LECTURE NOTES ON BABY BOOLEAN RELATION THEORY*

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Abstract. This is an introduction to the most primitive form of the new Boolean relation theory, where we work with only one function and one set. We give eight complete classifications. The thin set theorem (along with a slight variant), and the complementation theorem are the only substantial cases that arise in these classifications.

INTRODUCTION.

Let f be a multivariate function of arity k. Let A be a set. We write fA for $f[A^k]$.

Z = set of all integers, N = set of all nonnegative integers.

THEOREM. Let f be a multivariate function from Z into Z. There exists an infinite $A \subseteq Z$ such that ????.

QUIZ: Find four common mathematical symbols so that this is true and highly nontrivial.

Here is the answer to the quiz.

THIN SET THEOREM. Let f be a multivariate function from Z into Z. There exists an infinite $A \subseteq Z$ such that $fA \neq Z$.

DIGRESSION: TST is provable in $ACA_0' = RCA_0$ + "for all x,n, the n-th jump of x exists". We have shown that TST is not provable in ACA_0. See [FS00] or the proof below, which is better. It is open whether RCA_0 proves TST $\rightarrow$ ACA_0', or even whether RCA_0 proves TST $\rightarrow$ ACA_0. RCA_0 cannot prove TST $\rightarrow$ ACA_0 since ACA_0 is not finitely axiomatizable. We have also shown that TST for k = 2 is not provable in WKL_0. See [CGH00].

Proof of TST: Use RT = Ramsey's theorem. Wlog, replace Z by N. Let p be the number of order types of k-tuples. Set H(x) = f(x) if $f(x) \leq p$; p+1 otherwise. Let $A \subseteq N$ be infinite and H-homogeneous in the sense that the value of H at tuples from A depend only on their order types. Then the values of f at tuples of any given order type from A are either all the same number □ p, or all > p. In any event, at least one of the numbers {0,...,p} is omitted by f on A. QED

We can immediately ask what else can be put there other than the Boolean inequation fA ≠ Z. Think of Z as the universal set.

This is what BRT = Boolean relation theory is all about. In these lecture notes, we only consider "baby" BRT, where we have just one function and one set.

1. SOME BOOLEAN ALGEBRA.

The notion of Boolean algebra is a very robust concept with several different definitions. We give a particularly elegant definition.

By way of motivation, the clearest examples of Boolean algebras are the Boolean fields of sets. These are structures (W,∅,S,□,□,'), where W is a family of subsets of the set S closed under pairwise union, pairwise intersection, and complement relative to S. Here ' is complementation relative to S.

We let **2** be the structure ({0,1},0,1,+,•,-), where x+y = min(1,x+y), x•y is multiplication, and -x = 1-x.

A Boolean algebra is a structure B = (B,0,1,+,•,-), where 0,1 □ B, +,•:B2 □ B, -:B □ B, such that every equation that holds universally in **2** also holds universally in B.

THEOREM 1.1. (Stone's Representation Theorem). Every Boolean algebra is isomorphic to a field of sets.

The idea is to use the family of all ultrafilters on the Boolean algebra as the field of sets.

THEOREM 1.2. Let s = t be a Boolean equation. The following are equivalent.
i) s = t is universally true in all Boolean algebras;
ii) s = t is universally true in **2**;
iii) s = t becomes a tautology in the usual classical propositional calculus with 0 replaced by T, 1 replaced by F, + replaced by , • replaced by □, and = replaced by □ .

We say that s,t are Boolean equivalent if and only if s = t holds universally in all Boolean algebras.

We say that s,t are **2** equivalent if and only if s = t holds universally in **2**.

We say that two Boolean equations s = t and s′ = t′ are Boolean equivalent if and only if

s = t iff s′ = t′

holds universally in all Boolean algebras.

We say that two Boolean equations s = t and s′ = t′ are **2** equivalent if and only if

s = t iff s′ = t′

holds universally in **2**.

THEOREM 1.3. Let s,t,s′,t′ be Boolean terms. The following are equivalent.
i) s = t and s′ = t′ are Boolean equivalent;
ii) s = t and s′ = t′ are **2** equivalent;
iii) (-s + t)•(s + -t),(-s′ + t′)•(s′ + -t′) are Boolean equivalent.

Proof: Suppose iii). Let B be a Boolean algebra with an assignment to variables making s = t. Then (-s + t)•(s + -t) = 1, and so (-s′ + t′)•(s′ + -t′) = 1. Hence s′ = t′. This establishes i).

Obviously i) $\Rightarrow$ ii).

Suppose ii). We want to derive iii). Suppose iii) is false. Then (-s + t)•(s + -t),(-s′ + t′)•(s′ + -t′) are not {0,1} equivalent. So there is a **2** assignment which makes the first term 1 and the second term 0 (the other possibility is handled analogously). Hence this assignment makes (-s + t) = 1, (s + -t) = 1, and also makes (-s′ + t′) = 0 or (s′ + -t′) = 0. Therefore the assignment makes s = t and s′ ≠ t′. This contradicts ii). QED

COROLLARY 1.4. Let s,t be Boolean terms. Then s,t are Boolean equivalent if and only if s = 1, t = 1 are Boolean equivalent.

Proof: By Theorem 1.3. QED

THEOREM 1.5. The number of Boolean terms in $x_1,\ldots,x_n$ up to Boolean equivalence is the same as the number of Boolean equations in $x_1,\ldots,x_n$ up to Boolean equivalence, which is 2^2^n.

Proof: By Theorem 1.2, the first quantity is the same as the number of propositional formulas in n propositional letters up to tautological equivalence, which is well known to be 2^2^n. By Theorem 1.3, the second quantity is no greater than the first quantity. It follows from Corollary 1.4, that it is no smaller, either. QED

THEOREM 1.6. Any finite set of Boolean equations, interpreted conjunctively, is Boolean equivalent to a single Boolean equation.

Proof: We can write the equations in the form $s_1 = 1, \ldots , s_n = 1$. And then we can write this as $s_1 \bullet \ldots \bullet s_n = 1$. QED

It will be very useful to have a normal form for Boolean equations. We give what amounts to conjunctive normal form.

We will now assume that the Boolean algebra is given as a field of sets. This is the situation throughout Boolean relation theory.

We now use □ for •, □ for +, ∅ for 0, and U for 1. We can use □ with its usual meaning between sets.

A basic inclusion is an inclusion of the form

y_1 □ ... □ y_p □ z_1 □ ... □ z_q

where $p,q \geq 0$, and $y_1,\ldots,y_p,z_1,\ldots,z_q$ consists of p+q distinct variables.

If p = 0 and q > 0 then we write this basic inclusion as

z_1 □ ... □ z_q = U.

If q = 0 and p > 0 then we write this basic inclusion as

$$y_1 \cap \ldots \cap y_p = \emptyset.$$

If p = q = 0 then we write this basic inclusion as

$$U = \emptyset$$

THEOREM 1.7. Let Y be a finite set of Boolean equations. Then there is a finite set E of basic inclusions such that Y is equivalent to E in all fields of sets.

Proof: By Theorem 1.6, we can assume that Y consists of a single equation. We can write this equation in the form s = 1. We can move over to propositional calculus notation and put s into conjunctive normal form; i.e., a conjunction of disjunctions of variables and negated variables. Each of the conjuncts corresponds to a basic inclusion. There are the exceptional cases involving empty conjunctions or empty disjunctions, which are handled with empty E, or using U = ∅. QED

2. BABY BOOLEAN RELATION THEORY.

In "baby" BRT, or unary BRT, we start with a pair V,K, where V is a set of multivariate functions and K is a set of sets. To avoid ambiguities, a multivariate function is a pair (f,k), $k \geq 1$, where f is a function whose domain is of the form E^k.

All brands of BRT are based on forward imaging. Let f be a multivariate function and A be a set. Define

$$fA = \{f(x_1,\ldots,x_k): x_1,\ldots,x_k \in A\}$$

where the arity of f is k.

In equational unary BRT, we seek to analyze all statements of the form

"for all f ∈ V there exists A ∈ K such that a given Boolean equation holds of A,fA".

Here a Boolean equation is an equation between Boolean terms in the two letters A,fA. Thus fA is treated in BRT as a Boolean variable.

But what is the relevant field of sets used to interpret these Boolean equations?

We take the field of sets to be the power set of U, where U is the union of the ranges of f and the elements of K.

In typical situations, we have $U = \cup K$.

In inequational unary BRT, we seek to analyze all statements of the form

"for all $f \in V$ there exists $A \in K$ such that a given Boolean inequation holds of A,fA".

Here a Boolean inequation is just the denial of a Boolean equation.

THEOREM 2.1. There are exactly 16 Boolean equations in unary BRT.

Proof: Immediate from Theorem 1.5. QED

3. UNARY INEQUATIONAL BRT ON (MF(Z),INF(Z)).

MF(Z) is the set of all multivariate functions from Z into Z. INF(Z) is the set of all infinite subsets of Z.

Note that TST from the Introduction is an instance of unary inequational BRT on (MF(Z),INF(Z)).

We now analyze all 16 instances.

The first step is to analyze all of the basic inclusions.

Then one considers all sets of basic inclusions. This exponentiates, but in the inequational theory, if a set of basic equations is accepted, then all supersets are accepted.

Of course, in equational BRT, if a set of basic equations is accepted, then all subsets are accepted.

In practice, this makes many classifications manageable.

However, in the unary situation we are working in, there are only 16 things to look at anyways, so we could do things by brute force without even using basic inclusions. But this would be uninformative, and certainly will not be manageable when going to more functions and sets.

Here is the permanent set of basic inclusions. There is never a need to consider the degenerate inclusion U = ∅.

In our context, the universal set U is simply Z.

A = Z
A = ∅
fA = Z
fA = ∅
A □ fA
fA □ A
A □ fA = ∅
A □ fA = Z

We now mark these yes or no, according to the truth value of □f□A□□(A,fA), where □ is in the above list of basic inclusions.

A = Z yes. TST.
A = ∅ yes.
fA = Z yes.
fA = ∅ yes.
A □ fA no. f(x) = x.
fA □ A no f(x) = x.
A □ fA = ∅. yes
A □ fA = Z yes. Slight sharpening of TST.

As remarked earlier, we need not consider any set of these that includes at least one that is marked yes, because such a set will automatically be marked yes. I.e., we need only consider nonempty sets of these that are marked no.

So we need only consider subsets of

A □ fA
fA □ A

The two together are no because of f(x) = x.

Obviously if a set is rejected, then so is any subset. So we are done.

4. UNARY EQUATIONAL BRT ON (MF(Z),INF(Z)).

Again the basic inclusions. Here we are analyzing
□f□A□(A,fA).

A = Z yes.
A = ∅ no.
fA = Z no.
fA = ∅ no.
A □ fA no. $f(x) = x^2 + 1$.
fA □ A yes.
A □ fA = ∅ no.
A □ fA = Z yes.

Any set containing an element that is rejected is rejected. So we only have to consider subsets of those that have been accepted. I.e., subsets of

A = Z
fA □ A
A □ fA = Z

Any subset is yes because all three at once are yes.

5. UNARY EQUATIONAL BRT ON (MF(Z),BINF(Z)).

BINF(Z) is the set of all bi-infinite subsets of Z. These are the subsets of Z with infinitely many positive and infinitely many negative elements.

Bi-infinite subsets of Z are not only natural but turn out to be important for BRT. Let’s see how BINF affects the theory.

Again, our basic inclusions. We are analyzing □f□A□(A,fA).

A = Z yes.
A = ∅ no.
fA = Z no.
fA = ∅ no.
A □ fA no. $f(x) = x^2 + 1$.
fA □ A yes.
A □ fA = ∅ no.
A □ fA = Z yes.

No change from section 4.

6. UNARY INEQUATIONAL BRT ON (MF(Z),BINF(Z)).

What about the inequational theory?

A = Z yes.
A = ∅ yes.
fA = Z yes.
fA = ∅ yes.
A □ fA no. f(x) = x.
fA □ A no. f(x) = x.
A □ fA = ∅. yes.
A □ fA = Z. yes. slight sharpening of TST.

Again no change from section 4, because we can get a bi-infinite set in TST and its sharpening.

7. UNARY EQUATIONAL BRT ON (SD(Z),INF(Z)).

A multivariate f from Z into Z is strictly dominating iff we have |f(x)| > |x|, where | | is the sup norm. SD(Z) is the set of all strictly dominating multivariate functions from Z into Z.

A = Z yes.
A = ∅ no.
fA = Z no.
fA = ∅ no.
A □ fA no. $f(x) = x^2 + 1$.
fA □ A yes.
A □ fA = ∅ yes.
A □ fA = Z yes.

Here we have a change from sections 4,5 at fA □ A.

A = Z
fA □ A
A □ fA = ∅
A □ fA = Z

The first, second, and fourth of these are jointly accepted. So we need only to consider the subsets of these four that have A □ fA = ∅. Reject with 1, and reject with 2. It remains to look at {A □ fA = ∅, A □ fA = Z}.

COMPLEMENTATION THEOREM. Let f be a strictly dominating multivariate function from Z into Z. There exists infinite A □ Z such that fA = Z\A. There is a unique A □ Z such that fA = Z\A.

Proof: Let n ≥ 0 and suppose that for all |m| < n, it has been determined whether m □ A. Then put n in A if and only if n does not lie in the forward image of f on the numbers already thrown into A. Done. Uniqueness from the fact that we had no leeway in the construction. QED

8. UNARY INEQUATIONAL BRT ON (SD(Z),INF(Z)).

A = Z yes.
A = ∅ yes.
fA = Z yes.
fA = ∅ yes.
A □ fA yes.
fA □ A yes.
A □ fA = ∅. yes.
A □ fA = Z. yes. slight sharpening of TST.

Different at A □ fA and fA □ A from section 3. Any set containing an accepted one is automatically accepted, and so we do not have to look further.

9. UNARY EQUATIONAL BRT ON (SD(Z),BINF(Z)).

A = Z yes.
A = ∅ no.
fA = Z no.
fA = ∅ no.
A □ fA no. $f(x) = x^2 + 1$.
fA □ A yes.
A □ fA = ∅ yes.
A □ fA = Z yes.

No change from section 7 so far. We need only look at the subsets of the accepted ones.

A = Z
fA □ A
A □ fA = ∅
A □ fA = Z

As in section 7, the first, second, and fourth of these are jointly accepted. So we need only to consider the subsets of these four that have A □ fA = ∅. Reject with 1, and reject with 2. It remains to look at {A □ fA = ∅, A □ fA = Z}.

THEOREM 9.1. In the Complementation Theorem, we cannot sharpen "infinite" to "bi-infinite".

Proof: Counterexample in one dimension. Set $f(x) = |x|+1$. Let fA = Z\A. Then A includes all negative integers. Hence A excludes all integers ≥ 2. So A is not bi-infinite. QED

Exercise: Give a counterexample for f obeying $|f(x)| > |x|^2$. How many dimensions do you need? What about functions polynomially bounded, not polynomially bounded, etcetera?

10. UNARY INEQUATIONAL BRT ON (SD(Z),BINF(Z)).

A = Z yes.
A = ∅ yes.
fA = Z yes.
fA = ∅ yes.
A □ fA yes.
fA □ A yes.
A □ fA = ∅. yes.
A □ fA = Z. yes. slight sharpening of TST.

Same as section 8.

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