BOOLEAN RELATION THEORY NOTES

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Abstract. We give a detailed extended abstract reflecting what we know about Boolean relation theory. We follow this by a proof sketch of the main instances of Boolean relation theory, from Mahlo cardinals of finite order, starting at section 19. The proof sketch has been used in lectures.

1. MULTIVARIATE FUNCTIONS.

Boolean relation theory (BRT) concerns the relationship between sets and their images under multivariate functions.

To avoid ambiguities, we officially define a multivariate function to be a pair f = (g,k), where $k \geq 1$ (the arity of f) and dom(g) is a set of ordered k-tuples.

We take dom(f) to be dom(g), and write $f(x_1,...,x_k) = g(<x_1,...,x_k>)$.

In practice, we will not be so careful about multivariate functions.

BRT is based on the following CRUCIAL notion of forward image. Let f be a multivariate function and A be a set. $fA = \{f(x_1,...,x_k)$: k is the arity of f and $x_1,...,x_k \in A\}$.

We could have written $f[A^k]$, but want to suppress the arity and write fA. In this way, f defines a special kind of operator from sets to sets.

We say that f is a multivariate function from A into B if and only if f is a multivariate function with $dom(f) = A^k$ and $rng(f) \subseteq B$, where f has arity k.

2. EQUATIONAL, INEQUATIONAL, PROPOSITIONAL, BOOLEAN RELATION THEORY.

In BRT, we will use set variables $A_1,A_2,...$, and function variables $f_1,f_2,...$.

The BRT atoms consist of $\emptyset,U,A_i$, and f_iA_j, for any $i,j \geq 1$.

BRT expressions are defined as the least set containing the BRT atoms, where if s,t are BRT expressions then so are $s \cup t$, $s \cap t$, s'.

The idea here is that U is the universal set, and s' is U\s.

A BRT equation is s = t, where s,t are BRT expressions. A Boolean inequation is $s \neq t$, where s,t are BRT expressions.

Let V be a set of multivariate functions and K be a set of sets. Let $n,m \geq 1$.

Equational (inequational, propositional) BRT on (V,K) of type (n,m) analyzes statements of the form:

for all $f_1,...,f_n \in V$, there exists $A_1,...,A_m \in K$, such that a given BRT equation (BRT inequation, propositional combination of BRT equations) holds among the sets and their images under the functions.

More formally,

for all $f_1,...,f_n \in V$, there exists $A_1,...,A_m \in K$, such that a given BRT equation (BRT inequation, propositional combination of BRT equations) holds,

where we require that in the given BRT equation (BRT inequation, propositional combination of BRT equations), all BRT atoms that appear are among the m(n+1) BRT atoms

$A_1,...,A_m$
$f_1A_1,...,f_1A_m$
...
$f_nA_1,...,f_nA_m$.

For this purpose, we take the universal set U to be the union of the ranges of the elements of V and the elements of K. Most typically, this will simply be the largest element of K (which is not guaranteed to exist).

BRT of type (1,1) is called unary BRT. Here we just have one function and one set. Even equational BRT on (V,K) of type (1,1) can be deep. See section 18 below, and [Fr01], which scratches the surface of unary BRT.

There are 2^2^m(n+1) statements of equational (inequational) BRT on (V,K) of type (n,m), up to Boolean equivalence. There are 2^2^2^2^m(n+1) statements of propositional BRT on (V,K) of type (n,m) up to Boolean equivalence.

Any finite conjunction of BRT equations is Boolean equivalent to a single BRT equation, and inclusions between BRT terms are Boolean equivalent to a BRT equation.

3. EXTENDED EQUATIONAL, INEQUATIONAL, PROPOSITIONAL, BOOLEAN RELATION THEORY.

Extended BRT expressions are inductively defined:

i) BRT atoms are extended BRT expressions;
ii) if s,t are extended BRT expressions then so are $s \cup t$, $s \cap t$, s', $f_i s$.

An extended BRT equation is s = t, where s,t are extended BRT expressions.

An extended BRT inequation is s ≠ t, where s,t are extended Boolean expressions.

Let V be a set of multivariate functions and K be a set of sets. Let n,m ≥ 1.

Extended equational (inequational, propositional) BRT on (V,K) of type (n,m) analyzes statements

for all $f_1,...,f_n \in V$, there exists $A_1,...,A_m \in K$, such that a given extended BRT equation (BRT inequation, propositional combination of BRT equations) holds in the functions $f_1,...,f_n$ and the sets $A_1,...,A_m$.

More formally,

for all $f_1,...,f_n \in V$, there exists $A_1,...,A_m \in K$, such that a given BRT equation (BRT inequation, propositional combination of BRT equations) holds, where we require that all set variables are among $A_1,...,A_m$ and all function variables are among $f_1,...,f_n$.

There are infinitely many statements even in extended equational BRT on (V,K) of type (1,1). However, we can bound the number of occurrences of functions in the BRT atoms that appear, in order to create an appropriate hierarchy.

4. THE THIN SET THEOREM.

Z is the set of all integers. Let MF(**Z**) be the set of all functions f whose domain is a Cartesian power of Z and whose range is a subset of Z. Let INF(Z) be the set of all infinite subsets of Z.

THIN SET THEOREM. Every integral multivariate function omits a value over some infinite set.

More formally,

THIN SET THEOREM. $(\forall f \in MF(Z))(\exists A \in INF(Z))(fA \neq \mathbf{Z})$.

Obviously, TST (thin set theorem) is an instance of BRT on (MF(Z),INF(Z)) of type (1,1). TST is provable in $ACA_0' = RCA_0 + (\forall x,n)$(the n-th jump of x exists). We have shown that TST is not provable in ACA_0. See [FS00], p 139. It is open whether RCA_0 proves TST $\rightarrow$ ACA_0', or even whether RCA_0 proves TST $\rightarrow$ ACA_0. RCA_0 cannot prove TST $\leftrightarrow$ ACA_0 since ACA_0 is not finitely axiomatizable. We have also shown that TST for k = 2 is not provable in WKL_0. See [CGH00].

5. THE COMPLEMENTATION THEOREM.

For $x \in \mathbf{Z}^k$, $|x|$ is the sup norm of x. We say that $f \in MF(Z)$ is strictly dominating if and only if for all $x \in dom(f)$,

$$|f(x)| > |x|.$$

We write SD(**Z**) for the set of all strictly dominating integral multivariate functions.

The complementation theorem is provable in RCA_0, and is, in a sense, complete for bounded recursion.

COMPLEMENTATION THEOREM. $(\forall f \in SD(\mathbf{Z}))(\exists A \in INF(\mathbf{Z}))(fA = \mathbf{Z}\backslash A)$. $(\forall f \in SD(\mathbf{Z}))(\exists! A \subseteq \mathbf{Z})(fA = \mathbf{Z}\backslash A)$.

Obviously, the first statement is an instance of BRT on (SD(Z),INF(Z)) of type (1,1).

The structure of such unique fixpoints A is rather complex even for strictly dominating affine transformations from **Z** into **Z**. This can be studied systematically, also with versions living entirely within the nonnegative integers.

6. BABY BOOLEAN RELATION THEORY.

Baby BRT is just equational and inequational BRT of type (1,1). Or unary equational and inequational BRT. See [Fr01] for a gentle introduction to this theory. With any choice of V,K, there are 16 statements to be analyzed in the equational case as well as the inequational case.

In [Fr01], we give 8 complete classifications. We use

V = MF(Z)
V = SD(Z)
K = INF(Z)
K = BINF(Z)

Here BINF(Z) is the set of all bi-infinite subsets of Z; i.e., subsets of Z with infinitely many positive and infinitely many negative elements.

In the course of this analysis, the only interesting statements that arise are

a) the thin set theorem together with a straightforward refinement;
b) the complementation theorem.

The refinement of TST has the conclusion $fA \cup A \neq \mathbf{Z}$.

The following class of multivariate functions plays a crucial role in BRT.

ET(**Z**) = all expansively trapped elements f of MF(**Z**). I.e., there exists p,q > 1 such that

$$p|x| \leq |f(x)| \leq q|x|$$

for all $x \in \mathrm{dom}(f)$. Here | | is the sup norm.

We can extend the classifications in [Fr01] to include ET(Z). However, this turns out to be less interesting than one would expect, because the complementation theorem fails with ET(Z). This is because for f ∈ ET(Z), we have f(0) = 0.

ET(Z) becomes interesting and important when we consider extensions of the complementation theorem with more than one set.

7. EXTENSIONS OF THE COMPLEMENTATION THEOREM.

A is covered by B,C iff A ⊆ B ∪ C. A is disjointly covered by B,C if and only if A is covered by B,C, and B,C are disjoint.

Write this as A/B,C. We can restate the complementation theorem as follows.

THEOREM 7.1. (∀f ∈ SD(**Z**))(∃A ∈ INF(Z))(**Z**/A,fA). (∀f ∈ SD(**Z**))(∃!A ⊆ **Z**)(**Z**/A,fA).

We now strive for a bi-infinite disjoint cover theorem.

PROPOSITION 7.2. (∀g ∈ SD(**Z**))(∃A ∈ BINF(Z))(**Z**/A,gA).

Unfortunately this is refutable in RCA$_0$. Here is a fix.

PROPOSITION 7.3. (∀f,g ∈ SD(**Z**))(∃A ∈ BINF(**Z**)(fA/A,gA).

We weakened **Z**/A,gA to fA/A,gA. Unfortunately this, also, is refutable in RCA$_0$.

To fix this, we consider two sets A,B.

THEOREM 7.4. (∀f,g ∈ SD(**Z**))(∃A,B ∈ BINF(**Z**))(fA/B,gB).

Provable in RCA$_0$. We now extend to three sets.

THEOREM 7.5. (∀f,g ∈ SD(**Z**))(∃A,B,C ∈ BINF(**Z**))(fA/B,gB ∧ fB/C,gC).

This is also provable in RCA$_0$. We can also prove the following variant in RCA$_0$.

THEOREM 7.6. (∀f,g ∈ SD(**Z**))(∃A,B,C ∈ BINF(**Z**)(fA/B,gC ∧ fB/C,gC).

We now consider another variant.

PROPOSITION 7.7. (□f,g □ SD(**Z**))(□A,B,C □ BINF(**Z**))(fA/C,gB □ fB/C,gC).

More sharply,

PROPOSITION 7.8. (□f,g □ SD(**Z**))(□A_1,A_2,A_3 □ BINF(Z))(□i < j,k)(fA_i/A_j,gA_k).

Proposition 7.7 is refutable in RCA_0.

We can fix this by using ET(**Z**).

PROPOSITION 7.9. (□f,g □ ET(**Z**))(□A,B,C □ BINF(**Z**))(fA/C,gB □ fB/C,gC).

PROPOSITION 7.10. (□f,g □ ET(**Z**))(□A_1,A_2,A_3 □ BINF(**Z**))(□i < j,k)(fA_i/A_j,gA_k).

Propositions 7.9 and 7.10 can be proved, but only with small large cardinals. Let MAH = ZFC + {there exists an n-Mahlo cardinal}$_n$.

THEOREM 7.11. Propositions 7.9 and 7.10 are provably equivalent to 1-Con(MAH) over ACA.

We can extend further.

PROPOSITION 7.12. (□r ≥ 1)(□f,g □ ET(**Z**)(□A_1,...,A_r □ BINF(Z))(□i,j < k)(fA_i/A_j,gA_k).

Even further.

PROPOSITION 7.13. (□r ≥ 1)(□f,g □ ET(**Z**)(□A_1 □ ... □ A_k from BINF(**Z**))(□i,j < k)(fA_i/A_j,gA_k □ A_1 □ fA_r = ∅).

Once we bring in towers under inclusion, we can revisit Theorem 7.5:

PROPOSITION 7.14. (□f,g □ ET(**Z**))(□A □ B □ C □ from BINF(Z)(fA/B,gB □ fB/C,gC).

THEOREM 7.15. Propositions 7.12-7.14 are provably equivalent to 1-Con(MAH) over ACA.

8. EQUATIONAL BOOLEAN RELATION THEORY OF TYPE (2,3).

Note that our independent statements lie in equational BRT on (ET(**Z**),BINF(**Z**)) of type (2,3).

Conjecture: we can "classify" equational BRT on (ET(**Z**), BINF(**Z**)) of type (2,3). Let MAH+ = ZFC + (∃n)(∃κ)(κ is an n-Mahlo cardinal).

CONJECTURE 8.1. Every instance of equational BRT on (ET(**Z**), BINF(**Z**)) of type (2,3) is provable or refutable in MAH+.

We have seen that this conjecture is false with MAH+ replaced by MAH, assuming MAH is consistent. We encapsulate this conjecture by “it is necessary and sufficient to use Mahlo cardinals of finite order to classify equational BRT on (ET(**Z**),BINF(**Z**)) of type (2,3)”.

We run into independence results in equational BRT on (ET(**Z**),INF(**Z**)) of type (2,3).

PROPOSITION 8.2. (∀f,g ∈ ET(**Z**))(∃A,B,C ∈ INF(**Z**))(fA/C,gB ⊆ fB/C,gC ⊆ A ∩ fC = ∅).

THEOREM 8.3. Proposition 8.2 is provably equivalent to 1-Con(MAH) over ACA.

CONJECTURE 8.4. Every instance of equational BRT on (ET(Z), INF(Z)) of type (2,3) is provable or refutable in MAH+.

We encapsulate this conjecture with “it is necessary and sufficient to use Mahlo cardinals of finite order to classify equational BRT on (ET(Z),INF(Z)) of type (2,3)”.

CONJECTURE 8.5. Every instance of equational BRT on (ET(Z),BINF(Z)) of type (2,2) is provable or refutable in ACA. Every instance of equational BRT on (ET(Z),INF(Z)) of type (2,2) is provable or refutable in ACA.

9. DISJOINT COVER THEORY.

Equational BRT appears difficult, even of type (2,2), and even more so of type (2,3). There are 2^{512} instances in type (2,3).

Disjoint cover theory is a simplification of Boolean relation theory, and seems more tractable.

Disjoint cover theory (DCT) on (V,K) of type (n,m) analyzes statements

$(\forall f_1,\ldots,f_n \in V)(\exists A_1,\ldots,A_m \in K)$ such that a given set of disjoint cover conditions holds among the m(n+1) sets
$A_1,\ldots,A_m$
$f_1A_1,\ldots,f_1A_m$
...
$f_nA_1,\ldots,f_nA_m$.

THEOREM 9.1. Every statement of DCT on (ET(Z),BINF(Z)) of type (2,2) is either provable or refutable in RCA_0.

We haven't classified DCT on (ET(Z),BINF(Z)) of type (2,3), using large cardinals. But we have the following partial result.

THEOREM 9.2. Every statement of DCT with inclusion on (AET(Z),BINF(Z)) of type (2,3) with at most two disjoint cover conditions is either provable or refutable in RCA_0 or provably equivalent to 1-Con(MAH) over ACA.

We worked on this before we discovered our independent instances in section 7 that do not use inclusion; i.e., do not use $A \cup B \subseteq C$, and before we switched over to ET(Z) from asymptotically defined classes. I.e., AET(Z) is the same as ET(Z), except that finitely many exceptions are allowed. This is expected to carry over without inclusion for (ET(Z),BINF(Z)).

There are major obstacles in dealing with 3 disjoint cover conditions. We do not have enough of a theory to carry out such a classification, and have to rely too much on the enumeration and careful examination of many cases. With 3 disjoint cover conditions, the number of cases becomes unmanageable.

There are many other ways to put Proposition 7.9 into substantial and natural fragments of equational BRT on (ET(Z),BINF(Z)) of type (2,3). Hopefully these will be subject to complete analyses.

10. FINITENESS.

There is a striking fact that we have observed in all of our classifications to date. Let NFIN(**Z**) be the set of all nonempty finite subsets of **Z**.

In our classifications, any statement classified as true on (V,NFIN(**Z**)) is classified as true on (V,BINF(**Z**)). We call this finiteness. We conjecture that finiteness also holds for the various conjectured classifications.

In the case of the classification in Theorem 9.2, finiteness is provably equivalent to 1-Con(MAH) over ACA.

11. INTEGRAL PIECEWISE POLYNOMIALS.

An integral polynomial is a multivariate polynomial from **Z** into **Z**.

An integral piecewise polynomial is an integral multivariate function defined by possibly different integral polynomials in each of finitely many cases, each case given by a finite set of integral polynomial inequalities.

An integral multivariate function f is expansive if and only if for some $p > 1$, $|f(x)| \geq p|x|$ holds for all $x \in dom(f)$.

EPP(**Z**) = set of all expansive integral piecewise polynomials.
ETPP(**Z**) = set of all expansively trapped integral piecewise polynomials.

THEOREM 11.1. The status of our independent statements remain the same when stated for EPP(**Z**) or ETPP(**Z**).

In fact, when stated for EPP(**Z**) or ETPP(**Z**), we can require that the nine sets A,B,C,fA,fB,fC,gA,gB,gC be exponential Presburger; i.e., definable in $(\mathbf{Z},<,+,-,2^{|x|})$, which is well known to have quantifier elimination. This results in a Π^0_2 sentence which is provably equivalent to 1-Con(MAH) over PRA.

We can be more explicit about the form of A,B,C. We can take A,B,C to be images of integral piecewise polynomials on the set of double factorials, or alternatively, on the set of triple exponentials to base 2.

12. INTEGRAL POLYNOMIALS ON UPPER HALFSPACE.

Upper halfspace of $\mathbf{Z}^k$ is the set of all points whose final coordinate is nonnegative.

POLY*(**Z**) = the set of all integral polynomials P from the upper halfspace of a Cartesian power of Z into a Cartesian power of A, where each coordinate function of P is expansive on its domain.

For P ∈ POLY*(Z) and sets A, we define PA to be the set of all coordinates of all values of P at arguments from A. Thus we still have one dimensional images.

THEOREM 12.1. The status of our independent statements remain the same when stated for POLY*(**Z**).

We again get Π^0_2 sentences provably equivalent to 1-Con(MAH) over PRA, using exponential Presburger.

13. FINITE SETS.

We give two finite forms here. The first involves EPP(Z).

PROPOSITION 13.1. Let f,g ∈ EPP(**Z**) and E be a finite subset of N\{0,1} where the logarithms of successive elements have sufficiently large ratios. There exist finite A,B,C ⊆ **Z** containing E such that fA/C,gB and fB/C,gC.

THEOREM 13.2. Proposition 13.1 is provably equivalent to 1-Con(MAH) over ACA.

Proposition 13.1 is naturally in Π^0_3 form.

For the second finite form, we use a nonlinear power growth condition. Let NPT(Z) is the set of all nonlinear power trapped elements of MF(Z). I.e., there are real constants p,q > 1 such that $|x|^p \leq |f(x)| \leq |x|^q$.

The following is a variant of Proposition 7.9.

PROPOSITION 13.3. (∀f,g ∈ NPT(Z))(∃A,B,C ∈ BINF(Z))(fA/C,gB → fB/C,gC).

Proposition 13.3 is provably equivalent to 1-Con(MAH) over ACA.

Let $E \subseteq Z$ and $i \geq 1$. We write E[i] for the i-th smallest positive element of E. Here E[1] is the least positive element of E (assuming E is nonempty). If E has fewer than i positive elements then E[i] is undefined.

PROPOSITION 13.4. Let $k,r \geq 8$ and $f,g:Z^k \to Z$ lie in NPT(Z). There exist finite $A,B,C \subseteq Z$ such that fA/C,gB and fB/C,gC, where A = {2,C[k!!],C[(k+1)!!],...,C[r!!]}.

By a finitely branching tree argument, we see that a bound can be placed on the elements of A,B,C depending only on k,r, and the constants p,q, for $f,g \in$ NPT(Z), but not on f,g. This results in a demonstrably equivalent explicitly Π^0_2 sentence.

The numbers k!!,...,r!! and 8 are of course crude, but nice on the page.

THEOREM 13.5 Proposition 1 is provably equivalent to 1-Con(MAH) over ACA.

14. REAL PIECEWISE POLYNOMIALS.

We write EPP(**R**) for the set of all expansive real piecewise polynomials. We write EPP(**R**,**Z**) for the set of all expansive real piecewise polynomials with integer coefficients, where the polynomial inequalities also involve only integer coefficients.

We let POLY*(**R**,**Z**,k) be the set of all real polynomials P from the upper halfspace of $\mathbf{R}^k$ into $\mathbf{R}^k$, with integer coefficients, such that each coordinate function of P is expansive on its domain. Let POLY*(**R**,**Z**) be the union over k, of POLY*(**R**,**Z**,k).

PROPOSITION 14.1. Let f,g in EPP(**R**,**Z**) and E be a finite subset of **N**\{0,1} where the logarithms of successive elements have sufficiently large ratios. There exist finite sets A,B,C $\subseteq$ **R** containing E such that fA/C,gB and fB/C,gC.

THEOREM 14.2. Propositions 14.1 is provably equivalent to Con(MAH) over ACA.

We can use the elimination of quantifiers for the real field and double exponential bounds on how large the ratios of the logarithms need to be, together with bounds on the sizes of A,B,C, in order to put these Propositions in Π^0_1 form. We can of course quantify over algebraic real numbers with a bound on their presentations to make everything explicitly Π^0_1.

Instead of quantifying over algebraic real numbers, we can instead quantify over rationals, and use the notion of approximate disjoint cover. Write $A/_{\varepsilon}B,C$ iff B,C are disjoint and every element of A is within ε of some element of $B \cup C$.

We can modify these Propositions to state that for each $\varepsilon > 0$, there exist finite sets of rationals containing E such that the disjoint cover conditions hold with $/_{\varepsilon}$. We can bound everything involved double exponentially, and get explicitly Π^0_1 sentences equivalent to Con(MAH) this way.

15. INTEGRAL PIECEWISE LINEAR FUNCTIONS.

NOTE: The conjectures of this section are very strong in that we have proof sketches that need substantial checking, which will be done after the initial papers on BRT are finished.

EPL(**Z**) = the set of all expansive integral piecewise linear functions. ETPL(**Z**) = the set of all expansively trapped integral piecewise linear functions.

CONJECTURE 15.1. The status of our independent statements, when stated for EPL(**Z**) or ETPL(**Z**), is that they are equivalent to Con(MAH) over ACA.

We can take A,B,C to be images of integral piecewise linear functions on infinite geometric progressions, or on just the powers of 2. The representation can be bounded double exponentially in the data.

CONJECTURE 15.2. Let $f,g \in$ EPL(**Z**) and E be a finite subset of **N**\{0} where the ratios of successive elements are sufficiently large. There exist finite $A,B,C \subseteq$ **Z** containing E such that fA/C,gB and fB/C,gC.

CONJECTURE 15.3. Proposition 15.2 is provably equivalent, to Con(MAH) over ACA. The same holds for ETPL(**Z**). We can give a double exponential bound on "sufficiently large".

16. INTEGRAL AFFINE FUNCTIONS.

We let LIN*(**Z**,k) be the set of all integral affine functions T from a finite intersection of integral halfplanes in $\mathbf{Z}^k$ into

Z^k such that each coordinate function of T is expansive on its domain. Let LIN*(**Z**) be the union over k, of LIN*(**Z**,k).

The conjectures of section 15 are made for LIN*(**Z**), again as strong conjectures.

17. BOUNDED ARITY.

All independent statements thus far considered use multivariate functions without bounding the arity.

In each case other than those involving piecewise polynomials, polynomials, piecewise linear functions, and affine functions, the arity can be fixed to be a small number - perhaps even 2 - and one obtains the same results. Here one takes advantage of the constant(s) in the definition of expansive or expansively trapped. In the other cases, the arity can also be fixed, but it is not clear how small the fixed number can be.

18. SOME ILLUSTRATIVE EXAMPLES.

We now give some indication of the scope of BRT.

a. $(\forall f \in V)(\exists A \in K)(fA \subseteq A)$.
b. $(\forall f \in V)(\exists A \in K)(fA \neq U)$.

In a, if we take V to be the set of all bounded linear operators on Hilbert space and K to be the set of all non-trivial closed subspaces of Hilbert space, then we have a restatement of the invariant subspace problem for Hilbert space.

In b, take V = multivariate functions from ω_1 into ω_1, and K = subsets of ω_1 of cardinality $\aleph_1$. We have a restatement of the negation of a theorem of Todorcevic (even for just binary functions).

In b, if we take V to be the continuous functions from **R** into **R** and K to be the set of all dense open subsets of **R**, then we get a true theorem in elementary real analysis.

In b, if we take V to be the binary continuous functions from **R** into **R** and K to be the set of all dense open subsets of **R**, then we get a false theorem in elementary real analysis.

It is natural to use the set of all continuous functions from a topological space into itself, or the multivariate continuous functions from a topological space into itself, and K to be the open sets, or some variety of open sets. E.g., the nonempty open sets, the dense open sets, the open sets of the same cardinality as the space, the open sets that can be continuously mapped onto the space, etc. We call this topological BRT.

The thin set theorem and the examples discussed above for b are obviously in topological Boolean relation theory, using, respectively, the discrete topology on **Z**, the discrete topology on ω_1, and the usual topology on R.

One can consider algebraic Boolean relation theory. Here one takes V to be the polynomials over a ring, or some variety of polynomials over a ring, in several variables.

Or take V to be the rational functions over a field, or some variety of rational functions over a field (the singularities cause no problems in the definition of forward image), in general of several variables. Or one might have an ordered ring or ordered field, and use the order to define natural varieties of polynomials or rational functions, in general of several variables. The sets in K could be taken to be the zero sets, or infinite sets, or sets having some property involving the order, etc.

One can also consider geometric Boolean relation theory, where smoothness conditions are placed on the functions. Or analytic Boolean relation theory where analyticity conditions are placed on the functions.

Or linear Boolean relation theory, where V is the set of all linear or affine functions on a vector space, or a topological vector space, subject to boundedness or continuity conditions.

For the purist, there is finite combinatorial Boolean relation theory, where, in its purest form, one takes V = set of (multivariate) functions from a finite set into itself, and K to be set of subsets of the finite set. Also consider cardinality conditions on the elements of K. See [Fr01].

19. REVIEW OF STATEMENT TO BE PROVED.

Let ET(Z) be the set of all multivariate functions f from Z into Z of expansive linear growth. I.e., there exist rationals p,q > 1 such that

$$p|x| \leq |f(x)| \leq q|x|$$

holds for all $x \in dom(f)$. Here |x| is the sup norm of x.

Let BINF(Z) be the set of all bi-infinite subsets of Z.

Let A/B,C mean that "A is disjointly covered by B,C", which says that $A \subseteq B \cup C$ and $B \cap C = \emptyset$.

Recall this from section 7.

PROPOSITION 7.9. $(\forall f,g \in ET(\mathbf{Z}))(\exists A,B,C \in BINF(\mathbf{Z}))(fA/C,gB \wedge fB/C,gC)$.

We sketch a proof of Proposition 7.9 using suitable large cardinals.

We actually prove the following stronger result using suitable large cardinals.

PROPOSITION 7.13. $(\forall r \geq 1)(\forall f,g \in ET(\mathbf{Z})(\exists A_1 \subseteq ... \subseteq A_k$ from $BINF(\mathbf{Z}))(\forall i,j < k)(fA_i/A_j,gA_k \wedge A_1 \cap fA_r = \emptyset)$.

20. RELEVANT LARGE CARDINALS.

The Mahlo cardinals of order 0 are the strongly inaccessible cardinals.

The Mahlo cardinals of order k+1 are the Mahlo cardinals of order k in which every closed unbounded set contains a Mahlo cardinal of order k.

The relevant large cardinal axiom is $(\forall n)(\exists \kappa)(\kappa$ is an n-Mahlo cardinal).

We need the following very useful property of these large cardinals.

THEOREM 20.1. Let κ be n+1-Mahlo, $f_1,f_2,...:\kappa^n \to \kappa$, and $g:\kappa^n \to \kappa$ be bounded. There exists $E \subseteq \kappa$ of order type κ such that
i) each f_iE is of order type $< \kappa$;
ii) each f_iE has sup at most the sup of E;

iii) gE is finite.

We now fix f,g □ ET(Z), with constants p,q > 1. Without loss of generality we assume that p,q are rationals. We consider the discrete linearly ordered group with extra structure, M = (Z,<,0,1,+,-,f,g). Here - is always used as unary -.

21. A COUNTABLE INDISCERNIBLY GENERATED CORRELATE.

By standard model theoretic techniques, we can construct an associated discrete linearly ordered group with extra structure M* = (Z*,<*,0*,1*,+*,-*,f*,g*,c_0,c_1,...) with the following properties.

a. The c's (indexed by the nonnegative integers) generate M*.
b. The c's are greater than all positive standard integers.
c. The c's are atomic indiscernibles in M*.
d. Let t ≥ 1. Let B be the set of all elements of Z* that are generated from the c's,0*,1*,+*,-*,f*,g* using at most t applications of the four functions of M*. B defines the partial structure (B,<*|B,0*,1*,+*|B,-*|B,f*|B,g*|B,c_0,c_1,...). This structure is isomorphically embeddable into M.

It follows that every purely universal sentence that holds in M also holds in M*.

22. THE TRANSFINITE INDISCERNIBLY GENERATED CORRELATE.

Let □ be a suitable large cardinal. By a standard construction in model theory, there is a canonical extension M* to M** = (Z**,<**,0**,1**,+**,-**,f**,g**,c_0,c_1,...,$c_□$,...), □ < □, with the following properties.

a. The $c_□$'s (indexed by □) generate M**.
b. The $c_□$'s are greater than all positive standard integers.
c. The $c_□$'s are atomic indiscernibles in M**.
d. Let t ≥ 1. Let E be a set of the $c_□$'s that is of order type □. Let B be the set of all elements of Z** that are generated from the elements of E,0**,1**,+**,-**,f**,g** using at most t applications of the four functions of M**. The partial substructure of M** with domain B is isomorphically embeddable into M* and also into M.

23. THE CRUCIAL WELL FOUNDED PARTIAL ORDERING.

In general, <** is not well ordered. Let T be the set of elements of Z** that are generated by the $c_{\square}$'s using only f**,g**, and -**. In fact, <** may not be well ordered on T.

It is easily seen that T consists of nonstandard integers, and the inequality

$$p|x| \,\square\, |f^{**}(x)|,|g^{**}(x)| \,\square\, q|x|$$

holds for all tuples x from T. This makes sense because p,q are rationals.

LEMMA 23.1. The relation p|x| □ |y| on T is well founded.

This well foundedness will be good enough for our purposes.

24. THE CRUCIAL DEFINITION BY RECURSION.

We define W □ T as follows. For x □ T, let x □ W if and only if x □ g**W.

This definition is well defined since in order to determine if x □ W, we need only consider the values of g** at arguments $z_1,\ldots,z_k$ □ W, where $p|z_1,\ldots,z_k|$ □ |x|. Then apply Lemma 23.1.

Note that T/W,g**W. Also note that W contains every $c_{\square}$ and every $-c_{\square}$. Thus in a strong sense, W is bi-infinite.

25. THE SKOLEM HULL CONSTRUCTION.

We can start with E_1 = the set of all $c_{\square}$ and $-c_{\square}$, E_1 □ W. At the next stage, we can choose witnesses in W for the elements of $f^{**}E_1\backslash W$. Let E_2 □ W consist of E_1, these witnesses, and $f^{**}E_1$ □ W. Then we can choose witnesses in W for the elements of $f^{**}E_2\backslash W$. Let E_3 □ W consist of E_2, these witnesses, and $f^{**}E_2$ □ W. We can continue this process along the nonnegative integers. We then obtain E_1 □ E_2 □ ... □ W.

Note that by construction, each $f^{**}E_i/E_{i+1},g^{**}E_{i+1}$. To see this, since E_{i+1} □ W, E_{i+1} □ $g^{**}E_{i+1} = \varnothing$. Let x □ $f^{**}E_i$. If x □ W then x □ E_{i+1}. If x □ W, then witnesses for $f^{**}E_i\backslash W$ lie in E_{i+1}, and hence x □ $g^{**}E_{i+1}$.

It follows that for all i < j,k, $f^{**}E_i/E_{i+1},g^{**}E_{i+1}$.

Also note that E_1 □ $f^{**}E_r = \varnothing$ because of the inequality on f^{**}.

We make this construction only r times. Note that there is an obvious list $F_1,\ldots,F_s$ of n-ary functions F_i from the $\{c_{\square}$: □ < □$\}$ into terms representing elements of W (i.e., terms in the $c_{\square}$'s, $f^{**},g^{**},-^{**}$), whose values comprise exactly E_r. Here n is some positive integer.

We define the associated functions $G_{i,j}$ and H, where

i) $G_{i,j}(x)$ is the $c_{\square}$ that appears in $F_i(x)$ in position j, reading the term $F_i(x)$ from left to right, with default c_0;
ii) H(x) is the total number of symbols that appear in the terms $F_1(x),\ldots,F_s(x)$.

We can obviously view the $G_{i,j}$ as a single infinite list of n-ary functions on $\{c_{\square}$: □ < □$\}$.

We now apply the large cardinal property of □ to this construction, via Theorem 20.1. We obtain B □ $\{c_{\square}$: □ < □$\}$ of

order type □ (crucial!!),

where we use $D_1 = \pm B$ □ E_1 to start this same Skolem hull construction D_1 □ ... □ D_r □ W □ T. We will have

i) the total number of symbols that appear in the representations of any given element of D_r is bounded by a fixed integer;
ii) for every i, the set of $c_{\square}$'s that appear in the i-th position of representations of the elements of D_r, has order type □ □ and has sup □ $\sup(D_1)$.

We therefore have

iii) the set of $c_{\square}$'s that appear in the terms representing elements of D_r has order type □ □ and has sup □ $\sup(D_1)$.

(Here we use any choice of representations of elements of T by terms).

We thus see that the elements of D_r are generated from D_1 using a bounded finite number of $f^{**},g^{**},-^{**}$.

Therefore, we can stick this situation back into M* and finally back into M. We then obtain A_1 □ ... □ A_r □ Z via isomorphism.

We still have that for all i < j,k, $f^{**}D_i/D_{i+1},g^{**}D_{i+1}$, and also D_1 □ $f^{**}D_r = \emptyset$. By isomorphism, for all i < j,k, $fA_i/A_j,gA_k$, and A_1 □ $fA_r = \emptyset$.

Since D_1 is closed under -**, we see that A_1 is closed under -, and in particular $A_1,...,A_r$ are bi-infinite.

Since D_1 □ $f^{**}D_r = \emptyset$, we also have A_1 □ $fA_r = \emptyset$. QED

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