BOOLEAN RELATION THEORY

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Chapter 1 Introduction to BRT
1.1. General Formulation

We begin with two Theorems at the heart of BRT = Boolean Relation Theory.

THIN SET THEOREM. Let $k \geq 1$ and $f:N^k \rightarrow N$. There exists an infinite set $A \subseteq N$ such that $f[A^k] \neq N$.

COMPLEMENTATION THEOREM. Let $k \geq 1$ and $f:N^k \rightarrow N$. Suppose that for all $x \in N^k$, $f(x) > \max(x)$. There exists an infinite set $A \subseteq N$ such that $f[A^k] = N\backslash A$.

Proof of TST: Let $f:N^k \rightarrow N$. Color every $x \in N^k$ with $f(x)$ if $f(x) \in [0,ot(k)]$; $ot(k)+1$ otherwise. By Ramsey's theorem, let $A \subseteq N$ be infinite, where the color of $x \in A^k$ depends only on the order type of x. Then the possible colors of $x \in A^k$ form a subset of $[0,ot(k)]$ of cardinality $\leq ot(k)$, together with, possibly $ot(k)+1$. Hence $f[A^k]$ does not even include $[0,ot(k)]$. QED

TST IMPLIES INFINITE RAMSEY THEOREM?

This is still wide open. We showed that TST is not provable in ACA_0.

SHARP THIN SET THEOREM. Let $k \geq 1$ and $f:N^k \to N$. There exists an infinite set $A \subseteq N$ such that $f[A^k]$ does not include $[0,ot(k)]$.

THEOREM. Sharp TST implies infinite Ramsey theorem.

Proof: Assume TST. Let a 2-coloring of strictly increasing $x \in N^k$ be given, with colors 0,1. Color $x \in N^k$ by the order type of x if x is not strictly increasing (as an integer in $[2,ot(k)]$); the color of x otherwise. QED

We proved that TST in dimension 2 is not provable in WKL_0. It is open whether TST in dimension 3 implies ACA_0.

An old result of ours is that infinite Ramsey theorem is equivalent to ACA′ over RCA_0.

$ACA' = ACA_0 + (\forall n,x)$(the n-th Turing jump of x exists).

COMPLEMENTATION THEOREMS

COMPLEMENTATION THEOREM. Let $k \geq 1$ and $f:N^k \to N$. Suppose that for all $x \in N^k$, $f(x) > \max(x)$. There exists an infinite set $A \subseteq N$ such that $f[A^k] = N \setminus A$.

In fact, A is unique. Build A by recursion. Suppose we have decided membership in A for all $0 \leq i < n$. Put n in A if and only if $n \notin f[A]$ up to now. These decisions are stable because f is strictly dominating.

Every for simple f, the unique A may be complicated.

For a study of this, it is better to use the upper image. Let $f:N^k \to Z$. Write fA for $f[A^k]$, and $f_<A$ for $\{f(x) > \max(x): x \in A^k\}$.

COMPLEMENTATION THEOREM'. Let $k \geq 1$ and $f:N^k \to Z$. There exists an infinite set $A \subseteq N$ such that $f_<A = N \setminus A$. A is unique.

What is the structure of A if f is integral affine? Integral polynomial?

BACK TO BOOLEAN RELATION THEORY

We now use suggestive notation to restate the Thin Set Theorem and the Complementation Theorem.

Let MF = {f: dom(f) is some N^k and rng(f) ⊆ N}.
Let SD = {f ∈ MF: (∀x)(f(x) > max(x))}.
Let INF = {A ⊆ N: A is infinite}.

THIN SET THEOREM. (∀f ∈ MF)(∃A ∈ INF)(fA ≠ N).

COMPLEMENTATION THEOREM. (∀f ∈ SD)(∃A ∈ INF)(fA = N\A).

Here fA is the forward imaging used throughout BRT: fA = {f(x): x ∈ A^k}.

Formally, a multivariate function is a pair (k,f), k ≥ 1, where dom(f) consists of k-tuples. fA is defined using k.

Usually the k is clear from the f. But no matter what f is, we can write (1,f), since everything is considered a 1-tuple.

BRT SETTINGS

A BRT setting is a pair (V,K), where V is a set of multivariate functions, and K is a set of sets.

For the purposes of Boolean algebra, we extract the "universal set" U from (V,K). It is the least set U such that

i) for all $A \in K$, $A \subseteq U$;
ii) for all $f \in V$, $fU \subseteq U$.

Note that U may or may not lie in K. Recall:

THIN SET THEOREM. $(\forall f \in MF)(\exists A \in INF)(fA \neq N)$.

COMPLEMENTATION THEOREM. $(\forall f \in SD)(\exists A \in INF)(fA = N\backslash A)$.

Thin Set Theorem: IBRT in A,fA on (MF,INF).
Complementation Theorem: EBRT in A,fA on (SD,INF).

$(\forall f \in V)(\exists A \in K)$(Boolean inequation in A,fA).
$(\forall f \in V)(\exists A \in K)$(Boolean equation in A,fA).

This is BRT with one function and one set.

BRT SETTINGS CONSIDERED

The book works with five main BRT settings.

(MF,INF) and (SD,INF) have already been defined.

EVSD = {f ∈ MF: f is eventually strictly dominating}.

ELG = {f ∈ MF: f is of expansive linear growth}.

ELG ∩ SD
SD ELG
EVSD
MF

$f:N^k \to N$ is in ELG iff there exists c,d > 1 such that

$$c\max(x) \leq f(x) \leq d\max(x)$$

for all but finitely many $x \in N^k$.

BRT FRAGMENTS ANALYZED
ONE FUNCTION AND ONE SET

EBRT in A,fA on (MF,INF), (SD,INF), (ELG,INF), (EVSD,INF), (ELG ∩ SD,INF).

IBRT in A,fA on (MF,INF), (SD,INF), (ELG,INF), (EVSD,INF), (ELG ∩ SD,INF).

Then more intense:

EBRT in A,fA,fU on (MF,INF), (SD,INF), (ELG,INF), (EVSD,INF), (ELG ∩ SD,INF).

IBRT in A,fA,fU on (MF,INF), (SD,INF), (ELG,INF), (EVSD,INF), (ELG ∩ SD,INF).

A,fA gives rise to 16 cases on each setting.
A,fA,fU gives rise to 256 cases on each setting.

The EBRT classifications are conducted within RCA_0. The IBRT classifications are conducted within ACA′, where the Thin Set Theorem can be proved.

ONE FUNCTION AND ONE SET

From general Boolean algebra considerations:
There are four "elementary inclusions", and then we consider all subsets, for a total of 16.

$$A \cap fA = \emptyset$$

$$A \cup fA = U$$

$$A \subseteq fA$$

$$fA \subseteq A$$

For A,fA,fU, we consider subsets of the pruned list

$$A \cap fA = \emptyset$$

$$A \cup fA = U$$

$$A \subseteq fA$$

$$fA \subseteq A$$

$$fU \subseteq A \cup fA$$

$$A \cap fU \subseteq fA$$

For EBRT, we analyze $(\forall f \in V)(\exists A \in K)$(conjunction).
For IBRT, we analyze $(\exists f \in V)(\forall A \in K)$(conjunction).

This is the convenient dual of $(\forall f \in V)(\exists A \in K)$ $(\neg$conjunction).

ONE FUNCTION AND ONE SET

$(\forall f \in MF)(\exists A \in INF)(A \cup fA \neq N)$. This is just a trivial variant of the Thin Set Theorem.

The following lemmas arise in the classification:

THEOREM 2.3.1. Let $f \in SD$ and $B \in INF$. There exists $A \in INF$, $A \subseteq B$, such that $A \cap fA = \emptyset$ and $B \subseteq A \cup fA$. Moreover, this is provable in RCA_0.

THEOREM 2.3.2. For all $f \in EVSD$ there exists $A \in INF$ such that $A \cap fA = \emptyset$, $A \cup fN = N$. Moreover, this is provable in RCA_0.

THEOREM 2.3.3. Let $k \geq 2$. There exists k-ary $f \in ELG \cap SD$ such that $N \backslash fN = \{0\}$. There exists k-ary $f \in ELG$ such that $fN = N$.

THEOREM 2.3.7. Let $f \in ELG[1]$. Then $N \backslash fN$ is infinite.

LEMMA 2.3.8. No element of EVSD[1] is surjective.

Here [1] means 1-ary.

EBRT IN A,B,fA,fB,$\subseteq$ ON (SD,INF)

We analyze $(\forall f \in V)(\exists A \in K)(A \subseteq B$ and conjunction).
Full classification is given, within RCA_0.

A serious challenge: do this without $A \subseteq B$.

There are nine elementary inclusions.

$$A \cap fA = \emptyset.$$

$$B \cup fB = N.$$

$$B \subseteq A \cup fB.$$

$$fB \subseteq B \cup fA.$$

$$A \subseteq fB.$$

$$B \cap fB \subseteq A \cup fA.$$

$$fA \subseteq B.$$

$$A \cap fB \subseteq fA.$$

$$B \cap fA \subseteq A.$$

All subsets of these nine: $2^9 = 512$.

We use a treelike methodology in the book.

EBRT IN A,B,fA,fB,⊆ ON (SD,INF)

Here is what comes up:

LEMMA 2.4.1. Let f ∈ SD. There exist infinite A ⊆ B ⊆ N such that B ∪. fA = N and A = B ∩ fB.

LEMMA 2.4.2. Let f ∈ SD. There exist infinite A ⊆ B ⊆ N such that A ∪. fB = N, fA ⊆ B, and B ∩ fB ⊆ fA.

LEMMA 2.4.3. Let f ∈ SD and X ⊆ N. There exists a unique A such that A ⊆ X ⊆ A ∪. fA.

LEMMA 2.4.4. The following is false. For all f ∈ ELG ∩ SD there exist infinite A ⊆ B ⊆ N such that A ∩ fB = ∅ and fB ⊆ B.

LEMMA 2.4.5. Let f ∈ SD. There is no nonempty A ⊆ N such that A ⊆ fA.

THEOREM 2.4.6. EBRT in A,B,fA,fB, on (SD,INF) is RCA_0 secure.

Here (SD,INF) and (ELG ∩ SD,INF) behave the same here.

EBRT IN A,B,fA,fB,⊆ ON (ELG,INF)

Same nine elementary inclusions. But classification takes on a different, asymptotic, character. Identical to (EVSD,INF). This only seen by carrying out classification. Without ⊆, are they the same (test problem)?

Here is what comes up:

LEMMA 2.5.3. Let $f:[0,n]^k \rightarrow [0,n]$ be partial, $n \geq 0$. There exist $A \subseteq B \subseteq [0,n]$ such that $A = [0,n]\backslash fB$ and $B = [0,n]\backslash fA$.

LEMMA 2.5.4. For all $f \in$ EVSD there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = A \cup fB = N$.

LEMMA 2.5.5. For all $f \in$ EVSD there exist infinite $A \subseteq B \subseteq N$ such that $A \cup. fB = N$ and $B \cap fA = \emptyset$.

LEMMA 2.5.6. There exists $f \in$ ELG such that $f^{-1}(0) = \{(0,...,0)\}$, $f(N\backslash\{0\}) \subseteq 2N+1$, and for all $A \subseteq N$ containing 0, $fA \cap 2N \subseteq A \rightarrow fA$ is cofinite.

EBRT IN A,B,fA,fB,$\subseteq$ ON (ELG,INF)

More comes up:

LEMMA 2.5.7. The following is false. For all $f \in$ ELG there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fB = \emptyset$, $B \cup fB = N$, and $fB \subseteq B \cup fA$.

LEMMA 2.5.8. The following is false. For all $f \in$ ELG there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = N$ and $A \cap fB = \emptyset$.

LEMMA 2.5.9. For all $f \in$ EVSD there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = N$ and $A \subseteq fB$.

LEMMA 2.5.10. The following is false. For all $f \in$ ELG there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fA = \emptyset$, $B \cup fB = N$, $B \cap fB \subseteq A \cup fA$.

LEMMA 2.5.11. For all $f \in$ EVSD there exist infinite $A \subseteq B \subseteq N$ such that $A \cup. fB = N$ and $fA \subseteq B$.

LEMMA 2.5.12. Let $f \in$ EVSD. There exist infinite $A \subseteq B \subseteq N$ such that $fB \subseteq B \cup. fA$ and $A = B \cap fB$.

EBRT IN A,B,fA,fB,⊆ ON (ELG,INF)

And yet more comes up:

LEMMA 2.5.13. Let f ∈ EVSD. There exist infinite A ⊆ N such that A ∩ f(A ∪ fA) = ∅.

LEMMA 2.5.14. Let f ∈ EVSD and let X ⊆ N, where min(X) is sufficiently large. There exists a unique A such that A ⊆ X ⊆ A ∪. fA. If X is infinite then A is infinite.

THEOREM 2.5.15. EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) have the same correct formats. EBRT in A,B,fA,fB, on (ELG,INF) and (EVSD,INF) are RCA_0 secure.

EBRT IN $A_1,...,A_k,fA_1,...,fA_k,\subseteq$ on (MF,INF)

Classifications in EBRT on (MF,INF) are substantially easier than on (SD,INF), (ELG ∩ SD,INF), (ELG,INF), and (EVSD,INF), at least under ⊆.

We have been able to completely analyze one function and k sets under ⊆ on (MF,INF).

Again, the classification is conducted in RCA_0, and we see that again every instance of this BRT fragment is provable or refutable in RCA_0.

We begin with a listing of the elementary fifteen convenient types of inclusions based on simple inequalities on the subscripts. Five of these are easily eliminated, leaving a sublist of ten. The conjunction of all of these is accepted.

Without ⊆, we have an incomparably more difficult challenge, which we have not examined.

IBRT IN $A_1,...,A_k,fA_1,...,fA_k,\subseteq$ on (MF,INF)

Here we give a complete classification of IBRT in $A_1,...,A_k,fA_1,...,fA_k$, on (SD,INF), (ELG ∩ SD,INF), (ELG,INF), (EVSD,INF), (MF,INF). We work entirely within RCA_0, except for the BRT setting (MF,INF), where we work within ACA'. (MF,INF) presents difficulties; the rest are trivial.

Start with the $A_1,...,A_k,fA_1,...,fA_k$, elementary inclusions, grouped as before based on simple inequalities of subscripts.

For each of these elementary inclusions, ρ, we will provide a useful description of the witness set for ρ, in this sense: The set of all f ∈ MF such that

$$(\forall A_1,...,A_k \in \text{INF})(A_1 \subseteq ... \subseteq A_k \rightarrow \rho).$$

We then calculate the witness sets for the sets of elementary inclusions by taking intersections.

The corresponding IBRT statement is true if and only if the witness set is empty. This is given a decision procedure by inspection.

This is a much greater challenge without $\subseteq$.

MORE BRT SETTINGS

We conjecture that the behavior of BRT fragments in BRT settings depends very delicately on the choice of BRT setting. Thus the expectation that BRT is a mathematically fruitful problem generator of unprecedented magnitude and scope.

Even within MF, we can place a large variety of growth conditions.

Another line is Topological BRT, where we use multivariate continuous functions on various topological spaces, with the family of open subsets.

We can use, e.g., the set of all bounded linear operators on L^2, with the set of all nontrivial closed subspaces of L^2. The invariant subspace problem becomes

$$(\forall f \in V)(\exists A \in K)(fA = A).$$

In section 1.2, an estimated 1,000,000 significantly different BRT settings are discussed.

The book barely scratches the surface of only 5 BRT settings within (MF,INF).

THIN SET THEOREMS

The Thin Set Theorem in (V,K) asserts

$$(\forall f \in V)(\exists A \in K)(fA \neq U)$$

where U is the universal set for (V,K).

FCN(ℜ,ℜ). All functions from ℜ to ℜ.
BFCN(ℜ,ℜ). All Borel functions from ℜ to ℜ.
CFCN(ℜ,ℜ). All continuous functions from ℜ to ℜ.
C^1FCN(ℜ,ℜ). All C^1 functions from ℜ to ℜ.
C^∞FCN(ℜ,ℜ). All C^∞ functions from ℜ to ℜ.
RAFCN(ℜ,ℜ). All real analytic functions from ℜ to ℜ.
SAFCN(ℜ,ℜ). All semialgebraic functions from ℜ to ℜ.
CSAFCN(ℜ,ℜ). All continuous semialgebraic functions from ℜ to ℜ.

cSUB(ℜ). All subsets of ℜ of cardinality c.
UNCLSUB(ℜ). All uncountable closed subsets of ℜ.
NOPSUB(ℜ). All nonempty open subsets of ℜ.
UNOPSUB(ℜ). All unbounded open subsets of ℜ.
DEOPSUB(ℜ). All open dense subsets of ℜ.
FMOPESUB(ℜ). All open subsets of ℜ of full measure.
CCOPSUB(ℜ). All open subsets of ℜ whose complement is countable.
FCSUB(ℜ). All subsets of ℜ whose complement is finite.
≤1CSUB(ℜ). All subsets of ℜ whose complement has at most one element.

We determine status of TST in all 72 BRT settings.

MORE THIN SET THEOREMS

We also consider the corresponding 8 families of multivariate functions from $\Re$ to $\Re$. We use the same 9 families of subsets of $\Re$.

FCN($\Re^*$,$\Re$). All multivariate functions from $\Re$ to $\Re$.
BFCN($\Re^*$,$\Re$). All multivariate Borel functions from $\Re$ to $\Re$.
CFCN($\Re^*$,$\Re$). All multivariate continuous functions from $\Re$ to $\Re$.
C^1FCN($\Re^*$,$\Re$). All multivariate C^1 functions from $\Re$ to $\Re$.
C^∞FCN($\Re^*$,$\Re$). All multivariate C^∞ functions from $\Re$ to $\Re$.
RAFCN($\Re^*$,$\Re$). All multivariate real analytic functions from $\Re$ to $\Re$.
SAFCN($\Re^*$,$\Re$). All multivariate semialgebraic functions from $\Re$ to $\Re$.
CSAFCN($\Re^*$,$\Re$). All multivariate continuous semialgebraic functions from $\Re$ to $\Re$.

We again determine the status of TST in all 72 BRT settings.

CONJECTURES

Consider EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF). I.e., Equational BRT in two functions and three sets on (ELG,INF).

The number of elementary inclusions is $2^9 = 512$, and the number of BRT statements is 2^{512}.

CONJECTURE. Every instance of EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF) is provable or refutable from standard large cardinals.

QUESTION. Is EBRT in infinitely many functions and infinitely many sets on (ELG,INF) algorithmically solvable?

THEOREM. There is an instance of EBRT in A,B,C,fA,fB,gB,gC,⊆ that is provably equivalent to the 1-consistency of SMAH over ACA′.

Even a complete analysis of EBRT in A,B,C,fA,fB,gB,gC,⊆ seems out of reach at this time.

SMAH = ZFC + {there exists a strongly k-Mahlo cardinal}$_k$. ACA′ = ACA_0 + $(n,x)(x^{(n)}$ exists).

6561 CASES OF EQUATIONAL BOOLEAN RELATION THEORY

We have made a complete analysis of a natural fragment of EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF), rich enough to include an Exotic Case.

We write A $\cup.$ B for A $\cup$ B if A,B are disjoint; undefined otherwise. (Disjoint union).

TEMPLATE. For all f,g $\in$ ELG there exist A,B,C $\in$ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT.$$

Here X,Y,V,W,P,R,S,T are among the three letters A,B,C. There are obviously $3^8 = 6561$ instances.

There is symmetry: permute the three letters, and permute the two clauses.

Thus most of the equivalence classes have cardinality 12.

6561 CASES OF EQUATIONAL BOOLEAN RELATION THEORY

TEMPLATE. For all f,g ∈ ELG there exist A,B,C ∈ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT.$$

With exactly 12 exceptions, all of the 6561 instances are provable or refutable in RCA_0. A myriad of ad hoc combinatorial arguments are used to establish this.

The 12 exceptions are called the Exotic Cases. Here is the Principal Exotic Case - the other 11 obtained by symmetry.

PRINCIPAL EXOTIC CASE. For all f,g ∈ ELG there exist A,B,C ∈ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

THEOREM. The Exotic Cases are each provably equivalent to 1-Con(SMAH) over ACA'.

THE BRT TRANSFER THEOREM

BRT TRANSFER. Let X,Y,V,W,P,R,S,T be among the letters A,B,C. The following are equivalent.
i. for all f,g ∈ ELG and n ≥ 1, there exist finite A,B,C ⊆ N, each with at least n elements, such that X ∪. fY ⊆ V ∪. gW, P ∪. fR ⊆ S ∪. gT.
ii. for all f,g ∈ ELG, there exist infinite A,B,C ⊆ N, such that X ∪. fY ⊆ V ∪. gW, P ∪. fR ⊆ S ∪. gT.

I.e., arbitrarily large finite ⇔ infinite.

THEOREM. BRT Transfer is provably equivalent to 1-Con(SMAH) over ACA'.

This is shown as follows. The entire classification of the Template with INF, other than the 12 Exotic Cases, is checked to verify that it remains unchanged if INF is replaced by "arbitrarily large finite".

The 12 Exotic Cases are proved in RCA0 with INF replaced by "arbitrarily large finite".

This reduces BRT Transfer to merely the truth of the 12 Exotic Cases. I.e., to 1-Con(SMAH) over ACA'.

PROOF OF PRINCIPAL EXOTIC CASE

We actually prove the PEC in a sharper form:

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that
i. for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii. $A_1 \cap fA_n = \emptyset$.

Let $f,g \in ELG$, $n \geq 1$. Let κ be strongly Mahlo of sufficiently high finite order.

Form $M = (N,<,0,1,+,f,g)$. First extend M to a countable structure

$$M^* = (N^*,<,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$$

generated by the atomic indiscernibles c_i^*, $i \in N$. This uses the infinite Ramsey theorem, infinitely iterated.

We now extend M^* transfinitely to M^{**}.

PROOF OF PRINCIPAL EXOTIC CASE

$M = (N,<,0,1,+,f,g)$.
$M^* = (N^*,<,0^*,1^*,+^*,f^*,g^*,c_0^*,...)$

Extend M^* to

$$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},c_0^{**},...,c^{**},...)$$

where the c^{**}'s are indexed by the large cardinal κ. We verify that any partial substructure of M^{**} boundedly generated by 0^{**}, 1^{**}, and a set of c^{**}'s of order type ω, is embeddable back into M^* and M.

We verify that $2x < y$ is a well founded relation in M^{**}, using that N^{**} is generated by the c's.

We then apply the Complementation Theorem for well founded relations to obtain a unique set W of nonstandard elements of M^{**} such that for all nonstandard x in M^{**},

$$x \in W \Leftrightarrow x \notin g^{**}W.$$

There is a natural Skolem hull construction consisting entirely of elements of W. Start with the set of all c^{**}'s. Witnesses are thrown in from W that verify that values of f^{**} at elements thrown in at previous stages do not lie in W (if true). Only the first n stages of the construction will be used.

PROOF OF PRINCIPAL EXOTIC CASE

$M = (N,<,0,1,+,f,g)$.
$M^* = (N^*,<,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$
$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},c_0^{**},\ldots,c^{**},\ldots)$
For all nonstandard x in M**, $x \in W \Leftrightarrow x \notin g^{**}W$.

Skolem hull construction for n steps. Start with the set of all c**'s. Witnesses are thrown in from W that verify that values of f** at elements thrown in at previous stages do not lie in W (if true).

Let S be an appropriate set of indiscernibles of order type ω, using combinatorial properties of k. Redo the Skolem hull construction starting only with the c**'s whose indices lie in S.

By the indiscernibility, the n stage construction yields a subset of N** where the subscripts of all c**'s involved form a set of order type ω. Also, the c**'s in the first stage are not values of f** (or g**).

Transfer this length n tower back to M* and back to M. The result is the tower that verifies the PEC (in the sharper form of Proposition B).

REVERSAL OF PRINCIPAL EXOTIC CASE

We reverse PEC in more concrete form involving some simple functions. BAF (basic functions) are given by terms in 0,1,+,−,•,↑,log. These are functions from N (or N^2) into N.

1. 0,1,+,• are as usual.
2. x-y is usual if x ≥ y; 0 otherwise.
3. ↑ is base 2 exponentiation.
4. log(x) is the floor of the usual base 2 logarithm, with log(0) = 0.

BAF is closed under definition by cases using <.

PROPOSITION C. For all f,g ∈ ELG ∩ SD ∩ BAF, there exist A,B,C ∈ INF such that

$$A \,\dot{\cup}\, fA \subseteq C \,\dot{\cup}\, gB$$

$$A \,\dot{\cup}\, fB \subseteq C \,\dot{\cup}\, gC.$$

We construct, using C, a model of SMAH. The construction takes place in ACA'.

PEC, and hence Proposition C, is true in the arithmetic sets. This is by examining the proof.

QUESTION: Is Proposition C (or even A) true in the recursive sets?

PROPOSITION C AND LENGTH 3 TOWERS

We start with

PROPOSITION C. For all $f,g \in ELG \cap SD \cap BAF$, there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

Note that Proposition C does not tell us that $A \subseteq B \subseteq C$. This is a very important condition to have, as we want to extend length 3 chains to chains of arbitrary finite length, and then apply compactness to get a single structure.

So in section 5.1, we obtain a technical, weak form of Proposition C with $A \subseteq B \subseteq C$. This appears as Lemma 5.1.7.

FROM LENGTH 3 TOWERS TO LENGTH n TOWERS

In section 5.2, we obtain a variant of Lemma 5.1.7 involving length n towers rather than length 3 towers.

However, we have to pay a serious cost. As opposed to Lemma 5.1.7, we will only have that the sets in the length n towers have at least r elements, for any given $r \geq 1$.

So it is important to make sure that the first sets in these towers be a suitable set of indiscernibles before we relinquish that the first sets be infinite.

In order to accomplish this, we first apply the infinite Ramsey theorem to shrink the infinite first sets coming from Lemma 5.1.7 to infinite subsets are sets of indiscernibles of the right kind.

The indiscernibility has the flavor: given two r tuples of the same order type and the same min, they have the same atomic properties if we use the same parameters $\leq$ the common min.

COUNTABLE NONSTANDARD MODELS WITH LIMITED INDISCERNIBLES

Our basic standard structure is $(N,<,0,1,+,-,\cdot,\uparrow,\log)$ that provides the operations that generate BAF.

In section 5.3, we use Lemma 5.2.12 to create, for each $r \geq 3$, a structure $(N,<,0,1,+,-,\cdot,\uparrow,\log,E_1,...,E_r)$ with a related set of properties. This is Lemma 5.3.2, which frees us from any further consideration of BAF. Here $E_1 \subseteq ... \subseteq E_r$.

The next major step is to consolidate all of the structures given by Lemma 5.3.2 relative to each $r \geq 3$, to a single countable nonstandard structure based on a single tower $E_1 \subseteq E_2 \subseteq ...$ of infinite sets of infinite length.

This consolidation is accomplished first by using the compactness theorem. Then by taking only the first ω elements of E_1, and restricting to the associated cut.

After some development of the structure, we take E to be the union of the E's, and take $c_1,c_2,...$ to be the enumeration of the old E_1.

LIMITED FORMULAS, LIMITED INDISCERNIBLES, x-DEFINABILITY, NORMAL FORM - SNAPSHOT

LEMMA 5.4.17. There exists a countable structure $M = (A,<,0,1,+,-,\bullet,,\log,E,c_1,c_2,\ldots)$, and terms $t_1,t_2,\ldots$ of L, where for all i, t_i has variables among $v_1,\ldots,v_{i+8}$, such that the following holds.

i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0{}_1,L)$;

ii) $E \subseteq A\backslash\{0\}$;

iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;

iv) Let $r,n \geq 1$ and $t(v_1,\ldots,v_r)$ be a term of L, and $x_1,\ldots,x_r \leq c_n$. Then $t(x_1,\ldots,x_r) < c_{n+1}$;

v) $2\alpha(E;1,<\infty)+1,\ 3(E;1,<\infty)+1 \subseteq E$;

vi) Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. There exists $y_1,\ldots,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,\ldots,x_k) \in E^k \cap [0,c_n]^k$: $t_k(x_1,\ldots,x_k,y_1,\ldots,y_8) \in E\}$;

vii) Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula of L(E). Let $1 \leq i_1,\ldots,i_{2r} < n$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and the same min. Let $y_1,\ldots,y_r \in E$, $y_1,\ldots,y_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then

$\varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)^{c\_n} \Leftrightarrow$

$\varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)^{c\_n}$.

COMPREHENSION, INDISCERNIBLES

In section 5.5, we upgrade the bounded quantifier comprehension and indiscernibility to unbounded quantifier comprehension and indiscernibility. It is the indiscernibility itself that allows us to make this transition.

The comprehension produces bounded relations on E only.

A very robust and useful notion of internal relation emerges. These are the bounded relations on E that are definable with parameters from E and quantifiers ranging over E.

We pass to a second order structure where the internal relations are used to interpret the second order quantifiers.

We retain comprehension and indiscernibility in the appropriate forms.

Π^0_1 CORRECT INTERNAL ARITHMETIC, SIMPLIFICATION

In section 5.6, we derive a suitable form of the axiom of infinity. The axiom of infinity takes the form of the existence of an internal set containing 1, and closed under $+2c_1$.

We then define I to be the intersection of all internal sets containing 1, and closed under $+2c_1$. The set I will serve as the internal natural numbers.

It is important to link the arithmetic operations that are uniquely defined, internally, on I, with the arithmetic operations given by our complicated second order structure M*. This is required in order to be able to use the fact that M* satisfies the true Π^0_1 sentences. It allows us to conclude that the internal arithmetic on I satisfies the true Π^0_1 sentences.

We then pass to a convenient linearly ordered set theory K(Π). Thus section 5.6 closes with the following:

LINEARLY ORDERED SET THEORY

LEMMA 5.6.20. There exists a countable structure M# = $(D,<,,NAT,0,1,+,-,\bullet,\uparrow,\log,d_1,d_2,\ldots)$ such that

i) < is a linear ordering (irreflexive, transitive, connected);

ii) $x \in y \rightarrow x < y$;

iii) The d_n, $n \geq 1$, form a strictly increasing sequence of elements of D with no upper bound in D;

iv) Let φ be a formula of L# in which v_1 is not free. Then $(\exists v_1)(\forall v_2)(v_2 \in v_1 \Leftrightarrow (v_2 \leq v_3 \wedge \varphi))$;

v) Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula of L#. Let $1 \leq i_1,\ldots,i_{2r}$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. Let $y_1,\ldots,y_r \leq \min(d_{i_1},\ldots,d_{i_r})$. Then $\varphi(d_{i_1},\ldots,d_{i_r},y_1,\ldots,y_r) \Leftrightarrow$

$\varphi(d_{i_{r+1}},\ldots,d_{i_{2r}},y_1,\ldots,y_r)$;

vi) NAT defines a nonempty initial segment under <, with no greatest element, and no limit point, where all points are $< d_1$, and whose first two elements are 0,1, respectively;

vii) $(\forall x)$(if x has an element obeying NAT then x has a < least element);

viii) Let $\varphi \in TR(\Pi^0_1,L)$. The relativization of φ to NAT holds.

ix) $+,-,\bullet,\uparrow,\log$ have the default value 0 in case one or more arguments lie outside NAT.

TRANSFINITE INDUCTION, COMPREHENSION, INDISCERNIBLES, INFINITY, Π^0_1 CORRECTNESS

In M#, the < may not be internally well ordered. Moreover, we may not have extensionality.

In section 5.7, we create a structure like M# but with an internally well founded <. This is not a model of a set theory, but rather a second order structure. I.e., we will have a linearly ordered set of points, with a family of relations on the points, of each arity.

We have bounded comprehension, indiscernibility, infinity, well orderedness.

The idea is to use equivalence classes of well founded relations under isomorphism.

ZFC + V = L, INDISCERNIBLES, AND Π^0_1 CORRECT ARITHMETIC

We have a second order structure M^. In section 5.8, we move back to a model of set theory. This time, the model will be of ZFC + V = L + the true Π^0_1 sentences, with an unbounded infinite sequence of ordinals with indiscernibility.

We need to build the constructible hierarchy in order to fully utilize our indiscernibility. In particular, the definable well ordering arising from L is needed in order to derive power set from indiscernibility.

Because of the internal well foundedness, the points in M^ already behave like ordinals. In M^, we can perform various transfinite recursions, resulting in second objects in M^. Sometimes in order to accomplish this, we need to make use of the indiscernibles in M^.

ZFC + V = L + $(\exists\kappa)(\kappa$ IS STRONGLY k-MAHLO$)_k$ + TR(Π^0_1, L), AND 1-CON(SMAH)

James Schmerl in his thesis with Jack Silver was the first to establish the connection between the kind of indiscernibility we are using and strongly Mahlo cardinals of finite order. If we do not require the indiscernibles to be infinite ordinals, then this is just the indiscernibility in Paris/Harrington.

In fact, we need only consider finite sets of infinite indiscernibles.

We have, in essence, indiscernibles for all suitable light faced partitions. This can be turned into the existence of the same kind of indiscernibles for any partition, by taking the constructibly least partition without such indiscernibles.

Thus our model of ZFC + V = L has the large cardinals in it that we are looking for.

QED