1\},\{x_1,x_2\},\ldots,\{x_1,\ldots,x_n\}\} \subseteq W$ to obtain the desired $y \in W$.

$\square$

We now have access to ZF(rfn)\F.

It is convenient to define RED^+ as the following immediate strengthening of RED:

RED^+. $(x_1,\ldots,x_n \in W \ \&\ y \subseteq W \ \&\ \varphi) \rightarrow (\exists z,w \in W)(z \subseteq y \ \&\ \varphi[y/z])$, where φ is a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n,y,w$.

Here $x_1,\ldots,x_n,y,z,w$ are the variables $x_1,\ldots,x_{n+3}$, respectively.

LEMMA 2.6. $K_1(W)$ proves RED^+.

Proof: Let $x_1,\ldots,x_n \in W$ and $y \subseteq W$. Let $x' = y \bullet \{\emptyset\} \cup \langle x_1,\ldots,x_n\rangle$, and apply RED′ with an appropriate φ. $\square$

Let OW be the set of all ordinals $\beta \in W$ such that $\beta \subseteq W$. It is easy to see that OW is the least ordinal not in W.

THEOREM 2.7. $K_1(W)$ proves ZF(rfn)\F + "there exists an extremely indescribable ordinal." $K_1(W)$ proves the existence of a standard model of ZF + "there exists an extremely indescribable ordinal." $K_1(W)$ proves the existence of an $\in$

model of ZFC + "there exists an extremely indescribable cardinal."

Proof: We first prove in $K_1(W)$ that OW is an extremely indescribable ordinal. Let $R \subseteq V(OW)$ and φ be a sentence such that $(V(OW+OW),\in,R)$ satisfies φ. We must establish the conclusion of extreme indescribability.

Now by RED^+, we see that for all $\alpha \in OW$, $V(\alpha) \in W$. Hence $V(OW) \subseteq W$, and so $R \subseteq W$. If $R \in V(OW)$ then we are done. So we assume that $rk(R) = OW$. We let $S = \{R \cap V(\alpha): \alpha \in OW\}$. Note that $\cup S = R$. Now the hypothesis can be restated in the following form:

$$(\exists\beta)(\beta \text{ is a limit ordinal } \& \ (\forall x,y \in S)(x \cap V(\min(rk(x),rk(y))) = y \cap V(\min(rk(x),rk(y)))) \ \& \ rk(S) = \beta \ \& \ (V(\beta+\beta),\in,\cup S) \text{ satisfies } \varphi).$$

By RED^+, let $S' \in W$, $S' \subseteq S$, $\beta \in OW$, be such that

$$\beta \text{ is a limit ordinal } \& \ (\forall x,y \in S')(x \cap V(\min(rk(x),rk(y))) = y \cap V(\min(rk(x),rk(y)))) \ \& \ rk(S') = \beta \ \& \ (V(\beta+\beta),\in,\cup S') \text{ satisfies } \varphi.$$

Then it is easily verified that $S' = S \cap V(\beta)$.

Now that we know there is an extremely indescribable ordinal, we can apply RED^+ to obtain $V(\alpha)$, where $\alpha < OW$ is extremely indescribable. Then $V(OW)$ provides a standard model of ZF + "there exists an extremely indescribable ordinal." By Theorem 2.1, we obtain an $\in$ model of ZFC + "there exists an extremely indescribable cardinal." by passing to L.□

THEOREM 2.8. ZFC + "there exists a subtle cardinal" proves the existence of a standard model of $K_1(W)$; i.e., of the form $(V(\kappa),\in,V(\lambda))$.

Proof: Let κ be a subtle cardinal. Let $C = \{\alpha < \kappa: V(\alpha) \text{ is an elementary submodel of } V(\kappa)\}$. Since $V(\alpha)$ is strongly inaccessible, C is a closed and unbounded subset of κ.

Now define $f:\kappa \to V(\kappa)$ as follows. Let $\alpha \in C$ be given. Suppose that there exists a formula φ in $L(\in)$ with at most the free variables $x_1,\ldots,x_n,y$, and elements $x_1,\ldots,x_n \in V(\alpha)$, and $y \subseteq V(\alpha)$ such that in $V(\kappa)$,

$$\varphi(x_1,\ldots,x_n,y) \text{ holds, yet there is no } z \subseteq y \text{ with } z \in V(\alpha) \text{ such that } \varphi(x_1,\ldots,x_n,z) \text{ holds.}$$

Define $f(\alpha)$ to be $\{<\varphi,x_1,\ldots,x_n,u>: u \in y\}$, where $\varphi,x_1,\ldots,x_n,y$ is chosen as indicated. In all other cases, set $f(\alpha) = 0$.

By Lemma 2.2, let $\alpha < \beta$ be both from C, where $f(\alpha) = f(\beta) \cap V(\alpha)$. Note that α,β are limit ordinals.

Now clearly the main cases in the definitions of $f(\alpha),f(\beta)$ hold or fail together. Suppose that the main cases both hold. Then the y's are nonempty, and they use the same φ and $x_1,\ldots,x_n$. Let y be used for $f(\alpha)$ and y' for $f(\beta)$. Then $y \subseteq y'$ and $y \in V(\beta)$. Also $\varphi(x_1,\ldots,x_n,y)$ and $\varphi(x_1,\ldots,x_n,y')$ hold in $V(\kappa)$. But this contradicts the displayed condition for $f(\beta)$.

We have now shown that the main case fails in the definitions of $f(\alpha),f(\beta)$. We now verify that $(V(\kappa),\in,V(\beta))$ is a model of $K_1(W)$. RED^+ is the only nontrivial case.

Let φ be a formula in $L(\in)$ with at most the free variables $x_1,\ldots, x_{n+2}$, and let $x_1,\ldots,x_n \in V(\beta)$, and $x_{n+1} \subseteq V(\beta)$, and $\varphi(x_1,\ldots,x_{n+1},x_{n+2})$ holds in $V(\kappa)$. Then by the crucial failure of the displayed condition in the first case of the definition of $f(\beta)$, we see that $(\exists x_{n+3} \in V(\beta))(x_{n+3} \subseteq x_{n+1}\ \&\ (\exists x_{n+2} \in V(\kappa))(\varphi[x_{n+1}/x_{n+3}]))$. But since $\beta \in C$, we have $(\exists x_{n+2},x_{n+3} \in V(\beta))(x_{n+3} \subseteq x_{n+1}\ \&\ \varphi[x_{n+1}/x_{n+3}])$ as required by RED^+. □

3. Some useful preliminaries in EST

In the rest of the paper, there are contexts where we do not have ZF\F at our disposal as we did in sections 1 and 2. We present some useful elementary material with the system ES. ES is the following theory in the language $L(\in)$.

1. EXT.
2. Separation. $(\exists x)(\forall y)(y \in x \leftrightarrow y \in z\ \&\ \varphi)$, where φ is a formula in $L(\in)$ in which x is not free.
3. Tupling. $(\exists x)(y_1,\ldots,y_n \in x)$, where $n \geq 1$.

In 2, x,y,z are the variables x_1,x_2,x_3, respectively. In 3, $x,y_1,\ldots,y_n$ are the variables $x_1,\ldots,x_{n+1}$, respectively.

The development is directed towards the cumulative hierarchy.

We use x = y as an abbreviation for $(\forall z)(z \in x \leftrightarrow z \in y)$.

LEMMA 3.1. Let φ be a formula in $L(\in,W)$ which does not mention x_2. The following is provable in K(W). x = x. $x = y \rightarrow y = x$. $(x = y \;\&\; y = z) \rightarrow x = z$. $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$.

Proof: See Lemma 1.1. □

We make all of the usual definitions of rudimentary set theory such as $\{x_1,\ldots,x_n\}$, $<x_1,\ldots,x_n>$, $x_1 \cup \ldots \cup x_n$, $\cup x$, $x_1 \cap \ldots \cap x_n$, PS(x), $\emptyset$. Here we have to be worried about the existence of $x_1 \cup \ldots \cup x_n$, $\cup x$, and PS(x); the existence of the others follows from EST. Here PS(x) is the power set of x. We also define function, domain, range, one-one in the usual way. Again, we have to be worried about existence of domains and ranges. We write $f:x \rightarrow y$ to indicate that the domain is x and all values are elements of y.

We say that x is an ordinal if and only if x is transitive, $\in$-connected, and well founded. We use $x \in= y$ to abbreviate $(x \in y$ or $x = y)$.

LEMMA 3.2. The following is provable in EST. Every element of an ordinal is an ordinal. The ordinals are strictly linearly ordered under $\in$. Transfinite induction can be applied to the ordinals with respect to any formula with any parameters. Every transitive set of ordinals is an ordinal.

Proof: Let $x \in y$ where y is an ordinal. Obviously x is $\in$-connected. To see that x is transitive, let $a \in b \in x$. Then $a,b \in y$, and so $a \in x$ or $x \in a$ or $a = x$. Using Separation, the latter two disjuncts violate the well foundedness of y. So $a \in x$. Also x is clearly well founded since $x \subseteq y$. So x is an ordinal.

Assume the second claim is false. Let x be an ordinal which is not $\in$-comparable with every ordinal. Then either x is $\in$-minimal with this property, or some element of x is $\in$-minimal with this property (by using Separation to form the set of all elements of x with this property). Thus we can fix x to be an ∈-minimal ordinal which is not ∈-comparable with every ordinal.

Thus every element of x is ∈-comparable with every ordinal. Let y be an ordinal. Then every element of x is ∈-comparable with y. Hence either every element of x is in y, or y ∈ x. So we can assume that x ⊆ y. If x = y then we are done. Otherwise, let z be an ∈-minimal element of y which is not in x. Then z ⊆ x. It suffices to prove that x ⊆ z. Let b ∈ x. Then b,z are comparable. So b ∈ z or z ∈ b or b = z. If z ∈= b then z ∈ x, which is a contradiction. Hence b ∈ z as required.

Transfinite induction holds on the ordinals using Separation and the second claim.

Let x be a transitive set of ordinals. Then x is ∈-connected. And obviously any set of ordinals is well founded. □

For sets x, define x+1 = {y: y ∈ x or y = x}. We have to worry about existence.

LEMMA 3.3. The following is provable in EST. If x+1 exists, then x is an ordinal if and only if x+1 is an ordinal. If x,y are ordinals and x+1 = y+1 then x = y.

Proof: If x+1 is an ordinal then by Lemma 8.1, x is an ordinal. If x is an ordinal and x+ exists, then it is easy to verify that x+ is an ordinal. If x+1 = y+1 then x ∈= y and y ∈= x. If x,y are ordinals, the only possible case is x = y. □

A limit ordinal is a nonempty ordinal with an ∈-maximum element. We use 0 for the empty ordinal; i.e., ∅. A successor ordinal is an ordinal of the form x+1, in which case its predecessor is x. This is well defined by Lemma 3.3.

LEMMA 3.4. The following is provable in EST. Every ordinal is 0, a limit ordinal, or a successor ordinal, and these cases are mutually exclusive. If x ∈ y and y is an ordinal then x+1 ∈= y.

Proof: Suppose x ≠ 0 and has the ∈-maximum element y. By comparing any z ∈ x with y, we see that x = {z: z ∈= y} = y+1.

For the second claim, let $x \in y$, y is an ordinal. Let x' be the $\in$-least element of y such that $x \in x'$. If x' does not exist then x is the $\in$-maximum element of y, in which case $y = x+1$ by claim 1. So we can assume that x' exists, in which case x is the $\in$-maximum element of x'. So again by claim 1, $x' = x+1$, and so $x+1 \in y$. □

LEMMA 3.5. The following is provable in EST. Let x be a limit ordinal. Then $\emptyset \in x$ and $(\forall y \in x)(y+1 \in x)$.

Proof: Since $x \neq \emptyset$ and $x,\emptyset$ are $\in$-comparable, we have $\emptyset \in x$. Now let $y \in x$. By Lemma 3.3, $y+1 \in= x$. Since x is a limit ordinal, $y+1 \in x$. □

We are now prepared to define the cumulative hierarchy. We say that f is a cumulation function on x if and only if

i) x is an ordinal;
ii) f is a function with domain x;
iii) if $0 \in x$ then $f(0) = 0$;
iv) if $y+1 \in x$ then $f(y) = PS(f(x))$;
v) if $y \in x$ and y is a limit ordinal, then f(y) is the set of all elements of the f(z), $z \in y$.

We say that f is a cumulation function if and only if f is a cumulation function on some x.

LEMMA 3.6. The following is provable in EST. If f is a cumulation function on x and $y \in x$, then $f|y$ is a cumulation function on y. There is at most one cumulation function on any x. Any two cumulation functions are comparable under $\subseteq$.

Proof: The first claim is by inspection (note that by separation, $f|y$ exists). The remaining claims are by straightforward transfinite inductions. □

For ordinals α, we define $V(\alpha)$ to be the unique value of all cumulation functions with domain $\alpha+1$. In order for $V(\alpha)$ to exist, $\alpha+1$ must exist as well as a cumulation function on $\alpha+1$.

LEMMA 3.7. The following is provable in EST. If $\beta < \alpha$ and $V(\alpha)$ exists then $V(\beta)$ exists. $V(0) = 0$. If $V(\alpha+1)$ exists then

$V(\alpha+1) = PS(V(\alpha))$. If $V(\alpha)$ exists and α is a limit ordinal, then $V(\alpha)$ is the union of the $V(\beta)$, $\beta < \alpha$.

Proof: Obvious from the definitions and Lemma 3.5. □

LEMMA 3.8. The following is provable in EST. Each $V(\alpha)$ is transitive. If $V(\alpha+1)$ exists then $V(\alpha) \subseteq V(\alpha+1)$.

Proof: For the first claim, suppose that for all $\beta < \alpha$, $V(\beta)$ is transitive and $V(\alpha)$ exists. If α is a limit ordinal, let $x \in y \in V(\alpha)$. Then let $\beta < \alpha$, $y \in V(\beta)$. By the induction hypothesis, $x \in V(\beta)$, and so $x \in V(\alpha)$. If $\alpha = \beta+1$, then $V(\beta)$ is transitive. Let $x \in y \in V(\alpha)$. Then $y \subseteq V(\beta)$, and so $x \in V(\beta)$. Hence $x \subseteq V(\beta)$. Therefore $x \in V(\alpha)$. And obviously $V(0)$ is transitive.

For the second claim, let $x \in V(\alpha)$. Then $x \subseteq V(\alpha)$, and so $x \in V(\alpha+1)$. □

LEMMA 3.9. The following is provable in EST. If $\alpha < \beta$ and $V(\beta)$ exists, then $V(\alpha) \subseteq V(\beta)$ and $V(\alpha) \in V(\beta)$.

Proof: Let γ be an ordinal such that $V(\gamma)$ exists and for all $\beta < \gamma$, the claim holds. First suppose γ is a limit ordinal. Let $\alpha < \gamma$. We must check that $V(\alpha) \subseteq V(\gamma)$ and $V(\alpha) \in V(\gamma)$. To see the first, let $x \in V(\alpha)$. Then $x \in V(\gamma)$. To see the second, since the claim holds for $\alpha+1$, we have $V(\alpha) \in V(\alpha+1)$, and so $V(\alpha) \in V(\gamma)$.

Now suppose $\gamma = \delta+1$. Let $\alpha \leq \delta$. Then $V(\alpha) \subseteq V(\delta) \subseteq V(\gamma)$. And if $\alpha = \delta$ then $V(\alpha) \in V(\gamma)$. If $\alpha < \delta$ then $V(\alpha) \subseteq V(\delta)$, and so $V(\alpha) \in V(\gamma)$. □

LEMMA 3.10. The following is provable in EST. The union of every element of $V(\alpha)$ lies in $V(\alpha)$. If $V(\alpha+1)$ exists then the pair of any two elements of $V(\alpha)$ lies in $V(\alpha+1)$. If $V(\alpha+2)$ exists then the power set of any element of $V(\alpha)$ lies in $V(\alpha+2)$.

Proof: For the first claim, let $x \in V(\alpha)$. Now if α is a limit ordinal then let $x \in V(\beta)$, $\beta < \alpha$. Hence every element of an element of x lies in $V(\beta)$. Hence the union of x exists as a subset of $V(\beta)$, and hence as an element of $V(\alpha)$. Now let $\alpha = \beta+1$, and $x \in V(\alpha)$. Then $x \subseteq V(\beta)$, and so every element of an element of x lies in $V(\beta)$. Hence the union of x exists as a subset of $V(\beta)$, and so as an element of $V(\alpha)$.

The second claim is obvious. For the third claim, let $x \in V(\alpha)$. Then $x \subseteq V(\alpha)$, and so every subset of x lies in $V(\alpha+1)$. Hence the power set of x exists as a subset of $V(\alpha+1)$, and so lies in $V(\alpha+2)$. □

LEMMA 3.11. The following is provable in EST. If ω is the first limit ordinal and $V(\omega)$ exists, then $\varnothing \in V(\omega)$, and for all $x \in V(\omega)$, $x+1 \in V(\omega)$.

Proof: Clearly $\varnothing \in V(\omega)$ by Lemma 3.9. Now let $x \in V(\omega)$, and let $x \in V(\alpha)$, $\alpha < \omega$. By transitivity, x+1 exists as a subset of $V(\alpha)$, and so x+1 lies in $V(\alpha+1)$, and hence in $V(\omega)$. □

LEMMA 3.12. The following is provable in EST. Every subset of every $V(\alpha)$ lies in $V(\alpha)$. Every $V(\alpha)$ includes α as a subset.

Proof: For the first claim, suppose this is true for all $\beta < \alpha$ and $V(\alpha)$ exists. If α is a limit ordinal, then let $x \subseteq y \in V(\alpha)$. Let $x \subseteq y \in V(\beta)$, $\beta < \alpha$. Then $x \in V(\beta)$, and so $x \in V(\alpha)$. If $\alpha = \beta+1$, then let $x \subseteq y \in V(\alpha)$, and so $x \subseteq y \subseteq V(\beta)$. Then $x \in V(\alpha)$.

For the second claim, suppose this is true for all $\beta < \alpha$ and $V(\alpha)$ exists. The zero case and limit case is trivial using Lemma 3.9. Suppose $\alpha = \beta+1$. Then $\beta \subseteq V(\beta)$. Since every element of $\beta+1$ is therefore a subset of $V(\beta)$, we see that $\beta+1 \subseteq V(\beta+1)$. □

It is useful to define rk(x) to be the least ordinal α such that $x \in V(\alpha)$ provided this exists; undefined otherwise.

LEMMA 3.13. The following is provable in EST. If rk(x) exists then rk(x) is a limit ordinal. If $x \in y$ and rk(y) exists then rk(x) < rk(y). If rk(x) exists then rk(x) is the least successor ordinal greater than all rk(y), $y \in x$.

Proof: The first claim is obvious. For the second claim, let $x \in y$ and $rk(y) = \alpha+1$. Then $x \in y \subseteq V(\alpha)$, and so $x \in V(\alpha)$, in which case $rk(x) \leq \alpha < \alpha+1$. For the third claim, let $rk(x) = \alpha+1$. Then $\alpha+1$ is greater than all $rk(y)$, $y \in x$ by the second claim. On the other hand, suppose $\beta+1 < \alpha+1$ and for all $y \in x$, $rk(y) < \beta+1$. Then $x \subseteq V(\beta)$, and so $x \in V(\beta+1)$, violating $rk(x) = \alpha+1$. □

LEMMA 3.14. The following is provable in EST. Every nonempty subset of every $V(\alpha)$ has an $\in$-minimal element.

Proof: Let $x \subseteq V(\alpha)$ be nonempty. Let y be an element of x such that rk(y) is minimized. Suppose $z \in y,x$. Then by Lemma 3.13, $rk(z) < rk(y)$, contradicting the minimality of rk(y). □

4. Measurable cardinals

Let $K_2(W)$ be the following theory in the language $L(\in,W)$ with the constant symbol W.

1. EXT.
2. SS^-. $x \in W \rightarrow (\exists y \in W)(\forall z \in W)(z \in y \leftrightarrow (z \in x \;\&\; \varphi))$, where φ is a formula in $L(\in,W)$ in which y is not free.
3. RED.
4. TRANS. $(\exists x \in W)(\forall \text{ transitive } y \subseteq W)(y \subseteq x)$.

Here x,y,z are the variables x1,x2,x3, respectively.

Note that SS^- is weaker than SS in that the quantifier z is restricted to W. This is the natural formulation of Subworld Separation that accommodates the possibility that some elements of yesterday's world may pick up new elements from today's world. Another way of saying this is that yesterday's world may not be transitive today.

This form of Subworld Separation asserts that every property of elements of any element of W has a representation, y, in W that at least represents the property with regard to elements of W. We actually prove that this representation $y \in W$ is unique.

Note that we can formally derive the nontransitivity of W as follows. Suppose W is transitive. Then by TRANS, let $W \subseteq x \in W$. By SS^-, let $y \in W$ be such that for all $z \in W$, $z \in y \leftrightarrow (z \in x \ \&\ z \notin z)$. I.e., for all $z \in W$, $z \in y \leftrightarrow z \notin z$. Then $y \in y \leftrightarrow y \notin y$.

We remark that nothing in the axioms of $K_2(W)$ commits us to the existence of transitive closures or “every set is an element (or subset) of a transitive set,” and this cannot be proved in $K_2(W)$.

We will show that $K_2(W)$ proves the existence of a standard model of ZF + "there is a measurable rank." ZF + "there is a nontrivial elementary embedding from some rank into some rank" proves the existence of a standard model of $K_2(W)$.

Under the definitions made below, ZF + "there is a measurable rank" is $\in$ bi-interpretable with ZFC + "there is a measurable cardinal." In ZFC, V(alpha) is measurable if and only if alpha is a measurable cardinal.

As usual, we use $x = y$ to abbreviate $(\forall z)(z \in x \leftrightarrow z \in y)$.

LEMMA 4.1. Let φ be a formula in $L(\in,W)$ which does not mention x_2. The following is provable in $K_2(W)$. $x = x$. $x = y \rightarrow y = x$. $(x = y \ \&\ y = z) \rightarrow x = z$. $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$.

Proof: See Lemma 1.1. ∎

For any formula φ in $L(\in)$, let φ^W be the result of relativizing all quantifiers to W.

LEMMA 4.2. Let φ be a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n$, $n \geq 0$. The following is provable in $K_2(W)$. $x_1,\ldots,x_n \in W \rightarrow (\varphi \leftrightarrow \varphi^W)$.

Proof: See Lemma 1.2. ∎

LEMMA 4.3. $K_2(W)$ proves EST. In fact, axioms 1-3 of $K_2(W)$ suffice to prove EST.

Proof: For Separation, let φ be a formula in $L(\in,W)$ whose free variables are among $x_1,x_3,\ldots,x_n$, $n \geq 3$. By SS^-, $(\forall x_1,x_4,\ldots,x_n \in W)(\exists x_2 \in W)(\forall x_3 \in W)(x_3 \in x_2 \leftrightarrow (x_3 \in x_1 \ \&\ \varphi^W))$. By Lemma 4.2, $(\forall x_1,x_4,\ldots,x_n)(\exists x_2)(\forall x_3)(x_3 \in x_2 \leftrightarrow (x_3 \in x_1 \ \&\ \varphi))$.

For Tupling, let $x_1,\ldots,x_n \in W$. Then $(\exists y)(x_1,\ldots,x_n \in y)$. Hence by RED, $(\exists y \in W)(x_1,\ldots,x_n \in z)$. Thus we have shown that $(\forall x_1,\ldots,x_n \in W)(\exists y \in W)(x_1,\ldots,x_n \in z)$. Therefore by Lemma 4.2, $(\forall x_1,\ldots,x_n)(\exists y)(x_1,\ldots,x_n \in y)$. □

We are now in a position to freely use the results from section 3.

LEMMA 4.4. Let φ be a formula in $L(\in,W)$ The following is provable in $K_2(W)$. If $x,y \in W$ and every common element of x and W lies in y, then $x \subseteq y$. $x \in W \rightarrow [(\exists! y \in W)(\forall z \in W)(z \in y \leftrightarrow (z \in x \;\&\; \varphi)) \;\&\; (\exists y \in W \cap x)(\forall z \in W)(z \in y \leftrightarrow (z \in x \;\&\; \varphi))]$.

Proof: For the first claim, let $x,y \in W$ and $(\forall z \in W)(z \in x \rightarrow z \in y)$. Hence by RED, $(\forall z)(z \in x \rightarrow z \in y)$, and so $x \subseteq y$. For the second claim, let $x \in W$. The first conjunct follows from SS^- and the first claim. For the second claim, let y be given by the first conjunct. If $y \subseteq x$ is false then by reducibility, some element of $y \in W$ is not in x, which is impossible. □

LEMMA 4.5. The following is provable in $K_2(W)$. No set includes all ordinal as elements. The least ordinal, OW, that is not in W is a limit ordinal $> \omega$. There is an ordinal in $W\backslash OW$.

Proof: If all ordinals lie in x then by Separation, the set of all ordinals exists. By Lemma 3.2, this is a transitive set of ordinals, and hence an ordinal, which is a contradiction.

Now fix OW to be the least ordinal not in W. Suppose OW has a largest element, α. By Separation, $\alpha+1$ exists. Hence by RED, $\alpha+1 \in W$, and hence $\alpha+1 \in OW$, which is a contradiction. So OW is a limit ordinal. Now let ω be the least limit ordinal. By RED, $\omega \in W$. And by induction, using RED we see that $\omega \subseteq W$. Therefore $\omega \in OW$, and so $OW > \omega$.

Finally suppose that OW is the set of all ordinals in W. Note that OW is a transitive subset of W. By TRANS, fix $OW \subseteq x \in W$. By SS^-, let $y \in W$ be such that for all $z \in W$, $z \in y \leftrightarrow (z \in x$ & z is an ordinal). Then for all $z \in W$, $z \in y \leftrightarrow z$ is an ordinal. Using RED, we see that every element of y is an ordinal; if not, then there is an element of y in W that is not an ordinal, which is a contradiction. Hence y = OW. But OW $\notin$ W, and hence we have the desired contradiction. ∎

Fix λ to be the first ordinal in W that is greater than OW. λ exists by Lemma 4.5.

LEMMA 4.6. The following is provable in $K_2(W)$. λ is a limit ordinal. If $V(\alpha)$ exists then $V(\alpha) \in W \leftrightarrow \alpha \in W$. For all $\alpha <$ OW, $V(\alpha) \in\subseteq W$. V(OW) exists and is a transitive subset of W. $V(\lambda)$ exists.

Proof: If $\lambda = \alpha+1$ then by RED, $\alpha \in W$. Now $\alpha \leq$ OW. Since OW is a limit ordinal, $\alpha <$ OW, and hence $\alpha+1 = \lambda <$ OW, which is a contradiction.

For the second claim, we use RED. Clearly we can define $V(\alpha)$ from α. But we can also define α from $V(\alpha)$ since $V(\alpha) = V(\beta) \rightarrow \alpha = \beta$. The simplest way to see this is that otherwise some rank is an element of itself (Lemma 3.9), which contradicts Lemma 3.14.

We prove the third claim by transfinite induction. Suppose this is true for all $\beta < \alpha$, where $\alpha <$ OW. If $\alpha = 0$ then we are done. Suppose α is a limit ordinal. Then $V(\alpha)$ exists by Separation, and hence by RED, $V(\alpha) \in W$. And clearly $V(\alpha) \subseteq W$ by the induction hypothesis.

Finally, suppose $\alpha = \beta+1$. We have $V(\beta) \in\subseteq W$. Let $x \subseteq V(\beta)$. By SS-, there exists $y \in W$ whose elements in W are exactly the elements of x. We claim that $y \subseteq V(\beta)$. If this is false, then by RED, some element of y from W would not be in $V(\beta)$, which contradicts the choice of y. Now since $y \subseteq V(\beta)$, we see that y and x have the same elements, and so y = x.

Thus we have shown that every subset of $V(\beta)$ is an element of W. Hence $V(\beta+1)$ exists, and so by RED, $V(\beta+1) \in W$. We have already seen that $V(\beta+1) \subseteq W$.

For the fourth claim, note that V(OW) is obtained as a subset of W by Separation. By Lemma 3.8, V(OW) is transitive, and it is a subset of W by claim 3.

For the fifth claim, note that there exists an ordinal α such that $V(\alpha)$ exists and $\alpha \notin W$; namely α = OW. Hence by RED*, fix an ordinal $\alpha \in W$ such that $\alpha \neg\subseteq W$ and $V(\alpha)$ exists. Clearly $\alpha \geq OW$. Hence $\alpha \geq \lambda$. Therefore $V(\lambda)$ exists. □

Let $W' = \{x \in W: x \subseteq V(\lambda)\}$.

LEMMA 4.7. The following is provable in $K_2(W)$. If $x,y \in W'$ and $x \cap V(OW) \subseteq y$, then $x \subseteq y$. If $x,y \in W'$ and $x \cap V(OW) = y \cap V(OW)$ then $x = y$.

Proof: For the first claim, let $x,y \in W'$ be as given, and assume that $\neg x \subseteq y$. By RED, there exists $z \in W$ such that $z \in x$ and $z \notin y$. Choose z of least possible rank. By RED, this rank is in W, and is $< \lambda$. Therefore by the definition of λ, this rank is $< OW$. Hence $z \in x \cap V(OW)$ yet $z \notin y$, which is a contradiction. The second claim follows immediately from the first. □

LEMMA 4.8. The following is provable in $K_2(W)$. The correspondence that sends $x \in W'$ to $x \cap V(OW)$ is one-one onto $V(OW+1)$. Every element of $W'\backslash V(OW)$ has rank $\lambda+1$. The correspondence is the identity on $V(OW)$ and one-one from $W'\backslash V(OW)$ onto $V(OW+1)\backslash V(OW)$. This correspondence is an $\in$-isomorphism.

Proof: That the correspondence is one-one onto follows from Lemma 4.4. The rank of every element of W is an element of W by RED, and hence the rank of every element of $W'\backslash V(OW)$ is $\leq \lambda+1$ and $> OW$, and so is $\lambda+1$. Obviously the correspondence is the identity on $V(OW)$, and therefore must be one-one from $W'\backslash V(OW)$ onto $V(OW+1)\backslash V(OW)$. To see that this correspondence is an $\in$-isomorphism, let $x,y \in W'$. If $x \in y$ then by comparing ranks and using Lemma 3.13, we see that $x \in V(OW)$, and obviously the correspondence preserves $x \in y$. On the other hand, if $x \in y$ becomes true after the correspondence, then again we see that $x \in V(OW)$, and so $x \in y$. □

We are not claiming that this correspondence exists as an actual function. For $x \in V(OW+1)$, we let $H(x)$ be the unique $y \in W'$ such that $x = y \cap V(OW)$. Again, we emphasize that H does not necessarily exist as a function.

We say that m is a complete measure on $V(\alpha+1)$ if and only if

i) $m:V(\alpha+1) \to \{0,1\}$;

ii) for all $x \in V(\alpha+1)$, if $|x| \leq 1$ then $m(x) = 0$;

iii) for all $x \in V(\alpha+1)$, $m(V(\alpha)\backslash x) = 1-m(x)$;

iv) if $x \in V(\alpha)$, $f:x \to V(\alpha+1)$, and for all $b \in x$, $m(f(b)) = 0$, then $m(\cup rng(f)) = 0$.

Now define $m:V(OW+1) \to \{0,1\}$ by $m(x) = 1$ if $V(OW) \in H(x)$; 0 otherwise.

LEMMA 4.9. The following is provable in $K_2(W)$. m is a complete measure on $V(OW+1)$.

Proof: Since λ is a limit ordinal and $V(\lambda)$ exists, we see that $V(OW+10)$ exists. Now define $m:V(OW+1) \to \{0,1\}$ by $m(x) = 1$ if $V(\kappa) \in H(x)$; 0 otherwise.

The only condition that is nontrivial to verify is iv. Let x,f be as given. Let $z = \{\langle b,u\rangle: u \in f(b)\}$. We claim that $H(z) = \{\langle b,u\rangle: u \in H(f(b))\}$.

To establish this, we first claim that $H(z) \subseteq x \bullet V(\lambda)$. By Lemma 4.7, it suffices to observe that $H(z) \cap V(OW) = z \cap V(OW) \subseteq x \bullet V(\lambda)$, which is obvious.

So it suffices to prove that for all $b \in x$, the cross section $H(z)b = H(f(b))$. By Lemma 4.7, it suffices to prove that these sets have the same elements from $V(OW)$. But by the definition of H, $H(z)b$, zb, $f(b)$, and $H(f(b))$ must agree on elements of $V(OW)$, thereby establishing that $H(z) = \{\langle b,u\rangle: u \in H(f(b))\}$.

We now claim that $H(rng(z)) = rng(H(z))$. To see this, again note that the two sets agree on elements of $V(OW)$, and apply Lemma 4.7.

Now since for all $b \in x$, $V(OW) \notin H(f(b))$, we see that $V(OW) \notin rng(H(z))$. Hence $V(OW) \notin H(rng(z))$, and so $m(rng(z)) = m(\cup rng(f)) = 0$, as required. □

We say that $V(\alpha)$ is strongly inaccessible if and only if

i) α is a limit ordinal $> \omega$;

ii) every function from an element of $V(\alpha)$ into $V(\alpha)$ has range $\in V(\alpha)$.

LEMMA 4.10. The following is provable in $K_2(W)$. V(OW) is strongly inaccessible.

Proof: By Lemma 4.5, OW is a limit ordinal > ω. Now let f:x → V(OW), x ∈ V(OW). Then f ⊆ V(OW), and also f ⊆ H(f) ⊆ V(λ). We claim that H(f) is a function with domain x. This is seen by taking one or two elements of f of least possible rank that are counterexamples to H(f) being a function with domain x, and arguing that the rank lies in W, and therefore < OW, which implies that the counterexamples to H(f) being a function with domain lie in V(OW), which is a contradiction. Hence f = H(f). □

LEMMA 4.11. The following is provable in $K_2(W)$. For all α < OW there exists $\alpha < \beta$ < OW and a complete measure on some V(β+1).

Proof: Let α < OW be the least ordinal such that this is false. Then OW is defined from α as the least γ such that there is a complete measure on V(γ+1). By RED, OW ∈ W, which is a contradiction. □

It is convenient to use the following terminology: a measurable rank is a V(α) such that there is a complete measure on V(α+1).

THEOREM 4.12. The following is provable in $K_2(W)$. There is a standard model of ZF + "there exists arbitrarily large measurable ranks."

Proof: The standard model can be taken to be (V(OW),∈), according to Lemmas 4.10 and 4.11. □

THEOREM 4.13. ZFC + "there exists a nontrivial elementary embedding from a rank into a rank" proves the existence of a standard model of $K_2(W)$.

Proof: Let j be an elementary embedding from V(κ+1) into V(γ+1) with critical point κ. Then κ and γ are strongly inaccessible. Let H(κ) be the set of all elements of transitive sets of cardinality at most that of κ, and let H(γ) be the set of all elements of transitive sets of cardinality at most that of γ. Now extend j uniquely to an elementary embedding $j':H(\kappa) \to H(\gamma)$ with critical point κ. Then $rng(j') \in H(\gamma)$.

We claim that $(H(\gamma),\in,rng(j'))$ satisfies $K_2(W)$.

First we claim that the largest transitive subset of rng(j') is $V(\kappa)$. To see this, observe that if u is a transitive set and $\delta+1 < rk(u)$ then some element of u has rank $\delta+1$. This is proved by transfinite induction on rk(u). Now obviously no element of rng(j')can have rank $\kappa+1$, for otherwise κ would be fixed by j'. Thus any transitive subset of rng(j') must have rank at most $\kappa+1$; i.e., be a subset of $V(\kappa)$.

This verifies TRANS since $V(\kappa) \subseteq V(\gamma) \in H(\gamma)$.

Secondly, for RED, let $(\exists y)\varphi(x_1,\ldots,x_n,y)$ hold in $(H(\gamma),\in)$, where $x_1,\ldots,x_n \in rng(j')$ and φ is in $L(\in)$. Let $x_i = j'(u_i)$. By elementarity, $(\exists y)\varphi(u_1,\ldots,u_n,y)$ holds in $(H(\kappa),\in)$. Fix y. Then by isomorphism, $\varphi(j'u_1,\ldots,j'u_n,j'y)$ holds in $(H(\gamma),\in)$. I.e., $(\exists y \in rng(j'))\varphi(x_1,\ldots,x_n,y)$ holds in $(H(\gamma),\in)$.

It remains to verify SS^-. Let $j'(x)$ be given, where $x \in V(\alpha)$. Let $y \subseteq j'(x)$. Consider $j'(j'^{-1}[y])$. Then $j'(u) \in j'(j'^{-1}[y])$ if and only if $u \in j'^{-1}[y]$ if and only if $j'(u) \in y$. Thus we have shown that for all $v \in W$, $v \in j'(j'^{-1}[y]) \leftrightarrow v \in y$, which is what is required for SS^-. □

5. Elementary embeddings from $V(\alpha)$ into $V(\beta)$

Let K3(W) be the following theory in the language $L(\in,W)$ with the constant symbol W.

1. EXT.
2. SS^-
3. RED.
4. RED*. $\varphi \to (\exists x \subseteq W)(\varphi)$, where φ has at most the free variable x.
5. TRANS.

$K^3(W)$ is obtained from $K^2(W)$ by adding axiom 4, a new form of Reducibility. RED* can be written as:

$$(\exists x)(\varphi) \to (\exists x \subseteq W)(\varphi),$$

where φ has at most the free variable x_1.

We can also use the following sharper form:

RED*+. $(\exists x)(\varphi) \rightarrow (\exists x)(x \in^* W \;\&\; \varphi)$,

where φ is a formula in $L(\in)$ with at most the free variable x^1.

Here $x \in^* y$ means "x is an element of a transitive subset of y."

The results here apply if we use RED*+ instead of RED*. RED*+' is exclusively used in sections 6 - 8.

Since $K_2(W)$ is a subsystem of $K_3(W)$, we can continue from the development in section 4.

LEMMA 5.1. Let $n \geq 0$. The following is provable in $K_3(W)$. EST. $x_1,\ldots,x_n \in W \rightarrow \{x_1,\ldots,x_n\} \in W$.

Proof: EST is from Lemma 4.3. The second claim is by RED. ∎

LEMMA 5.2. The following is provable in $K_3(W)$. For all ordinals α, $\alpha+1$ and $V(\alpha)$ exists.

Proof: Let α be the least ordinal such that $\alpha+1$ or $V(\alpha)$ does not exist. Using RED and RED*, $\alpha \in\subseteq W$. Hence $\alpha < OW$. But by Lemmas 4.5 and 4.6, $\alpha+1$ and $V(\alpha)$ exist. ∎

We say that x is in the cumulative hierarchy if and only if there exists $V(\alpha)$ such that $x \in V(\alpha)$. The rank of x is the least α such that $x \in V(\alpha)$. A set x has a rank if and only if it lies in the cumulative hierarchy. The cumulative hierarchy is treated as a proper class.

LEMMA 5.3. The relativization of the universal closure of each axiom of ZF to the cumulative hierarchy is provable in $K_3(W)$.

Proof: It suffices to verify Replacement. Fix a formula for Replacement, and assume that it fails in the cumulative hierarchy. Then choose α least such that it fails with domain $= V(\alpha)$ with this formula. Using both forms of Reducibility, we see that $\alpha \in\subseteq W$, and so $\alpha < OW$. Hence by Lemma 4.6, $V(\alpha)$

$\in\subseteq$ W. Therefore by RED, the unique values in this failed instance of Replacement all lie in W. But then they can be collected up by Separation, and so we have a contradiction. □

It may be surprising that we cannot prove in $K_3(W)$ that every subset of the cumulative hierarchy lies in the cumulative hierarchy. We can't get a bound on the levels of the cumulative hierarchy that the members of a set have. Of course, there is no problem proving in $K_3(W)$ that every subset of every element of the cumulative hierarchy lies in the cumulative hierarchy.

Recall λ from section 4 as the least ordinal in W greater than OW.

By Lemma 5.3, we now fix μ to be the first ordinal > λ such that $V(\mu)$ is a $\sum_{10}$ elementary submodel of the cumulative hierarchy.

LEMMA 5.4. The following is provable in $K_3(W)$. μ, $V(\mu) \in W$. $V(\mu)$ satisfies ZF_5. For all φ in $L(\in)$ with at most the free variables $x_1,\ldots,x_n$, we have $x_1,\ldots,x_n \in V(\mu) \cap W \rightarrow ((\exists y \in V(\mu))(\varphi^{V(\mu)}) \rightarrow (\exists y \in V(\mu) \cap W)(\varphi^{V(\mu)}))$. $V(\mu) \cap W$ is an elementary submodel of $V(\mu)$. (These claims are formulated as single sentences, and not as schemes).

Proof: The first claim follows from RED, since $\lambda \in W$. The second claim is from the definition of μ, the fact that the cumulative hierarchy satisfies ZF (formulated as a scheme), and that the axioms of ZF_5 are $\sum_{10}$.

To verify the second claim, let $x_1,\ldots,x_n \in V(\mu) \cap W$, and suppose that $(\exists y \in V(\mu))(\varphi^{V(\mu)})$. This can be put in the form $(\exists y)(y \in V(\mu)\ \&\ \varphi^{V(\mu)})$, where this statement is viewed as having the parameters $x_1,\ldots,x_n,\varphi,V(\mu)$. Hence by RED, $(\exists y \in W)(y \in V(\mu)\ \&\ \varphi^{V(\mu)})$. The final claim follows from the third claim. □

We let $W^* = W \cap V(\mu)$.

LEMMA 5.5. The following is provable in $K_3(W)$. W^* and $(W^*,\in)$ are in the cumulative hierarchy. $(W^*,\in)$ is a well founded structure satisfying ZF_5. For all $x \in W^*$ and $y \subseteq x \cap W^*$, there exists $z \in W^*$ such that $y = z \cap W^*$.

Proof: W* and $(W^*,\in)$ are in the cumulative hierarchy, and thereby subject to treatment using ZF according to Lemma 5.3, since any subset of any rank lies in the next rank. Also clearly $(W^*,\in)$ is in the next few ranks even if it is construed as an ordered pair. Also $(W^*,\in)$ is well founded by Lemma 3.14. By Lemma 5.4, we see that $(W^*,\in)$ satisfies ZF_5. Now let $x \in W^*$ and $y \subseteq x \cap W^*$. By SS-, there exists $z \in W$ such that $y = z \cap W$. By Lemma 4.4, $z \subseteq x$. Hence $z \in W^*$ and $z \subseteq W^*$. Therefore $y = z \cap W^*$. □

LEMMA 5.6. The following is provable in $K_3(W)$. There is a unique $\in$-isomorphism from W* onto a transitive set. The $\in$-isomorphism lies in the cumulative hierarchy. The transitive set is a rank $V(\gamma)$ satisfying ZF_5. The $\in$-isomorphism is the identity on V(OW) and sends λ to OW.

Proof: We intensively use Lemma 5.5. Since $(W^*,\in)$ lies in the cumulative hierarchy, it is subject to the classical theorem that every extensional well founded relation has a unique isomorphism onto a transitive set under $\in$. Because of the well foundedness of W*, the isomorphism is unique (not just unique within the cumulative hierarchy). Since this proof is carried out in the cumulative hierarchy, the isomorphism lies in the cumulative hierarchy, as well as its range, B.

Now since isomorphisms preserve sentences, we see that B is a transitive set satisfying ZF_5. We show by transfinite induction that each level of the cumulative hierarchy in B is an actual rank. The successor case, $\alpha+1$, is the only nontrivial case. Let f be the unique isomorphism, and consider $f^{-1}(V(\alpha))$. Clearly f maps $f^{-1}(V(\alpha)) \cap W^*$ one-one onto $V(\alpha)$. Using the last claim in Lemma 5.5, we see that every subset of $V(\alpha)$ is the value of f at some element of W′. Hence $V(\alpha) \subseteq B$. Since B satisfies ZF_5, clearly $V(\alpha) \in B$ and $V(\alpha)$ is the rank on α in B.

We have thus shown that all levels of the cumulative hierarchy in B are actual ranks, and therefore $B = V(\gamma)$.

Now since $V(OW) \subseteq W^*$ and V(OW) is transitive, we see that f is the identity on V(OW). Since λ is the first ordinal in W* after OW, and ordinals must go to ordinals, we see that f(λ) = OW. □

We fix γ as in Lemma 5.6.

LEMMA 5.7. The following is provable in K_3(W). There is an elementary embedding j from V(γ) into V(μ) with critical point OW.

Proof: Let f be the ∈-isomorphism from W* onto V(γ) from Lemma 5.6. Let j be the inverse of f. By Lemma 5.4, j is an elementary embedding from V(γ) into V(μ). By Lemma 5.6, j is the identity on V(OW) and j(OW) = λ. Hence OW is the critical point of j. □

LEMMA 5.8. The following is provable in K3(W). There are arbitrarily large inaccessible ranks. There are arbitrarily large α < γ such that V(α) is strongly inaccessible.

Proof: Let β be the least strict upper bound for the inaccessible ranks. Then by RED*, β ⊆ W, and so β ≤ OW. But by Lemma 4.10, V(OW) is strongly inaccessible.

For the second claim, note that there are arbitrarily large strongly inaccessible ranks in V(μ), since V(μ) is a $\sum_{10}$ elementary substructure of the cumulative hierarchy. Hence there are arbitrarily large strongly inaccessible ranks in V(α). □

THEOREM 5.9. The following is provable in K_3(W). There is a nontrivial elementary embedding from some strongly inaccessible rank into some strongly inaccessible rank.

Proof: By Lemma 5.8, let OW < α < γ, where V(α) is strongly inaccessible. Then V(j(α)) = V(β) is also strongly inaccessible. Hence j|V(α) is the required elementary embedding from some strongly inaccessible rank into some strongly inaccessible rank. □

It is not clear just how strong T = ZF + “there is a nontrivial elementary embedding from some strongly inaccessible rank into some strongly inaccessible rank” is. However the existence of a standard model of “there exists a supercompact cardinal” is provable in ZFC + T.

According to Woodin, his forcing methods will not yield the consistency of ZFC + "there exists a supercompact cardinal" relative to T. Nevertheless, there is the belief that this can be done, but may require a more advanced stage of the inner model program for large cardinals.

If we use the normal ultrafilter definition of supercompactness, we will see below that $K_3(W)$ proves the existence of arbitrarily large supercompact cardinals. However ZF does not (seem to) prove the equivalence of the ultrafilter definition of supercompactness with the elementary embedding definition of supercompactness.

Nevertheless, according to Woodin, one can at least derive projective determinacy (PD) from T, or even from ZF + "there is a nontrivial elementary embedding from some rank into some rank." And one obtains a standard model of ZFC + $AD^{L(R)}$.

THEOREM 5.10. PD is provable in $K_3(W)$. $K_3(W)$ proves the existence of a standard model of ZFC + $AD^{L(R)}$.

Proof: By Theorem 5.9 and the results attributed above to Woodin. ∎

We now turn to the normal ultrafilter approach to supercompactness as formalized in ZF. The reader must bear in mind that for the rest of this section, we are using only the normal ultrafilter definition of supercompactness, which is (apparently) not equivalent to the elementary embedding definition in ZF.

Let κ be a cardinal and $\mu \geq \kappa$ be an ordinal. We say that κ is $V(\mu)$-supercompact if and only if there is a normal ultrafilter over $P_\kappa V(\mu)$ = the set of all subsets of $V(\mu)$ of cardinality $< \kappa$. We say that κ is supercompact if and only if κ is $V(\mu)$-supercompact for all ordinals $\mu \geq \kappa$.

(See Solovay, Reinhardt; and also [Ka94], p. 301-302 for a discussion of normal ultrafilters in this context).

LEMMA 5.11. The following is provable in ZF. Let j be an elementary embedding from $V(\kappa+\eta)$ into $V(\varsigma)$ with critical point κ and $\eta < j(\kappa)$. Let $\delta+5 \leq \eta$. Then κ is $V(\kappa+\delta)$-supercompact.

Proof: This is Proposition 23.6 from [Ka94], p. 316, adapted to ZF without choice. ∎

We now return to the elementary embedding j from $V(\gamma)$ into $V(\mu)$ with critical point OW afforded by Lemma 5.7. Recall that μ is the least ordinal > λ = j(OW) such that V(μ) is a Σ10 elementary submodel of the cumulative hierarchy.

LEMMA 5.12. The following is provable in K_3(W). If β < min(γ,λ) then OW is V(β)-supercompact. OW is V(γ)-supercompact or V(λ)-supercompact.

Proof: Obviously j is an elementary embedding from V(OW+β+5) into V(j(OW+δ+4)) with critical point OW and β+5 < λ = j(OW). Now apply Lemma 5.11. The second claim follows immediately. □

LEMMA 5.13. The following is provable in K_3(W). If OW is V(γ)-supercompact then λ is supercompact, and there are arbitrarily large ordinals < λ that are V(λ)-supercompact, and there are arbitrarily large ordinals that are supercompact. If OW is V(λ)-supercompact then there are arbitrarily large ordinals < OW that are V(OW)-supercompact and arbitrarily large ordinals < λ that are V(λ)-supercompact.

Proof: Suppose firstly that OW is V(γ)-supercompact. Then using j, we see that λ is V(μ)-supercompact. Now since V(μ) is a Σ_{10} elementary submodel of the cumulative hierarchy, we see that λ is supercompact. Now if there is a bound to the supercompact cardinals, then by RED*, such a bound must be $\subseteq$ OW, and hence $\leq$ OW, which contradicts the supercompactness of λ. Hence there are arbitrarily large ordinals that are supercompact. But then V(μ) must also satisfy that there are arbitrarily large supercompact cardinals.

Now suppose that OW is V(λ)-supercompact. Suppose there is a bound α on the ordinals < OW that are V(OW)-supercompact. Using j, we see that α is also a bound on the ordinals < λ that are V(λ)-supercompact. This is a contradiction. Hence there are arbitrarily large ordinals < OW that are V(OW)-supercompact. Using j, there are arbitrarily large ordinals < λ that are V(λ)-compact. □

THEOREM 5.14. The following is provable in K_3(W). There is a standard model of ZF + "there are arbitrarily large supercompact cardinals" under the normal ultrafilter definition of supercompactness.

Proof: By Lemma 4.10, V(OW) and V(λ) are strongly inaccessible. Apply Lemma 5.13. □

THEOREM 5.15. ZFC + "there exists an extendible cardinal" proves the existence of a standard model of K_3(W). This holds even if we use RED^{*+} instead of RED*.

Proof: Let κ be an extendible cardinal. Let j be an elementary embedding from V(κ+1) into some V(α+1) with critical point κ. Then κ and α are strongly inaccessible cardinals, and V(κ) is an elementary submodel of V(α).

Let j' be an elementary embedding from V(α+1) into some V(β+1) with critical point κ. Then j'(α) = β and β is a strongly inaccessible cardinals. Therefore j'[V(α)] ∈ V(β).

We claim that (V(β),∈,rng(j')) is a model of K_3(W). By examination of the proof of Theorem 4.13, we have only to verify RED^{*+}:

$(\exists x_1)(\varphi) \rightarrow (\exists x_1)(x1 \in^* W \ \&\ \varphi)$, where φ is a formula in L(∈) with at most the free variable x1.

As in the proof of Theorem 4.13, V(κ) is the largest transitive subset of rng(j'). Thus it suffices to show that if an ordinal δ is definable in V(β) then δ < κ. Let δ be definable in V(β). Then $j^{-1}(\delta)$ is definable in V(α). But V(κ) is an elementary submodel of V(α). Hence $j^{-1}(\delta) < \kappa$, and so δ < κ. □

6. ZF\Foundation again

In order to treat the large large cardinals in the next two sections, we use a stronger second form of Reducibility which we referred to as 4' in the previous section. This leads to an axiomatization of ZF(col)\F. Here ZF(col) is ZF formulated with the Collection scheme instead of the Replacement scheme. Of course, ZF(col) = ZF(rep) = ZF(rfn), but ZF(rep)\F ⊄ ZF(col)\F ⊄ ZF(rfn)\F.

Accordingly, let K'(W) be the following system in the language L(∈,W):

1. EXT.
2. SS^-.
3. RED.
4. RED^{*+}. $\varphi \rightarrow (\exists x)(x \in^* W \& \varphi)$, where φ is a formula in $L(\in)$ with at most the free variable x.

Note that K'(W) is a subsystem of K(W). This is because K(W) proves the transitivity of W, and so axiom 4 can be derived from axiom 3 using W as the transitive subset of W.

Actually, we accomplish this with the weaker system K*(W):

1. EXT.
2. SS^-.
3. RED.
4. RED**. $\varphi \rightarrow (\exists x_1)(\varphi$, where x_1, every element of x_1, and every element of every element of x_1 lie in W), where φ is a formula in $L(\in)$ with at most the free variable x1.

Note that RED** is a direct weakening of RED^{*+}, and a strengthening of RED*.

For any formula φ in $L(\in)$, let φ^W be the result of relativizing all quantifiers to W.

Define $x = y$ if and only if $(\forall z)(z \in x \leftrightarrow z \in y)$. Define $x \in\subseteq y$ if and only if $x \in y \& x \subseteq y$.

LEMMA 6.1. Let φ be a formula in $L(\in,W)$ which does not mention y. The following is provable in K(W). $x = x$. $x = y \rightarrow y = x$. $(x = y \& y = z) \rightarrow x = z$. $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$.

Proof: These are a version of the axioms for equality. Use EXT to derive these. □

LEMMA 6.2. Let φ be a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n$, $n \geq 0$. The following is provable in K*(W). $x1,\ldots,xn \in W \rightarrow (\varphi \leftrightarrow \varphi^W)$.

Proof: As in Lemma 1.2. □

LEMMA 6.3. The following is provable in K*(W). $(\exists x)(x \in W)$.

Proof: Apply RED in the case n = 0 to the statement $(\exists x)(x \in x \rightarrow x \in x)$. □

LEMMA 6.4. The following is provable in K*(W). $(x \in\subseteq W \ \& \ y \subseteq x) \rightarrow y \in W$.

Proof: Let $x \in\subseteq W$ and $y \subseteq x$. By SS^-, $(\exists z \in W)(\forall w \in W)(w \in z \leftrightarrow (w \in x \ \& \ w \in y))$. Let $z \in W$ have this property. Then $(\forall w \in W)(w \in z \rightarrow w \in x)$. Hence by RED, $z \subseteq x$, and in particular, $z \subseteq W$. Hence $(\forall w)(w \in z \leftrightarrow (w \in x \ \& \ w \in y))$. Therefore $(\forall w)(w \in z \leftrightarrow w \in y)$. □

We now show that each of the axioms of ZF(rfn)\F is provable in K2(W).

Before doing this, we use the following simplification of the axioms of ZF(col)\F:

1. Extensionality. $(\forall x)(x \in y \leftrightarrow x \in z) \rightarrow (\forall x)(y \in x \leftrightarrow z \in x)$.
2. Pairing. $(\exists x)(y,z \in x)$.
3. Union. $(\exists x)(\forall y \in w)(\forall z \in y)(z \in x)$.
4. Separation. $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \ \& \ \varphi))$, where φ is a formula in $L(\in)$ in which x is not free.
5. Power set. $(\exists x)(\forall y)(y \subseteq z \rightarrow y \in x)$.
6. Collection. $(\forall x \in y)(\exists z)(\varphi) \rightarrow (\exists w)(\forall x \in y)(\exists z \in w)(\varphi)$, where φ is a formula in $L(\in)$ in which w is not free.
7. Infinity. $(\exists x)(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x))$.

LEMMA 6.5. Pairing is provable in K*(W).

Proof: Obviously $(\forall y,z \in W)(\exists x)(y,z \in x)$. Hence by RED, $(\forall y,z)(\exists x)(y,z \in x)$. □

LEMMA 6.6. Union is provable in K*(W).

Proof: By RED**, it suffices to let $x \in W$ be such that every element and every element of every element of x lies in W, and prove the existence of a set y such that every element of every element of x lies in y. Just choose y = W. □

LEMMA 6.7. Separation is provable in K*(W).

Proof: Let φ be a formula in $L(\in)$ whose free variables are among $x_1,\ldots,x_n$, $n \geq 1$. By SS^-, we have $(\forall x_1,\ldots,x_n,w \in W)(\exists y \in W)(\forall z \in W)(z \in y \leftrightarrow (z \in w \ \& \ \varphi^W))$. By Lemma 6.2, $(\forall x_1,\ldots,x_n,w)(\exists y)(\forall z)(z \in y \leftrightarrow (z \in w \ \& \ \varphi))$. □

LEMMA 6.8. Power set is provable in K*(W).

Proof: By RED, it suffices to let x ∈⊆ W and prove the existence of a set y such that every subset of x lies in y. By Lemma 6.4, we can set y = W. □

LEMMA 6.9. Collection is provable in K*(W).

Proof: By RED**, it suffices to prove that for all x ∈⊆ W and any choice of parameters $u_1,\dots,u_k$, if (∀y ∈ x)(∃z)(φ) then (∃w)(∀y ∈ x)(∃z ∈ w)(φ). Here $u_1,\dots,u_k$ is a complete list of the parameters involved. Hence by RED, it suffices to prove that for all x ∈⊆ W and any choice of parameters $u_1,\dots,u_k$ ∈ W, if (∀y ∈ x)(∃z)(φ) then (∃w)(∀y ∈ x)(∃z ∈ w)(φ). Fix x ∈⊆ W and $u_1,\dots,u_k$ ∈ W. Assume (∀y ∈ x)(∃z)(φ). Then by RED, (∀y ∈ x)(∃z ∈ W)(φ). □

LEMMA 6.10. The following is provable in K*(W). For all x ∈⊆ W, x ∪ {x} ∈⊆ W.

Proof: Let x ∈⊆ W. By Separation, x ∪ {x} exists. By RED, x ∪ {x} ∈ W. But x ∪ {x} ⊆ W. Therefore x ∪ {x} ∈⊆ W. □

LEMMA 6.11. Infinity is provable in K*(W).

Proof: By Separation, let x = {y ∈ W: y ⊆ W}. By Lemma 6.10, ∅ ∈ x & (∀y ∈ x)(y ∪ {y} ∈ x). □

THEOREM 6.12. Every theorem of ZF(col)\F is a theorem of K*(W).

Proof: We have verified the derivability of all of the axioms of ZF(col)\F in K(W). □

Since K*(W) ⊆ K′(W) ⊆ K(W), we can lift the results about K(W) in the other direction. E.g., every theorem of K*(W) (or K′(W)) in L(∈), is a theorem of ZF.

We close this section with some results about K*(W) that will be useful later.

Let ν be the least ordinal not in W.

LEMMA 6.13. The following is provable in K*(W). $V(\nu)$ is an elementary submodel of the cumulative hierarchy. $V(\nu)$ satisfies ZF. For all φ in $L(\in)$ with at most the free variables x1,…,xn, we have $x_1,\ldots,x_n \in V(\nu) \cap W \rightarrow ((\exists y \in V(\nu))(\varphi^{V(\nu)}) \rightarrow (\exists y \in V(\nu) \cap W)(\varphi^{V(\nu)}))$. $V(\nu) \cap W$ is an elementary submodel of $V(\nu)$. (All four claims are formulated as schemes).

Proof: For the first claim, $\varphi(x_1,\ldots,x_k,y)$ be a formula in $L(\in)$ with all free variables shown. By RED, ν is a limit ordinal, and so let $x_1,\ldots,x_k \in V(\alpha)$, $\alpha \in W$. By ZF(col)\F, there exists β such that for all $x_1',\ldots,x_k' \in V(\alpha)$, if there exists y in the cumulative hierarchy such that $\varphi(x_1,\ldots,x_k,y)$, then there exists $y \in V(\beta)$ such that $\varphi(x_1,\ldots,x_k,y)$. Choose β to be least with this property. Then by RED, $\beta \in W$. Thus if there exists y in the cumulative hierarchy such that $\varphi(x_1,\ldots,x_k,y)$ then y can be chosen in $V(\nu)$.

The second claim follows immediately from the first claim using ZF(col)\F.

The last two claims are proved as in Lemma 5.4. ∎

LEMMA 6.14. The following is provable in K*(W). There is a unique $\in$-isomorphism from $V(\nu) \cap W$ onto some transitive set B. And B is of the form $V(\gamma)$, $\gamma \leq \nu$.

Proof: Since we have ZF(col)\F at our disposal, let h be the unique $\in$-isomorphism from $V(\nu) \cap W$ onto a transitive set B. By Lemma 6.13, $V(\nu) \cap W$ satisfies ZF (as a scheme), and hence B also satisfies ZF (as a scheme). Now if $x,y \in V(\nu) \cap W$ have the same rank then h(x),h(y) have the same rank.

Obviously h maps every rank in $V(\nu) \cap W$ onto an internal rank in B. But we prove by transfinite induction that h maps every rank in $V(\nu) \cap W$ onto an actual rank. The only nontrivial case is the successor case. This is handled easily by SS^-. Thus the internal ranks in B are actual ranks. Hence B is an actual rank.

Now since for ordinals $\alpha \in V(\nu) \cap W$, $h(\alpha) \geq \alpha$, we see that $\gamma \leq \nu$. ∎

7. Elementary embeddings from $V(\alpha)$ into $V(\alpha)$

We let $K^4(W)$ be the following theory in the language $L(\in,W)$.

1. EXT.
2. SS^-.
3. RED.
4. RED^{*+}.
5. TRANS.
6. Largeness. $(\forall x \in W)(\exists y \in W)(\exists f)$(f is a one-one function from x into $y \cap W$).

Here x,y are the variables x_1,x_2, respectively.

Note that axioms 1-4 constitute K'(W), which already implies ZF(col)\F. This supports the statement of Largeness. And note that $K_4(W) = K'(W) + K_3(W)$ + Largeness.

The axiom of Largeness asserts that although an element of yesterday's world may gain new elements from today's world, any element of yesterday's world can be one-one mapped back into the elements of yesterday's world that lie in some element of yesterday's world. This puts a limit on the number of new elements that members of yesterday's world gain.

We shall see that K(W) proves that the existence of a nontrivial elementary embedding from some $V(\lambda)$ into $V(\lambda)$, and in fact that there exists a standard model of ZF + "there is a nontrivial elementary embedding from some $V(\lambda)$ into $V(\lambda)$."

We will actually accomplish this using only the system $K_4^-(W)$ in which RED^{*+} is replaced by RED** from section 6.

1. EXT.
2. SS^-.
3. RED*.
4. RED**.
5. TRANS.
6. Largeness.

Woodin has used his forcing technique to show that ZF + "there is a nontrivial elementary embedding from some $V(\lambda)$ into $V(\lambda)$" proves the existence of an $\in$-model of ZFC + "there is a cardinal which is n-huge for all $n < \omega$."

Note that $K_4^-(W) = K_3(W) + K^*(W)$ + Largeness, and so we can use the development of sections 5 and 6. In particular, we let OW and λ be as in section 5; in fact, OW and λ go back to section 4. Also let ν be as in section 6. We also have ZF(col)\F.

LEMMA 7.1. The following is provable in $K_4^-(W)$. For all $\alpha \in W$, there exists $\beta \in W$ and an $\in$-isomorphism from α onto $\beta \cap W$. The order type of $\nu \cap W$ is ν.

Proof: Let α be an ordinal in W. By Largeness, let $x \in W$ and $f:\alpha \to x \cap W$ be one-one. Let $R = \{<a,b>: a,b \in x \cap W \ \& \ f^{-1}(a) < f^{-1}(b)\}$. Then f is an isomorphism from α onto R.

By RED, $x \bullet x \in W$, and $R \subseteq x \bullet x$. By SS^-, let $R' \in W$, where $R = R' \cap W$. By RED, $R' \subseteq x \bullet x$. Also by RED, $W \cap x \bullet x = (x \cap W) \bullet (x \cap W)$.

We claim that R′ is a well ordering. Firstly, R′ is a linear ordering as far as elements of R′ in W are concerned. Therefore by Reducibility, R′ is a linear ordering. Secondly, suppose R′ is not a well ordering. By Reducibility, let $y \in W$ be a nonempty subset of dom(R′) which has no R′-least element.

By RED, $dom(R) = dom(R') \cap W$. By RED, $y \cap dom(R) = y \cap W$ is a nonempty subset of dom(R). Let b be the R-least element of $y \cap dom(R)$. Since $dom(R) \subseteq W$, we have $b \in W$.

Now $(\forall u \in y \cap fld(R))(\neg R(u,b))$. Hence $(\forall u \in y \cap W)(\neg R'(u,b))$. By RED, $(\forall u \in y)(\neg R'(u,b))$. Hence b is the R-least element of y.

Now since $R' \in W$ is a well ordering, we see by RED that W satisfies that R′ is a well ordering. Since W satisfies ZF(col)\F, let h be the unique element of W that is satisfied in W to be the isomorphism from R′ onto an ordinal, $\beta \in W$. By RED, h must map $dom(R) = dom(R') \cap W$ onto $\beta \cap W$. Then $f \circ h$ is as required by the first claim.

For the second claim, map $\alpha \in W$ to the unique $\beta \in W$ such that α is isomorphic to $\beta \cap W$. Note that $\alpha \leq \beta$. Hence this mapping is onto $\nu \cap W$. ∎

LEMMA 7.2. The following is provable in $K_4^-(W)$. There is a unique $\in$-isomorphism from $V(\nu) \cap W$ onto $V(\nu)$. The $\in$-isomorphism is the identity on V(OW) and sends λ to OW.

Proof: By Lemma 6.14, there is a unique $\in$-isomorphism from $V(\nu) \cap W$ onto $V(\gamma)$, for some $\gamma \leq \nu$. Arguing as in Lemma 5.6, we see that h is the identity on V(OW) and sends λ to OW. The order type of the ordinals in $W \cap V(\nu)$ must be the same as the order type of the ordinals in $V(\gamma)$. Hence by Lemma 7.1, we have $\gamma = \nu$. □

LEMMA 7.3. The following is provable in $K_4^-(W)$. There is a nontrivial elementary embedding from $V(\nu)$ into $V(\nu)$ with critical point OW, mapping OW to λ (formulated as a scheme). In particular, there is a nontrivial Σ_0 elementary embedding from $V(\nu)$ into $V(\nu)$.

Proof: From Lemmas 6.13 and 7.2 as in the proof of Lemma 5.7. The second claim immediately follows by the truth definition for Σ_0 formulas (i.e., formulas with only bounded quantifiers) and coding of tuples. □

LEMMA 7.4. The following is provable in $K_4^-(W)$. Suppose j is a Σ_0 elementary embedding from $V(\alpha)$ into $V(\alpha)$ such that α is the limit of the iterates of j at the critical point. Then j is an elementary embedding from $V(\alpha)$ into $V(\alpha)$. Also for all $n \geq 0$, $V(j_n(\kappa))$ is an elementary submodel of $V(\alpha)$.

Proof: First note that $V(\kappa) < V(j(\kappa))$ since j is the identity on $V(\kappa)$. (Here < indicates “elementary submodel”). For all $n \geq 0$, let SAT(n) be the satisfaction relation for $V(j_n(\kappa))$. Then $V(\kappa) < V(j(\kappa))$ as formalized using SAT(1). Therefore for all $n \geq 0$, $V(j_n(\kappa)) < V(j_{n+1}(\kappa))$ as formalized using SAT(n+1). This establishes the second claim.

Continuing our proof of the first claim, let $\varphi(x_1,\ldots,x_k)$ be given, $x_1,\ldots,x_k \in V(j_r(\kappa))$, $r \geq 0$. Suppose $\varphi(x_1,\ldots,x_k)$ holds in $V(\alpha)$. Then $\varphi(x_1,\ldots,x_k)$ holds in $V(j_r(\kappa))$. Now formalize this using SAT(r). Hence $\varphi(jx_1,\ldots,jx_k)$ holds in $V(j_{r+1}(\kappa))$ as formalized using SAT(r+1). Therefore $\varphi(jx_1,\ldots,jx_k)$ holds in $V(\alpha)$. □

THEOREM 7.5. $K_4^-(W)$ proves ZF(col)\F + “there exists a nontrivial elementary embedding from some rank into itself.”

Proof: By Lemma 7.3, we have a nontrivial Σ_0 elementary embedding j from $V(\nu)$ into $V(\nu)$. Let α be the limit of the iterates of j at the critical point κ. Then $j|V(\alpha)$ satisfies the hypothesis of Lemma 7.4. □

LEMMA 7.6. The following is provable in $K_4^-(W)$. There are arbitrarily large critical points of nontrivial elementary embeddings from ranks into themselves. This claim holds in V(OW).

Proof: Suppose this is false, and let δ be the least strict upper bound. Then by RED**, δ < OW. This is a contradiction. For the second claim, note that $V(\nu)$ is an elementary submodel of the cumulative hierarchy. By Lemma 7.4, V(OW) is an elementary submodel of $V(\nu)$ and hence of the cumulative hierarchy. Hence the first claim holds in V(OW). □

THEOREM 7.7. $K_4^-(W)$ proves ZF(col)\F + “there exists nontrivial elementary embeddings from ranks into themselves with arbitrarily large critical points.” $K_4^-(W)$ proves the existence of a standard model of ZF + “there exists nontrivial elementary embeddings from ranks into themselves with arbitrarily large critical points.”

Proof: The first claim is from Lemma 7.6. The second claim is also from Lemma 7.6, using that $V(\kappa)$ is strongly inaccessible, and therefore satisfies ZF. □

THEOREM 7.8. ZF + “there exists $\alpha < \beta$ and a nontrivial elementary embedding from $V(\alpha)$ into $V(\alpha)$, where $V(\alpha)$ is an elementary submodel of $V(\beta)$” proves the existence of a standard model of $K_4(W)$.

Proof: Let $\alpha < \beta$ and j be given, with critical point κ. We claim that $(V(\beta),\in,j[V(\alpha)])$ satisfies $K_4(W)$. This uses the fact that $V(\kappa)$ is an elementary submodel of $V(\alpha)$, and hence of $V(\beta)$. □

The elementary embedding hypothesis used in Theorem 7.8 to derive standard models of $K_4(W)$ is not quite one of the standard ones. However, it follows from one of the standard ones:

THEOREM 7.9. In VBC + “there exists an elementary embedding j from V into a transitive class M, with $V(\mu) \subseteq M$, where μ is the limit of the iterates of j at the critical point of j” proves “there exists $\alpha < \beta$ and a nontrivial elementary embedding from $V(\alpha)$ into $V(\alpha)$, where $V(\alpha)$ is an elementary submodel of $V(\beta)$” and therefore the existence of a standard model of $K_4(W)$.

Proof: Let j,M,μ be as given, and let κ be the critical point of j. Form a well known tree T whose nodes represent pieces of nontrivial elementary embeddings from ranks into higher ranks, where the ranks are below μ, and are elementary submodels of $V(\mu)$. This is done so that the infinite paths represent nontrivial elementary embeddings from a rank into itself, where the rank is an elementary submodel of $V(\mu)$, as well as the ranks on the iterates at the critical point. Obviously T has an infinite path since we can simply use the pieces of j. Now we can start with a node whose critical point is minimized, and whose target rank is then minimized, with the property that this node can be extended to an infinite path. Because of absoluteness of well foundedness of trees in this context, we see that this minimized critical point and minimized target rank must be fixed under j. Hence they are below κ. We can continue in this manner for ω stages and thereby obtain an infinite path through T which lies in $V(\kappa)$. Thus we have produced a nontrivial elementary embedding from a rank into itself, where that rank is an elementary submodel of a higher rank, namely $V(\mu)$. □

We now relate Theorem 7.7 to extensions of ZFC. Hugh Woodin has proved that ZFC + “there exists a nontrivial elementary embedding from a rank into itself” proves the existence of $\in$ models of ZFC + “there exists a cardinal which is simultaneously n-huge for all $n < \omega$.”, and also that ZFC + “there exists nontrivial elementary embeddings from ranks into themselves with arbitrarily large critical points” proves the existence of a standard model of ZF + “there exists arbitrarily large cardinals which are simultaneously n-huge for all $n < \omega$.”

THEOREM 7.10. $K_4^-(W)$ proves the existence of a $\in$ model of ZFC + “there exists arbitrarily large cardinals which are simultaneously n-huge for all $n < \omega$.”

Proof: By Theorem 7.7 and the quoted result of Woodin. □

8. Elementary embeddings incompatible with choice

Let $K_5(W)$ be the following system in $L(\in,W)$.

1. EXT.
2. SS^-.
3. RED.
4. RED^{*+}.
5. TRANS.
6. Subworld Collection. $(x \in W \;\&\; (\forall y \in x)(\exists z \in W)(\varphi)) \rightarrow (\exists w \in W)((\forall y \in x)(\exists z \in w)(\varphi)$, where φ is a formula in $L(\in,W)$ in which w is not free.

Here x,y,z,w are the variables x_1,x_2,x_3,x_4, respectively.

We can again consider the subsystem $K_5^-(W)$:

1. EXT.
2. SS^-.
3. RED.
4. RED**.
5. TRANS.
6. Subworld Collection.

We can follow the development of section 7 until the axiom of Largeness is used there; we don't have an axiom of Largeness in $K_5^-(W)$.

In summary, we let OW be the least ordinal not in W, λ be the least ordinal in W that is greater than OW, and ν be the least ordinal not in W.

LEMMA 8.1. The following is provable in $K_5^-(W)$. $V(\nu)$ is strongly inaccessible. $\nu \cap W$ has order type ν.

Proof: Let $f:V(\alpha) \rightarrow V(\nu)$, $\alpha < \nu$. Apply Subworld Collection in the obvious way. The second claim follows immediately. □

LEMMA 8.2. The following is provable in $K_5^-(W)$. There is an elementary embedding j from $V(\nu)$ into $V(\nu)$ with critical point OW, $j(OW) = \lambda$, and $rng(j) = V(\nu) \cap W$.

Proof: Same as the proof of Theorem 7.5. □

Let MK be the Morse-Kelley theory of classes.

THEOREM 8.3. $K_5^-(W)$ proves ZF(col)\F + “there exists a nontrival elementary embedding from some strongly inaccessible rank into itself.” $K_5^-(W)$ proves the existence of a standard model of ZF + “there exists a nontrivial elementary embedding from some strongly inaccessible rank into itself,” and a standard model of MK + “there exists a nontrivial elementary embedding from V into V.”

Proof: This follows from the fact that strongly inaccessible ranks provably satisfy ZF, and in fact if $V(\alpha)$ is strongly inaccessible, then $V(\alpha+1),V(\alpha)$ provably satisfies MK. ∎

THEOREM 8.4. ZF + “there exists a nontrivial elementary embedding from some strongly Mahlo rank into itself” proves the existence of a standard model of $K_5(W)$.

9. Axiomatic elementary embeddings: huge cardinals and V(λ) into V(λ)

The direct axiomatization of an elementary embedding leads to some particularly elegant axiomatizations corresponding to n-hugeness and elementary embeddings from $V(\lambda)$ into $V(\lambda)$, as well as from V into V. The Subworld approach can be viewed as axiomatizing the image of an elementary embedding from a rank into a (possibly different) rank.

Let K(J) be the following theory in the language $L(\in,=,J)$, where J is a unary function symbol. We use the classical predicate calculus for $L(\in,=,J)$, which includes the axioms of identity.

1. EXT.
2. Separation. $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \ \&\ \varphi))$, where φ is a formula in $L(\in,=,J)$ in which x is not free.
3. Elementarity. $\varphi \leftrightarrow \varphi[x_1/J(x_1),\ldots,x_n/J(x_n)]$, where $n \geq 0$ and φ is a formula in $L(\in,=)$ whose free variables are among $x_1,\ldots,x_n$.
4. Fixed points. $(\exists x)(\forall y)(y \in x \leftrightarrow J(y) = y)$.

Here x,y,z are the variables x1,x2,x3, respectively.

K(J) proves the existence of a standard model of ZF + "there is a cardinal which is n-huge for all n." ZF + "there is a nontrivial elementary embedding from some rank into itself" proves the existence of a standard model of K(J).

Woodin's forcing technique shows that ZF + "there is a nontrivial elementary embedding from some rank into itself" proves the existence of an $\in$ model of ZFC + "there is a cardinal which is simultaneously n-huge for all $n < \omega$."

Recall the development in section 3, which we freely use here.

LEMMA 9.1. The following is provable in K(J). J maps ordinals to ordinals. J is strictly increasing on the ordinals. $J(x) \geq x$ on the ordinals. There is an ordinal which is moved by J.

Proof: For the first two claims, note that by Elementarity, J maps ordinals to ordinals and $x \in y \leftrightarrow J(x) \in J(y)$. The third claim follows by transfinite induction. For the fourth claim, if all ordinals are fixed by J then by Fixed Points, the set of all ordinals exists. This is a transitive set of ordinals, and so is an ordinal by Lemma 3.2, which is a member of itself. But this contradicts Lemma 3.2. ∎

We continue to use κ for the first ordinal moved by J.

LEMMA 9.2. The following is provable in K(J). For all $\alpha < \kappa$, $V(\alpha)$ exists, and J fixes $V(\alpha)$ and all elements of $V(\alpha)$. κ is a limit ordinal $> \omega$.

Proof: We prove the first claim by transfinite induction on $\alpha < \kappa$. The case $\alpha = 0$ is obvious. Suppose true for $\alpha < \kappa$. Then since $\alpha+1$ is definable from α, we see that $\alpha+1 < \kappa$. Now we claim that for all $x \subseteq V(\alpha)$, $J(x) = x$. To see this, fix $x \subseteq V(\alpha)$ and $y \in x$. Then $J(y) = y \in x$. On the other hand suppose $y \in J(x)$. Now since $x \subseteq V(\alpha)$, we see that $J(x) \subseteq J(V(\alpha)) = V(\alpha)$. Hence $y \in V(\alpha)$, and so $J(y) = y$. Therefore from $y \in J(x)$, we obtain $y \in x$. This establishes the claim.

By Axiom 4, there is a set that includes all subsets of $V(\alpha)$ as elements. Hence $V(\alpha+1)$ exists, and J fixes all elements of $V(\alpha+1)$. Also J fixes $V(\alpha+1)$ since it is definable from α.

Finally, let $\alpha < \kappa$ be a limit ordinal. By Axiom 4, there is a set that includes all elements of all $V(\beta)$, $\beta < \alpha$. Hence $V(\alpha)$ exists. Obviously J fixes all elements of $V(\alpha)$ by the induction hypothesis. Also J fixes $V(\alpha)$ because it is definable from α.

If $\kappa = \alpha+1$ then $J(\alpha) = \alpha$, and so $J(\kappa) = \kappa$, since κ is definable from α, which is a contradiction. So κ is a limit ordinal. Hence $\omega \leq \kappa$. But ω is definable. Hence $J(\omega) = \omega$. Therefore $\omega < \kappa$. □

LEMMA 9.3. The following is provable in K(J). $V(\kappa)$ satisfies ZF.

Proof: By the results of section 4, and that κ is a limit ordinal $> \omega$, it suffices to verify replacement in $V(\kappa)$.

Let $\alpha < \kappa$ and fix an instance of replacement with domain $V(\alpha)$ for which the hypothesis holds in $V(\kappa)$. Let β be least such that each unique associate lies in $V(\beta)$. Clearly $\beta \leq \kappa$ since κ has this property. Now β is definable (from an element of $\omega < \kappa$), and so $J(\beta) = \beta$. Hence $\beta < \kappa$ as required. □

Now let λ be the first fixed point of J that is greater than κ if it exists; ∞ otherwise.

LEMMA 9.4. The following is provable in K(J). λ is a limit ordinal. For all $\alpha < \lambda$, $V(\alpha)$ exists.

Proof: First suppose $\lambda = \infty$. The first claim amounts to there not being a greatest ordinal α. If α is the greatest ordinal then α is definable, and so $J(\alpha) = \alpha$, in which case $\lambda < \infty$. If $\lambda < \infty$ then if $\lambda = \alpha+1$, then α is definable from λ, and hence $J(\alpha) = \alpha$, again contradicting the choice of λ.

For the second claim, let α be least such that $V(\alpha)$ does not exist. Then α is definable, and so $J(\alpha) = \alpha$. If $\alpha < \lambda$ then $\alpha < \kappa$, in which case $V(\alpha)$ exists. So $\lambda \geq \kappa$. □

THEOREM 9.5. K(J) proves ZF + "for all $n \geq 0$, there exists a nontrivial elementary embedding j from a rank into a rank such that the n-th iterate of j at the critical point exists." K(J) proves the existence of a standard model of ZF + "for all $n \geq 0$, there exists a nontrivial elementary embedding j from a rank into a rank such that the n-th iterate of j at the critical point exists."

Proof: To begin with, the formalization of $Jn(\kappa)$ can be carried out in the obvious way within K(J), since we have full Separation in K(J). Since $V(\kappa)$ satisfies ZF, clearly $V(J^{n+1}(\kappa))$ satisfies ZF. Also, we see that $V(J^{n+1}(\kappa))$ also satisfies that $J|V(J^{n-1}(\kappa))$ is an elementary embedding from $V(J^{n-1}(\kappa))$ into $V(J^{n}(\kappa))$ with critical point κ which sends each $J^{m}(\kappa)$ to $J^{m+1}(\kappa)$, $m \leq n-2$. Here by convention, $J^{0}(\kappa) = \kappa$. □

THEOREM 9.6. ZF + "there exists a nontrivial elementary embedding from a rank into itself" proves the existence of a standard model of K(J).

Proof: Let J be a nontrivial elementary embedding from some $V(\alpha)$ into $V(\alpha)$, where α has been minimized. It is well known that the set of all fixed points of J is exactly $V(\kappa)$, where κ is the critical point. The well founded transitive standard model of K(J) is $(V(\alpha),\in,J)$. □

10. Axiomatic elementary embeddings: V into V.

Let K*(J) be the following system in $L(\in,=,J)$.

1. EXT
2. Replacement. $(\forall x \in y)(\exists! z)(\varphi) \rightarrow (\exists w)(\forall z)(z \in w \leftrightarrow (\exists x \in y)(\varphi))$, where φ is a formula in $L(\in,=,J)$ in which x4 is not free.
3. Elementarity.
4. Initial fixed points. $(\exists x)(\forall y)(y \in x \leftrightarrow (J(y) = y \;\&\; (\forall z \in y)(J(z) = z)))$.

Here x,y,z are the variables x1,x2,x3, respectively.

There is a fairly exact correspondence of K*(J) with VB + "there exists a nontrivial elementary embedding from V into V. Firstly, these theories are equiconsistent. Secondly, each one proves the existence of (well founded transitive) standard models of every finite fragment of the other.

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ADDENDUM

October 28, 1997

We are in the process of making significant improvements and simplifications in this manuscript. We have completed a new treatment of measurable cardinals below, which is to replace section 4. We are continuing these improvements for the higher cardinals.

4. Measurable cardinals

Let $K_2(W)$ be the following theory in the language $L(\in,W)$ with the constant symbol W. We use $\subseteq^*$ to represent core inclusion; i.e., $x \subseteq^* y$ is read "x is a core subset of y." Informally, $x \subseteq^* y$ means that every element of x is an element of y, every element of every element of x is an element of y, etcetera. This is formalized in $K_2(W)$ by saying that x is a subset of some transitive subset of y. We could, alternatively, introduce $\subseteq^*$ as primitive, with the defining axiom

$$x \subseteq^* y \leftrightarrow (x \subseteq y \ \&\ (\forall y \in x)(y \subseteq^* x)).$$

However, we will not pursue this approach here, preferring in this paper to adhere to the language $L(\in,W)$.

1. EXT.
2. SS′. $(\exists x \in W)(\forall y \in W)(y \in x \leftrightarrow (y \subseteq^* W \ \&\ \varphi))$, where φ is a formula in $L(\in,W)$ in which x is not free.
3. RED.

Here x,y are the variables x_1,x_2, respectively.

Note that SS′ skirts the Russell Paradox in a similar way to SS; i.e., if we remove $\subseteq^*$ then we get the familiar Russell Paradox.

Recall that K(W) and $K_1(W)$ prove that W is transitive. It is easy to see that W is not transitive in $K_2(W)$. This is because if W is transitive then $(\forall y \in W)(y \subseteq^* W)$, and so we can remove $\subseteq^*$ in SS′, resulting in Russell's Paradox.

We remark that nothing in the axioms of $K_2(W)$ commits us to the existence of transitive closures or "every set is a subset of a transitive set," and this cannot be proved in $K_2(W)$.

We will show that $K_2(W)$ proves the existence of a standard model of ZF + "there is a measurable rank." ZF + "there is a nontrivial elementary embedding from some rank into some rank" proves the existence of a standard model of $K_2(W)$.

Under the definitions made below, ZF + "there is a measurable rank" is ∈ bi-interpretable with ZFC + "there is a measurable cardinal." In ZFC, V(alpha) is measurable if and only if alpha is a measurable cardinal.

As usual, we use x = y to abbreviate (∀z)(z ∈ x ↔ z ∈ y).

LEMMA 4.1. Let φ be a formula in L(∈,W) which does not mention x2. The following is provable in $K_2(W)$. x = x. x = y → y = x. (x = y & y = z) → x = z. x = y → (φ ↔ φ[x/y]).

Proof: See Lemma 1.1. ∎

For any formula φ in L(∈), let φ^W be the result of relativizing all quantifiers to W.

LEMMA 4.2. Let φ be a formula in L(∈) whose free variables are among x1,…,xn, n ≥ 0. The following is provable in $K_2(W)$. $x_1,\ldots,x_n \in W \rightarrow (\varphi \leftrightarrow \varphi^W)$.

Proof: See Lemma 1.2. ∎

We let W* be the class of all x ∈ W such that x ⊆* W. We are specifically not claiming that W* is a set in K2(W). We use x ∈⊆ y for x ∈ y & x ⊆ y. We say that x is the transitive closure of y if and only if y ⊆ x, x is transitive, and every transitive set that includes y as a subset must include x as a subset.

LEMMA 4.3. The following is provable in $K_2(W)$. The transitive closure of every element of W* exists and is an element of W*. W* is transitive. W* satisfies Extensionality.

Proof: Let x ∈ W*. By SS′ let y ∈ W be such that the elements u of y from W are exactly the u ⊆* W that all transitive supersets of x have in common. Let x ⊆ u ⊆ W, u transitive. Then u ⊆ W*. Hence any common element of all transitive supersets of x must be ⊆* W. Hence the elements of y from W are exactly the elements that lie in all transitive supersets of x. In particular, every element of y from W lies in all transitive supersets of x. By RED, every element of y lies in all transitive supersets of x. Hence the elements of y are exactly the common elements of all transitive supersets of x; i.e., y is the transitive closure of x. But $y \subseteq u \subseteq W$ and y is transitive and $y \in W$. Hence $y \in W^*$.

For the second claim, suppose $x \in y \in W^*$. Then $y \in W$ and y is a subset of some transitive subset z of W. Hence $y \subseteq W$, and so $x \in W$. Also since $y \subseteq z$, we see that $x \in z$, and hence $x \subseteq z$. Hence $x \in W^*$, establishing the transitivity of W*. The final claim follows immediately.

LEMMA 4.4. The following is provable in $K_2(W)$. Let $x,y,z,w \in W^*$. Then $x \cup y \cup \{z,w\}$ exists and lies in W.

Proof: Let $x,y,z,w \in W^*$. By SS′ let $u \in W$ be such that the elements v of u from W are exactly the $v \subseteq^* W$ such that ($v \in x$ or $v \in y$ or $v = z$ or $v = w$). Hence the elements of u from W are exactly the v such that ($v \in x$ or $v \in y$ or $v = z$ or $v = w$). By RED, u is as desired.

LEMMA 4.5. The following is provable in $K_2(W)$. The unordered pair of any two elements of W* lies in W*.

Proof: Let $x,y \in W^*$. By SS′ let $z \in W$ be such that the elements of z from W are exactly x and y. Then by RED, the elements of z are exactly x and y; i.e., $z = \{x,y\}$. By Lemma 4.3, let $x \subseteq x' \in W$ and $y \subseteq y' \in W$, where x′,y′ are transitive. Then $x',y' \in W^*$. By Lemma 4.4, $x' \cup y' \cup \{x,y\}$ exists and is a transitive subset of W. Hence $\{x,y\} \in W^*$.

LEMMA 4.6. Let φ be a formula in $L(\in,W)$. The following is provable in $K_2(W)$. For all $x \in W^*$, $\{y \in x: \varphi\}$ exists and lies in W*. The union of any element of W* is an element of W*. The power set of any element of W* is an element of W*.

Proof: For the first claim, let $x \subseteq y \in W^*$. Then x is a subset of a transitive subset of W since y is. Also note that $y \subseteq W^*$. By SS′ let $z \in W$ be such that the elements of z from W are exactly the elements of x. Now apply RED to the statement "every element of z from W lies in y" to obtain $z \subseteq y$. Hence $z \subseteq W$, in which case $z = x \in W$. Hence $z = x \in W^*$.

For the second claim, we can assume that φ is in L($\in$), since we can absorb W as a parameter. Let $x \in W^*$ and $z_1,...,z_k$ be sets. We first want to show that $\{y \in x: \varphi(y,z_1,...,z_k)\}$ exists. By RED, it suffices to prove that for all $z_1,...,z_k \in W$, $\{y \in x: \varphi(y,z_1,...,z_k)\}$ exists.

By SS′, let $w \in W$ be such that the elements of w from W are exactly the $y \subseteq^* W$ such that $y \in x$ and $\varphi(y,z_1,...,z_k)$. I.e., the elements of w from W are exactly the $y \in x$ such that $\varphi(y,z_1,...,z_k)$. In particular, the elements of w from W all lie in x. By RED, $w \subseteq x$. Hence $w \subseteq W$. Therefore the elements of w are exactly the $y \in x$ such that $\varphi(y,z_1,...,z_k)$.

For Union, let $x \in W^*$. Then every element of every element of x is in W*, since W* is transitive. Let $y \in W$ be such that the elements of y from W are exactly the elements of the elements of x. Hence every element of y from W is an element of an element of x. Therefore by RED, every element of y is an element of an element of x. So $y \subseteq W$. Hence $y = \cup x$.

For power set, let $x \in W^*$. By SS′ let $y \in W$ be such that the elements of y from W are exactly the subsets of x. We can do this by the first claim. Now every element of y from W is a subset of x. Hence by RED, every element of y is a subset of x. Hence y is the power set of x.

Note that external parameters are allowed in the first claim of Lemma 4.6. As a consequence, we see that every subset of any element of W* is an element of W*.

Thus we have verified the following within W*:

1. Extensionality.
2. Pairing.
3. Union.
4. Power set.
5. Every set has a transitive closure.
6. Separation with respect to any external formula with any external parameters.

We refer to 6 as External Separation for W*.

In $K_2(W)$ we say that x is an ordinal if and only if x is $\in$-connected, transitive, and every nonempty subset of x has an $\in$-least element. We use $x \in\subseteq y$ for $x \in y$ & $x \subseteq y$.

LEMMA 4.7. The following is provable in $_{K2}$(W). If x is an ordinal then x ∈ W* ↔ x ∈⊆ W. Every element of every ordinal in W* is an ordinal in W*.

Proof: For the first claim let x be an ordinal, x ∈⊆ W. Then x ⊆* W since x is transitive. Hence x ∈* W.

For the second claim, let x ∈ y ∈ W*, y an ordinal. Since W* is transitive, x ∈* W. To see that x is an ordinal, clearly x is ∈-connected. To see that x is transitive, let u ∈ v ∈ x. Then u ∈ x or x ∈ u or u = x. The latter two cases are impossible by the definition of ordinal and External Separation for W*. The well foundedness condition on x follows since x ⊆ y.

LEMMA 4.8. The following is provable in K_2(W). For all ordinals x,y ∈ W*, we have x ⊆ y or y ∈ x. For all ordinals x,y ∈ W*, we have x ∈ y or y ∈ x or x = y exclusively.

Proof: Suppose x ¬⊆ y. By External Separation for W*, let z be the ∈-least element of x that is not in y. Then z ⊆ y. Now let u ∈ y. We have u ∈ z or z ∈ u or u = z. If z ∈ u then z ∈ u ∈ y, and so z ∈ y which is a contradiction. If z = u then z ∈ y. Hence u ∈ z. What we have shown is that y ⊆ z. Hence y = z ∈ x.

For the second claim, suppose x ⊆ y. If x ∉ y then by the first claim y ⊆ x, and so x = y. For the exclusivity, note that if x ∈ x then x ∈ x ∈ x, is a contradiction.

LEMMA 4.9. The following is provable in K_2(W). Every nonempty set of ordinals from W* has an ∈-least element. We can apply transfinite induction with respect to any formula with any parameters on the ordinals in W*.

Proof: Let A be a nonempty set of ordinals from W*. Let x ∈ A. If x and A are disjoint then we are done. Otherwise, there is an ∈-least element in common with x and A, which is what we want.

The second claim follows from the first claim using External Separation for W*.

For sets x, define x+1 = {y: y ∈ x or y = x}. We have to worry about existence. We let x ∈= y mean x ∈ y or x = y.

LEMMA 4.10. The following is provable in $K_2(W)$. If x,y are ordinals in W* and x+1 = y+1 then x = y. If x is an ordinal in W* then x+1 exists and is an ordinal in W*.

Proof: If x+1 = y+1 then x ∈= y and y ∈= x. If x,y are ordinals, the only possible case is x = y.

Let x be an ordinal in W*. By Pairing and Union in W*, we obtain x+1 = ∪({x,{x}}) ∈ W*. It is easy to verify that x+1 is an ordinal using External Separation for W*. □

A limit ordinal is a nonempty ordinal with an ∈-maximum element. We use 0 for the empty ordinal; i.e., ∅. A successor ordinal is an ordinal of the form x+1, in which case its predecessor is x. The predecessor is unique by Lemma 4.10.

LEMMA 4.11. The following is provable in $K_2(W)$. If x ∈ W* is a nonzero ordinal with the ∈-maximum element y, then x = y+1. Every ordinal in W* is 0, a limit ordinal, or a successor ordinal, and these cases are mutually exclusive. If x ∈ y ∈ W* and y is an ordinal then x+1 ∈= y. Every transitive set of ordinals from W* is an ordinal.

Proof: For the first claim, let x,y be as given. By comparing any z ∈ x with y, we see that x = {z: z ∈= y} = y+1.

The second claim follows immediately from the first claim.

For the third claim, let x,y be as given. By External Separation for W*, we can form the set of all u ∈ y such that x ∈ u. Let x′ be the ∈-least element of u. If x′ does not exist then x is the ∈-maximum element of y, in which case y = x+1 by claim 1. So we can assume that x′ exists, in which case x is the ∈-maximum element of x′. So again by claim 1, x′ = x+1, and so x+1 ∈ y.

For the final claim, let x be a transitive set of ordinals from W*. Then clearly x is ∈-connected. Well foundedness follows External Separation for W*.

We say that an ordinal is finite if and only if it is not a limit ordinal, and no element is a limit ordinal.

LEMMA 4.12. There is a formula $\varphi(x)$ in $L(\in)$ with at most the free variable x such that the following is provable in $K_2(W)$. For all x, x is a finite ordinal in W* ↔ $\varphi(x)$.

Proof: Take $\varphi(x)$ if and only if

i) x is a finite ordinal;
ii) every element of x is either ∅, uniquely a successor of some element of x, or a limit ordinal, exclusively;
iii) y, if there is an element of x not in y then there is an ∈-least element of x not in y). Let $\varphi(x)$. We must show that x is a finite ordinal in W*.

First suppose x ¬⊆ W, and let y be the ∈-least element of x that is not in W. Then y ⊆ W and y is nonzero. If y = z+1 then by RED, y ∈ W, which is a contradiction. Therefore y is a limit ordinal. This contradicts that x is a finite ordinal.

Hence x ⊆ W, and so x ⊆* W. If x = 0 then x ∈ W*. Otherwise, let x = y+1. Then y ∈ W, and so by RED, x ∈ W, in which case x ∈* W.

LEMMA 4.13. The following is provable in $K_2(W)$. The set of all finite ordinals in W* exists and lies in W*. It is the least limit ordinal.

Proof: Let x ∈ W be such that the elements of x from W are exactly the finite ordinals in W*. By Lemma 4.12, every element of x from W has $\varphi(x)$. Therefore by RED, every element of x has $\varphi(x)$. By Lemma 4.12, the elements of x are exactly the finite ordinals in W*. Thus the set ω of all finite ordinals in W* lies in W. Obviously ω is transitive and is a subset of W. Hence ω ∈* W. Also ω is a transitive set of ordinals from W*. Hence by Lemma 4.11, ω is an ordinal.

Clearly ω is a limit ordinal since the successor of every finite ordinal in W* is a finite ordinal in W*. And ω must be the least limit ordinal since its elements are finite ordinals in W*, which cannot be limit ordinals.

We can relate the least limit ordinal, which we write as ω, to the usual axiom of infinity.

LEMMA 4.14. The following is provable in $K_2(W)$. ω is the least set satisfying $\emptyset \in \omega$ & $(\forall x \in \omega)(x \cup \{x\} \in \omega)$. $\omega \in W^*$.

Proof: Left to the reader.

We are now prepared to define the cumulative hierarchy. We say that f is a cumulation function on x if and only if

i) x is an ordinal;
ii) f is a function with domain x;
iii) if $0 \in x$ then $f(0) = 0$;
iv) if $y+1 \in x$ then $f(y) = PS(f(x))$;
v) if $y \in x$ and y is a limit ordinal, then f(y) is the set of all elements of the f(z), $z \in y$.

We say that f is a cumulation function if and only if f is a cumulation function on some x. Here functions are treated in terms of sets of ordered pairs, and order pairs are treated with the usual $\langle x,y\rangle = \{\{x\},\{x,y\}\}$.

LEMMA 4.15. The following is provable in $K_2(W)$. If $f \in W^*$ is a cumulation function on x then $x \in W^*$. If $f \in W^*$ is a cumulation function on x and $y \in x$ then $f|y$ is a cumulation function on y and $f|y \in W^*$. For any $x \in W^*$, there is at most one cumulation function on x. Any two cumulation functions lying in W^* are comparable under $\subseteq$.

Proof: For the first claim, let $f \in W^*$ be a cumulation function on x. In particular $f \in W$, and so by RED, $dom(f) = x \in W$. Now let $y \in x$. Then y is the first coordinate of an element of f, and so is the first coordinate of an element of W. So again by RED, $y \in W$. Thus $x \subseteq W$. Since x is transitive, $x \in W^*$.

For the second claim, let $f \in W^*$ be a cumulation function on x and $y \in x$. By External Separation for W^*, $f|y$ exists and lies in W^*, and by inspection, $f|y$ is a cumulation function on y.

For the last claim, use transfinite induction on the ordinals in W^* (Lemma 4.9).

LEMMA 4.16. The following is provable in $K_2(W)$. There is a cumulation function in W^* on every ordinal in W^*.

Proof: We prove this by transfinite induction on the ordinals in W*. The case $\alpha = 0$ is trivial, taking the cumulation funciton to be $\emptyset \in W^*$. Suppose $f \in W^*$ is a cumulation function on the ordinal $\alpha \in W^*$. We can obviously extend f to a cumulation function on $\alpha+1$ using union, pairing, and power set in W*.

Suppose that for all $\beta < \alpha$, there is a cumulation function f $\in W^*$ on β. By Lemma 4.15, these are the unique cumulation functions on the $\beta < \alpha$, and they are comparable under $\subseteq$.

At this point, we would like to use External Separation for W* in order to form the union of these cumulation functions. The trouble is that we can't yet place all of the elements of this union inside one big element of W*. So we have to argue more carefully.

Let $x \in W$ be such that the elements of x from W are exactly the elements of values of the cumulation functions on the $\beta < \alpha$. Because of uniqueness, the cumulation functions on the β **<** α are unique if they exist, they must all be from W*.

By RED, every element of x is an element of a value of a cumulation function on some $\beta < \alpha$. Hence x is exactly the set of elements of values of the cumulation functions on the $\beta < \alpha$. And since these cumulation functions lie in W*, we see that $x \subseteq W^*$, and so $x \in W^*$. Also obviously x is the actual cumulation function on the ordinal α.

For ordinals α in W*, we define $V(\alpha)$ to be the unique value of the cumulation function with domain $\alpha+1$ at α.

LEMMA 4.17. The following is provable in $K_2(W)$. $V(0) = 0$. If α is an ordinal in W* then $V(\alpha+1) = PS(V(\alpha)$. If α is a limit ordinal in W* then $V(\alpha)$ is the union of the $V(\beta)$, $\beta < \alpha$. For all ordinals in W*, $V(\alpha) \in W^*$.

Proof: Obvious from the definitions and Lemma 4.16. The last claim is from the fact that the cumulation function on $\alpha+1$ lies in W*. □

LEMMA 4.18. The following is provable in $K_2(W)$. Let α be an ordinal in W*. $V(\alpha)$ is transitive. $V(\alpha) \subseteq V(\alpha+1)$.

Proof: For the first claim, we use transfinite induction on the ordinals in W*. Suppose that for all $\beta < \alpha$, $V(\beta)$ is transitive and $V(\alpha)$ exists. If α is a limit ordinal, let $x \in y \in V(\alpha)$. Then let $\beta < \alpha$, $y \in V(\beta)$. By the induction hypothesis, $x \in V(\beta)$, and so $x \in V(\alpha)$. If $\alpha = \beta+1$, then $V(\beta)$ is transitive. Let $x \in y \in V(\alpha)$. Then $y \subseteq V(\beta)$, and so $x \in V(\beta)$. Hence $x \subseteq V(\beta)$. Therefore $x \in V(\alpha)$. And obviously $V(0)$ is transitive.

For the second claim, let $x \in V(\alpha)$. Then $x \subseteq V(\alpha)$, and so $x \in V(\alpha+1)$. □

LEMMA 4.19. The following is provable in $K_2(W)$. If $\alpha < \beta \in W^*$ then $V(\alpha) \in\subseteq V(\beta)$.

Proof: By trasnfinite induction on ordinals in W*. Let γ be an ordinal such that $V(\gamma)$ exists and for all $\beta < \gamma$, the claim holds. First suppose γ is a limit ordinal. Let $\alpha < \gamma$. We must check that $V(\alpha) \subseteq V(\gamma)$ and $V(\alpha) \in V(\gamma)$. To see the first, let $x \in V(\alpha)$. Then $x \in V(\gamma)$. To see the second, since the claim holds for $\alpha+1$, we have $V(\alpha) \in V(\alpha+1)$, and so $V(\alpha) \in V(\gamma)$.

Now suppose $\gamma = \delta+1$. Let $\alpha \leq \delta$. Then $V(\alpha) \subseteq V(\delta) \subseteq V(\gamma)$. And if $\alpha = \delta$ then $V(\alpha) \in V(\gamma)$. If $\alpha < \delta$ then $V(\alpha) \subseteq V(\delta)$, and so $V(\alpha) \in V(\gamma)$. □

LEMMA 4.20. The following is provable in $K_2(W)$. Let α be an ordinal in W*. The union of every element of $V(\alpha)$ lies in $V(\alpha)$. The unordered pair of any two elements of $V(\alpha)$ lies in $V(\alpha+1)$. The power set of any element of $V(\alpha)$ lies in $V(\alpha+2)$.

Proof: For the first claim, let $x \in V(\alpha)$. Now if α is a limit ordinal then let $x \in V(\beta)$, $\beta < \alpha$. Hence every element of an element of x lies in $V(\beta)$. Hence the union of x exists as a subset of $V(\beta)$, and hence as an element of $V(\alpha)$. Now let $\alpha = \beta+1$, and $x \in V(\alpha)$. Then $x \subseteq V(\beta)$, and so every element of an element of x lies in $V(\beta)$. Hence the union of x exists as a subset of $V(\beta)$, and so as an element of $V(\alpha)$.

The second claim is obvious. For the third claim, let $x \in V(\alpha)$. Then $x \subseteq V(\alpha)$, and so every subset of x lies in $V(\alpha+1)$. Hence the power set of x exists as a subset of $V(\alpha+1)$, and so lies in $V(\alpha+2)$. □

LEMMA 4.21. The following is provable in $K_2(W)$. Let α be an ordinal in W*. Every subset of every $V(\alpha)$ lies in $V(\alpha)$. Every $V(\alpha)$ includes α as a subset.

Proof: For the first claim, suppose this is true for all $\beta < \alpha$ and $V(\alpha)$ exists. If α is a limit ordinal, then let $x \subseteq y \in V(\alpha)$. Let $x \subseteq y \in V(\beta)$, $\beta < \alpha$. Then $x \in V(\beta)$, and so $x \in V(\alpha)$. If $\alpha = \beta+1$, then let $x \subseteq y \in V(\alpha)$, and so $x \subseteq y \subseteq V(\beta)$. Then $x \in V(\alpha)$.

For the second claim, suppose this is true for all $\beta < \alpha$ and $V(\alpha)$ exists. The zero case and limit case is trivial using Lemma 3.9. Suppose $\alpha = \beta+1$. Then $\beta \subseteq V(\beta)$. Since every element of $\beta+1$ is therefore a subset of $V(\beta)$, we see that $\beta+1 \subseteq V(\beta+1)$. □

It is useful to define rk(x) to be the least ordinal α in W* such that $x \in V(\alpha)$ provided this exists; undefined otherwise.

LEMMA 4.22. The following is provable in EST. If rk(x) exists then rk(x) is not a limit ordinal. If $x \in y$ and rk(y) exists then rk(x) < rk(y). If rk(x) exists then rk(x) is the least successor ordinal greater than all rk(y), $y \in x$.

Proof: The first claim is obvious. For the second claim, let $x \in y$ and $rk(y) = \alpha+1$. Then $x \in y \subseteq V(\alpha)$, and so $x \in V(\alpha)$, in which case $rk(x) \leq \alpha < \alpha+1$. For the third claim, let $rk(x) = \alpha+1$. Then $\alpha+1$ is greater than all rk(y), $y \in x$ by the second claim. On the other hand, suppose $\beta+1 < \alpha+1$ and for all $y \in x$, $rk(y) < \beta+1$. Then $x \subseteq V(\beta)$, and so $x \in V(\beta+1)$, violating $rk(x) = \alpha+1$. □

LEMMA 4.23. The following is provable in EST. Every nonempty subset of every $V(\alpha)$, $\alpha \in W^*$, has an $\in$-minimal element.

Proof: Let $x \subseteq V(\alpha)$ be nonempty. Let y be an element of x such that rk(y) is minimized. We can do this by External Separation for W*. Suppose $z \in y,x$. Then by Lemma 4.22, rk(z) < rk(y), contradicting the minimality of rk(y).

LEMMA 4.24. The following is provable in $K_2(W)$. The cumulation function on any ordinal $\alpha \in W^*$ lies in $V(\alpha+3)$.

Proof: By transfinte induction on ordinals in W*.

At this point we let $W^\#$ be the class of all elements of the $V(\alpha)$, for ordinals $\alpha \in W^*$. We emphasize that we are not claiming that $W^\#$ is a set in $K_2(W)$ - at least not at this time.

LEMMA 4.25. The following is provable in $K_2(W)$. $W^\#$ is transitive. Every subset of every element of $W^\#$ is in $W^\#$ (the subset condition). $W^\#$ satisfies the following first order axioms, which we call T_0:

1. Extensionality.
2. Pairing.
3. Union.
4. Power set.
5. Separation.
6. Infinity.
7. Every set has an $\in$-minimal element.
8. There is a cumulation function on every ordinal.
9. Every set is an element of the value of some cumulation function on some ordinal.

Furthermore, External Separation holds for $W^\#$ in the sense that for all $x \in W\#$ and formulas φ in $L(\in,W)$ with any set parameters anywhere, we have $\{b \in x: \varphi\} \in W^\#$.

Proof: The transitivity of $W^\#$ follows from Lemma 4.18. The subset condition follows from Lemma 4.21. Extensionality follows from transitivity. Pairing follows from Lemma 4.20. Power set follows from Lemma 4.20 and the subset condition. Infinity follows from Lemmas 4.21 and 4.14. Foundation (7) follows from Lemma 4.23. 8 follows from Lemma 4.24 and the fact that ordinals in the sense of W* are exactly the ordinals that lie in W* (by the subset condition). 9 is immediate - cumulation functions look like cumulation functions in W#. Finally, Extended Separation for W# follows from transitivity, the subset condition, and External Separation for W*.

Note that we have not yet proved that $W^\#$ exists as a set. We now aim for this, as well as proving that we can get lots of subsets of $W^\#$.

LEMMA 4.26. The following is provable in $K_2(W)$. If $x,y \in W$ and every element of x in W is an element of y, then $x \subseteq y$. In particular, if $x,y \in W$ and x,y have the same elements from W, then $x = y$.

Proof: By RED.

LEMMA 4.27. The following is provable in $K_2(W)$. There is a unique set $A \in W$ such that $A \cap W = W^\#$. A is transitive. $W^\#$ is an elementary submodel of A. In particular, A satisfies each instance of T_0. Furthermore, A satisfies each instance of External Separation.

Proof: By SS' let $A \in W$ such that the elements of x from W are exactly the elements of $W^\#$. By Lemma 4.26, A is unique. And $W^\# \subseteq A$.

To see that A is transitive, note that $(\forall x,y \in W)(x \in y \in A \rightarrow x \in A)$. By RED, $(\forall x,y)(x \in y \in A \rightarrow x \in A)$.

To see that $W^\#$ is an elementary submodel of A, suppose $\varphi(x1,...,xk)$ holds in $W^\#$, where $x_1,...,x_k \in W^\#$. Let $\varphi'(x_1,...,x_k)$ be the result of relatizing the quantifiers in φ to A, and then expanding this out where the "$v \in A$" appears as conjuncts or antecedents of implicants. Finally consider $\varphi'(x_1,...,x_k)^W$. This has the same menaing as $\varphi(x_1,...,x_k)$ holding in $W^\#$, and so is true. Now by RED, we have $\varphi'(x_1,...,x_k)$. I.e., $\varphi(x_1,...,x_k)$ holds in A.

Finally, we show that A satisfies each instance of External Separation. We can assume that the formula does not mention W since we can absorb that parameter. We need to prove that $(\forall x \in A)(\forall y^1,...,y^k)(\exists z \in A)(\forall w \in A)(w \in z \leftrightarrow \varphi^A(x,y^1,...,y^k,w))$. It suffices to prove that $(\forall x \in A,W)(\forall y^1,...,y^k \in W)(\exists z \in A,W)(\forall w \in A,W)(w \in z \leftrightarrow \varphi^{A,W}(x,y_1,...,y_k,w))$. I.e., $(\forall x \in W^\#)(\forall y_1,...,y_k \in W)(\exists z \in W^\#)(\forall w \in W^\#)(w \in z \leftrightarrow \varphi^{W\#}(x,y_1,...,y_k,w))$. But this follows from External Separation for $W^\#$.

We write the A in Lemma 4.27 as $W^{\#\#}$. Since $W^\#$ is an elementary submodel of $W^{\#\#}$, and so $W^{\#\#}$ satisfies T_0, we can use the notation $V(\alpha)$ for ordinals α in the sense of $W^{\#\#}$ without conflicting with the earlier use of the notation V(α) for ordinals α in $W^{\#}$.

LEMMA 4.28. The following is provable in $K_2(W)$. The ordinals in $W^{\#\#}$ are exactly the ordinals in the sense of $W^{\#\#}$. Transfinite induction can be applied to the ordinals in $W^{\#\#}$ with respect to any formula in L($\in$,W) using any set parameters. The V(α)'s in $W^{\#\#}$ obey the standard defining conditions. There is a least ordinal in $W^{\#}$# that is not in $W^{\#}$. This is the same as the least ordinal not in W. It is a limit ordinal.

Proof: The first two claims follows from the transitivity of $W^{\#\#}$ and External Separation for $W^{\#\#}$. The third claim follows from the fact that $W^{\#\#}$ satisfies T_0.

The fourth claim follows from transfinite induction on the ordinals in $W^{\#\#}$, provided we can show that there is an ordinal in $W^{\#\#}$ that is not in $W^{\#}$. Suppose that all ordinals in $W^{\#\#}$ are in $W^{\#}$. Let x be the unique element of W such that the elements of x from W are exactly the ordinals in $W^{\#}$. Then every element of x from W is an element of $W^{\#\#}$. Hence $x \subseteq W^{\#\#}$. Now every element of x from W is an ordinal in $W^{\#\#}$. Hence every element of x is an ordinal in $W^{\#\#}$. Therefore every element of x is an ordinal in $W^{\#}$. Hence $x \in W$ is exactly the set of all ordinals in $W^{\#}$. So x is a transitive subset of W. Hence $x \in W^*$, and x is a transitive set of ordinals. Therefore x is an ordinal in $W^{\#}$. Hence $x \in x$, which is a contradiction.

Let α be the least ordinal in $W^{\#\#}$ that is not in $W^{\#}$. Then $\alpha \subseteq W^{\#}$. Hence α is the least ordinal not in W. We write it as OW. It is a limit ordinal by RED, since otherwise it could be defined from its successor.

As indicated in the proof of Lemma 4.28, we let OW be the least ordinal not in W. It is in $W^{\#\#}$ but not in $W^{\#}$.

LEMMA 4.28. The following is provable in $K_2(W)$. $V(OW) = W^{\#}$, and hence $W^{\#}$ exists and lies in $W^{\#\#}$.

Proof: Both V(OW) and $W^{\#}$ consist of the elements of the V(α) for which α < OW.

We will need to use V(OW+n) for small n, and we will need to know that these iterated power sets over V(OW) have real substance. This is afforded by External Separation for $W^{\#\#}$.

LEMMA 4.29. The following is provable in $K_2(W)$. For all $x \subseteq V(OW)$ there is a unique $y \in W$ such that $x = y \cap W$. This y is a subset of $W^{\#\#}$.

Proof: By SS′ let $y \in W$ be such that the elements of y from W are exactly the elements of x. there is such a $y \in W$. Then $x = y \cap W$. Let $y' \in W$ be an alternative. Then y and y′ have the same elements from W. Therefore by RED, $y = y'$.

Clearly every element of y from W is an element of $W^{\#\#}$. Hence by RED, $y \subseteq W^{\#\#}$.

For $x \subseteq V(OW)$, we let H(x) be the unique y afforded by Lemma 4.29. We are not claiming that H exists as a function.

LEMMA 4.30. Let φ be a formula in $L(\in)$ with at most the free variables x1,...,xk. The following is provable in $K_2(W)$. The fixed points of H are exactly the elements of V(OW). For all $x_1,\ldots,x_k \subseteq V(OW)$, φ holds in $(V(OW),\in,x_1,\ldots,x_k)$ if and only if φ holds in $(W^{\#\#},\in,H(x_1),\ldots,H(x_k))$. Here we use $x_1,\ldots,x_k$ as monadic predicates.

Proof: Suppose $x \in V(OW)$. Then $x \in W$ and $x = x \cap W$. So by Lemma 4.29, $H(x) = x$. On the other hand, let $H(x) = x$. Then $x \in W$ and $x \subseteq V(OW)$. By RED, the rank of x must be in W. However it is at most OW. Hence the rank of x is $<$ OW. Therefore $x \in V(OW)$ as required.

For the second claim, expand out "if φ holds in $(W^{\#\#},\in,H(x_1),\ldots,H(x_k))$" so that the quantifiers range over everything. Then by RED, relativize to W. This does not change the truth value. But this has the same meaning as "φ holds in $(V(OW),\in,x_1,\ldots,x_k)$.

We say that m is a complete measure on $V(\alpha+1)$ if and only if

i) $m:V(\alpha+1) \to \{0,1\}$;
ii) for all $x \in V(\alpha+1)$, if $|x| \leq 1$ then $m(x) = 0$;
iii) for all $x \in V(\alpha+1)$, $m(V(\alpha)\backslash x) = 1-m(x)$;
iv) if $x \in V(\alpha)$, $f:x \to V(\alpha+1)$, and for all $b \in x$, $m(f(b)) = 0$, then $m(\cup rng(f)) = 0$.

Now define $m:V(OW+1) \to \{0,1\}$ by $m(x) = 1$ if $V(OW) \in H(x)$; 0 otherwise.

LEMMA 4.31. The following is provable in $K_2(W)$. m is a complete measure on V(OW+1), and exists as a set.

Proof: We can construct m within $W^{\#\#}$ as an element of V(OW+9). Recall that we have External Separation for $W^{\#}\#$, which is needed because W is used in the definition of m (W is used in the definition of the correspondence H).

To verify condition ii), let x ⊆ V(OW) have at most 1 element. Then x ∈ V(OW) and so H(x) = x. Hence m(x) = 0.

To verify condition iii), let x ⊆ V(OW), and let y = V(OW)\x. Then “x,y partition V(OW).” Hence “H(x),H(y) partition $W^{\#\#}$.” Hence m(x) = 0 ↔ m(y) = 1 as required.

To verify condition iv), let x ∈ V(OW) and f:x → V(OW+1). Let z = {<b,u>: u ∈ f(b)}. We claim that H(z) = {<b,u>: u ∈ H(f(b))}. To see this, we first show that for any b ∈ x, (∀u ∈ $W^{\#\#}$)(<b,u> ∈ H(z) ↔ u ∈ H(f(b))). Fix b ∈ x. Then this is equivalent to (∀u ∈ V(OW))(<b,u> ∈ z ↔ u ∈ f(b)), which is clearly true. It now merely suffices to prove that every element of H(z) is an ordered pair whose first coordinate is in x. This follows from the fact that every element of z is an ordered pair whose first coordinate is in x.

We say that V(α) is strongly inaccessible if and only if

i) α is a limit ordinal > ω;
ii) every function from an element of V(α) into V(α) has range ∈ V(α).

LEMMA 4.32. The following is provable in $K_2(W)$. V(OW) is strongly inaccessible.

Proof: By Lemmas 4.25 and 4.28, OW is a limit ordinal > ω. Now let f:x → V(OW), x ∈ V(OW). Then f ⊆ V(OW), and also f ⊆ H(f) ⊆ $W^{\#\#}$. Now H(f) is a function with domain x since f is a function with domain x. Hence H(f) = f. But then f ∈ V(OW) by Lemma 4.30. Hence rng(f) ∈ V(OW). □

LEMMA 4.33. The following is provable in $K_2(W)$. For all α < OW there exists α < β < OW and a complete measure on V(β+1).

Proof: Let α < OW be the least ordinal such that this is false. Then OW is defined from α within $W^{\#\#}$ as the least γ such that there is a complete measure on V(γ+1). We can use RED since α and $W^{\#\#}$ lie in W. By RED, OW $\in$ W, which is a contradiction. ∎

It is convenient to use the following terminology: a measurable rank is a V(α) such that there is a complete measure on V(α+1).

THEOREM 4.34. The following is provable in K_2(W). There is a standard model of ZF + “there exists arbitrarily large measurable ranks.”

Proof: The standard model can be taken to be (V(OW),$\in$), according to Lemmas 4.10 and 4.11. ∎

THEOREM 4.35. ZFC + “there exists a nontrivial elementary embedding from a rank into a rank” proves the existence of a standard model of K_2(W).

Proof: Let j be an elementary embedding from V(κ+1) into V(γ+1) with critical point κ. Then κ and γ are strongly inaccessible. Let H(κ) be the set of all elements of transitive sets of cardinality at most κ, and let H(γ) be the set of all elements of transitive sets of cardinality at most γ. Now extend j uniquely to an elementary embedding j’:H(κ) $\rightarrow$ H(γ) with critical point κ. Then rng(j’) $\in$ H(γ).

We claim that (H(γ),$\in$,rng(j’)) satisfies K_2(W).

First we claim that the largest transitive subset of rng(j’) is V(κ). To see this, observe that if u is a transitive set and δ+1 < rk(u) then some element of u has rank δ+1. This is proved by transfinite induction on rk(u). Now obviously no element of rng(j')can have rank κ+1, for otherwise κ would be fixed by j’. Thus any transitive subset of rng(j’) must have rank at most κ+1; i.e., be a subset of V(κ).

Note that SS’ now is clear because every subset of V(κ) is the intersection of some element of rng(j’) with V(κ) - namely its image under j’.

For RED, let $(\exists y)\varphi(x_1,\ldots,x_n,y)$ hold in $(H(\gamma),\in)$, where $x_1,\ldots,x_n \in \mathrm{rng}(j')$ and φ is in $L(\in)$. Let $x_i = j'(u_i)$. By elementarity, $(\exists y)\varphi(u_1,\ldots,u_n,y)$ holds in $(H(\kappa),\in)$. Fix y. Then by isomorphism, $\varphi(j'u_1,\ldots,j'u_n,j'y)$ holds in $(H(\gamma),\in)$. I.e., $(\exists y \in \mathrm{rng}(j'))\varphi(x_1,\ldots,{}_{xn},y)$ holds in $(H(\gamma),\in)$.