SELECTION FOR BOREL RELATIONS

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Abstract. We present several selection theorems for Borel relations, involving only Borel sets and functions, all of which can be obtained as consequences of closely related theorems proved in [DSR 96,99,01,01X] involving coanalytic sets. The relevant proofs given there use substantial set theoretic methods, which were also shown to be necessary. We show that none of our Borel consequences can be proved without substantial set theoretic methods. The results are established for Baire space. We give equivalents of some of the main results for the reals.

Introduction.

Let S be a set of ordered pairs and A be a set. We say that f is a selection for S on A if and only if dom(f) = A and for all x $\in$ A, (x,f(x)) $\in$ S. We say that f is a selection for S if and only if f is a selection for S on {x: ($\exists$y)((x,y) $\in$ S)}.

Let N be the set of all nonnegative integers. $2^N = \{0,1\}^N$ is Cantor space, where {0,1} is given the discrete topology. N^N is Baire space, where N is given the discrete topology.

We use ¬ for the reals with the usual topology. All results in sections 1-5 are formulated on N^N. This is most convenient for the proofs. In section 6 we give equivalent formulations on ¬ of some of the main results.

The following result from [DSR01X] led to this research.

PROPOSITION I. Let S $\subseteq$ N^N x N^N be coanalytic and E $\subseteq$ N^N be Borel. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

Proposition I is proved in [DSR01X] using set theoretic assumptions going beyond ZFC. In fact, [DSR01X] shows that Proposition I is provably equivalent to

COUNT. For all $f \in N^N$, $N^N \cap L[f]$ is countable.

over ZFC.

[DSR01X] also gives the comparatively simple proof of Proposition I using analytic determinacy, which works for all coanalytic E. Moreoever, [DSR01X] shows that Proposition I for coanalytic E is equivalent to analytic determinacy.

The proof of Proposition I from COUNT in [DSR01X] is rather complicated, and we conjecture that the proof can be considerably simplified using additional methods from modern set theory.

In section 1, we present the simple [DSR01X] proof of Proposition I from analytic determinacy. It is obvious that if "coanalytic" is replaced by "Borel", then the proof goes through unmodified with Borel determinacy instead of analytic determinacy. Therefore we have the following theorem of ZFC.

THEOREM II. Let $S \subseteq N^N \times N^N$ be Borel and $E \subseteq N^N$ be Borel. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

In section 4, we show that Theorem II cannot be proved using only countably many iterations of the power set operation.

Moreoever, we show that Theorem II with $E = N^N$ cannot be proved using only countably many iterations of the power set operation.

In fact, we go further and show that the following theorem cannot be proved using only countably many iterations of the power set operation.

THEOREM III. Let $S \subseteq N^N \times N^N$ be Borel. If there is a constant selection for S on every compact set, then there is a Borel selection for S.

We now discuss strenthenings of Theorem III involving only Borel selection.

PROPOSITION IV. Let $S \subseteq N^N \times N^N$ be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

This turns out to be independent of ZFC.

Moreover, we show in sections 3 and 5 that Proposition IV is equivalent to the following principle DOM over ZFC.

DOM. For all $f \in N^N$ there exists $g \in N^N$ such that every $h \in N^N \setminus L[f]$ is eventually strictly dominated by g.

That Proposition IV is provable in ZFC + DOM follows from more general results in [DSR99] and [DSR01X], using the fact that the set of all Borel selectors of a Borel relation can be coded by a Π^1_1 set of reals.

The "reason" Proposition IV is independent of ZFC relates to the unbounded levels of the hypothesized Borel selections. Consider the following variants involving Borel selections of bounded ranks.

THEOREM V. Let $S \subseteq N^N \times N^N$ be Borel, and λ be a countable limit ordinal. If there is a $<\lambda$ Borel selection for S on every compact set, then there is a Borel selection for S.

Theorem V follows from Theorem VII (see below), which we prove in ZFC using Borel determinacy.

The two versions of Theorem V obviously imply Theorem III, and hence cannot be proved with only countably many iterations of the power set operation.

We now bring the Borel set $E \subseteq N^N$ back into the discussion.

It follows from [DSR99] and [DSR01X] that ZFC + COUNT proves the following, again using the observation that the set of all Borel selectors of a Borel relation can be coded by a Π^1_1 set of reals.

PROPOSITION VI. Let $S \subseteq N^N \times N^N$ be Borel and $E \subseteq N^N$ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

Obviously Proposition VI implies Proposition IV, and hence Proposition IV implies DOM over ZFC. However, we don’t know if Proposition VI can be proved in ZFC + DOM.

Finally, we come to the analog of Proposition VI with E.

THEOREM VII. Let S □ N^N x N^N be Borel, E □ N^N be Borel, and □ be a countable limit ordinal. If there is a <□ Borel selection for S on every compact subset of E, then there is a □□ Borel selection for S on E.

In section 2, we give a proof of Proposition VII in ZFC using Borel determinacy. This result is implicit in [DSR01X]. In fact, using the more precise arguments there, it follows that we can replace <□ and □□ by □□ and □□, where □ is any countable ordinal.

In section 6 we give versions of some of the main results on the reals. The equivalences are rather straightforward.

THEOREM VIII. Let S □ ¬ x ¬ be Borel and E □ ¬ be Borel with empty interior. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

In section 6 we show that Theorem VIII is provably equivalent to Theorem II over ATR_0. Here ATR0 is one of the basic systems of reverse mathematics (see [Si99], p. 37).

PROPOSITION IX. Let S □ ¬ x ¬ be Borel and E □ ¬ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

In section 6 we show that Proposition IX is provably equivalent to Proposition VI over ATR_0.

THEOREM X. Let S □ ¬ x ¬ be Borel. If there is a constant selection for S on every compact set of irrationals, then there is a Borel selection for S on the irrationals.

In section 6 we show that Theorem X is provably equivalent to Theorem III over ATR_0.

1. Proofs from analytic determinacy.

We present the proof in [DSR01X] of

PROPOSITION I. Let S $\subseteq$ N^N x N^N be coanalytic and E $\subseteq$ N^N be Borel. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

from analytic determinacy.

As remarked in the Introduction, [DSR01X] also shows that Proposition 1 for is provably equivalent to COUNT by a much more difficult argument.

The proof using analytic determinacy works for coanalytic E. In fact, in [DSR01], Proposition 1 for coanalytic E is shown to be equivalent to analytic determinacy over ZFC.

Proof: Let S,E be coanalytic. Assume there is a continuous selection for S on every compact subset of E.

For each g $\in$ 2^N with infinitely many 1's, let g* $\in$ N^N be the successive positions of the 1's in x. Thus g* is strictly increasing. Let g** $\in$ N^N be given by g**(n) = the exponent of 2 in x*(n).

Let G be the following game, where player I,II successively play elements of N. Let I play f $\in$ N^N and II play g $\in$ N^N.

II wins the game if and only if

i) g $\in$ 2^N and g has infinitely many 1's;
ii) f $\notin$ S or (f,g**) $\in$ S.

Note that the set of wins for player II is the union of an analytic set and a coanalytic set. Hence by an argument of Donald Martin in [Du90], this game is determined just using analytic determinacy.

Suppose I wins this game with winning strategy T. Let V $\subseteq$ N^N be the set of all plays of I made using strategy T, where II plays elements of 2^N. Then obviously V is compact. Let J be a continuous selelction for S on V.

We now show how player II can defeat player I even though player I uses winning strategy T. Player II can accomplish this by

1) first playing enough 0's until the value of J(f)(0) has been determined;
2) then playing 0's followed by a 1 so that g**(0) = J(f)(0);
3) then playing enough 0's until the value of J(f)(1) has been determined;
4) then playing 0's followed by a 1 so that g**(1) = J(f)(1);
5) continuing to play in this way.

This describes II's plays even if the values of the J(f)(n) are not all determined during the game. This is because II would, according to 1) - 5), simply play a tail of 0's.

However, it is clear that the values of the J(f)(n) do get determined because we know that I must play an element of S. (We know this because II obviously plays an element of 2^N, and I is using a winning strategy).

In fact, if player I plays $f \in N^N$ then $f \in S$ and player II plays some $g \in 2^N$ such that g** = J(f). This is a contradiction since player II wins this run of the game.

We have shown that I does not win the game, and hence II wins the game. A winning strategy for II defines a continuous selection for S on E as follows. Let $f \in E$. Have player I play f. Then II plays $g \in 2^N$, where there are infinitely many 1's in g and (f,g**) $\in$ S. The map that sends f to g** is obviously continuous. QED

We now give a proof of another result implicit in [DSR01X], using analytic determinacy.

PROPOSITION VI. Let $S \subseteq N^N \times N^N$ be Borel and $E \subseteq N^N$ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

In fact, [DSR01X] has a much more difficult proof of Proposition VI even for coanalytic S.

The proof using analytic determinacy works for coanalytic X,E. In fact, in [DSR01], Proposition VI for coanalytic S,E is shown to be equivalent to analytic determinacy over ZFC.

Proof: Let S,E be coanalytic. Assume there is a Borel selection for S on every compact subset of E.

Consider the following game G. Players I,II successively play elements of N. Let I play $f \in N^N$ and II play $g \in N^N$.

II wins if and only if

i) $g \in 2^N$ codes a Borel function $H:N^N \to N^N$;
ii) $f \notin E$ or $(f,H(f)) \in S$.

The set of wins for II is the intersection of a coanalytic set with the union of an analytic set and a coanalytic set. Hence by a result of Donald Martin in [Du90], this game is determined assuming analytic determinacy.

The proof then follows that of Proposition I above, with less to verify. Again, assume I wins with winning strategy T, and let V be the set of plays of I using T, where II plays in 2^N. Then V is again compact, and let $J:V \to N^N$ be a Borel selection for S on V. Let $g \in 2^N$ be a Borel code for J.

Have player II play g no matter what player I plays. Let player I play his winning strategy T. Then player I will play $f \in S$. But then player II will win this run of the game since $(f,J(f)) \in S$.

Again we conlude that player II wins the game. Hence player II must play a Borel code no matter what player I plays. The set of plays of II is analytic. Hence let α be a countable limit ordinal such that player II always plays a rank $<\alpha$ Borel code.

We now obtain the desired Borel selection for S on E using a winning strategy for player II in the game. For each $f \in N^N$, have player I play f, and let F(f) be the response of player II using his winning strategy. According to the previous paragraph, F(f) is a $<\alpha$ rank Borel code. Finally, let F'(f) be the value of the Borel function coded by F(f) at the argument f. Note that if $f \in E$ then $(f,F'(f)) \in S$. Hence F' is a Borel selection for S on E. Furthermore, F' is [illegible] Borel. QED

2. Proofs from Borel determinacy.

THEOREM II. Let $S \subseteq N^N \times N^N$ be Borel and $E \subseteq N^N$ be Borel. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

Proof: The proof is identical to the proof of Proposition I, except that Borel determinacy suffices. This is because the relevant game is Borel. QED

THEOREM VII. Let $S \subseteq N^N \times N^N$ be Borel, $E \subseteq N^N$ be Borel, and λ be a countable limit ordinal. If there is a $<\lambda$ Borel selection for S on every compact subset of E, then there is a $\leq\lambda$ Borel selection for S on E.

Proof: The proof is identical to the proof of Proposition VI, except that Borel determinacy suffices. This is because the relevant game is Borel. QED

3. Proof from ZFC + DOM.

For $f_1,\ldots,f_k \in N^N$ write $\omega_1(f_1,\ldots,f_k)$ for the first uncountable ordinal in the sense of $L[f_1,\ldots,f_k]$.

LEMMA 3.1. Assume DOM and let $f \in N^N$. There exists $g,h \in N^N$ such that
ii) $\omega_1(f,g,h) = \omega_1(f,g)$.
iii) h eventually strictly dominates every $x \in N^N \cap L[f,g]$;

Proof: Assume DOM and let $f \in N^N$. If there exists $g \in N^N$ such that $\omega_1(g) = \omega_1$, then apply DOM to (f,g) to obtain the desired h.

Now assume that for all $g \in N^N$, $\omega_1(g) < \omega_1$. We set g = f and force over $L_{\omega_1}[f]$.

We use a notion of forcing in $L_{\omega_1}[f,F]$ that has the countable chain condition in $L_{\omega_1}[f]$, where the generic object eventually strictly dominates all elements of N^N lying in $L_{\omega_1}[f]$. A convenient choice is Hechler forcing (see [He74]), where the conditions are (s,G), where $G \in N^N \cap L_{\omega_1}[f]$, and s is a finite sequence indexed from 0, and G strictly dominates s. We take $(s,G) \leq (s',G')$ if and only if s is an initial segment of s' and G' strictly dominates G.

Note that generic objects exist for this notion of forcing over $L_{\omega_1}[f]$ since the number of dense sets of conditions is countable. Let $h \in N^N$ be generic with respect to this notion of forcing. Then $L_{\omega_1}[f,h]$ preserves cardinals over $L_{\omega_1}[f]$, and h eventually strictly dominates every $x \in N^N \cap L[f]$. Since $L_{\omega_1}[f,h]$ contains all elements of N^N constructible in (f,h), we see that $\omega_1(f,h) = \omega_1(f)$. QED

PROPOSITION IV. Let $S \subseteq N^N \times N^N$ be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

We now prove Proposition IV from ZFC + DOM. This result follows from more general results in [DSR99] and [DSR01X], using the fact that the set of all Borel selectors of a Borel relation can be coded by a Π^1_1 set of reals.

Proof: Let $S \subseteq N^N \times N^N$ be Borel with Borel code $f \in N^N$. Assume there is a Borel selection for S on every compact set. By DOM and Lemma 3.1, let $g,h \in N^N$, $\omega_1(f,g,h) = \omega_1(f,g)$, and h eventually strictly dominate every $x \in N^N \cap L[f,g]$.

Let $V = \{x \in N^N$: x is eventually strictly dominated by h$\}$. Then V is a countable union of compact subsets of N^N and $N^N \cap L[f,g] \subseteq V$. Furthermore, the sequence of presentations of these compact subsets as trees all lie in L[f,g,h].

There is a Borel selection for S on V. This is a Σ^1_2 statement. Hence by Shoenfield absoluteness (Sh[61]), there is a Borel selection for S on V whose code lies in L[f,g,h]. Let λ be the rank of some Borel selection for S on V whose code lies in L[f,g,h]. Then $\lambda < \omega_1\{f,g,h) = \omega_1(f,g)$.

We now claim that the following holds in L[f,g]:

for every compact set B there is a Borel selection for S on B of rank λ.

To see this, note that λ is countable in L[f,g], and for every compact set B coded in L[f,g] there is in fact a Borel selection for S on B of rank λ. (This is because $B \subseteq V$). The claim follows by absoluteness.

We now can apply Theorem V (section 2) to L[f,g] and obtain a □□+□ Borel selection for S in the sense of L[f,g]. We then obtain an actual □□+□ Borel selection for S by absoluteness. QED

We will find it useful to summarize what we have proved as follows. Let ZFC\P be ZFC without the power set axiom.

THEOREM 3.2. Theorem V → Theorem IV is provable in ZFC\P + DOM.

4. **Obtaining Borel determinacy.**

In this section, we focus on the following.

THEOREM III. Let S □ N^N x N^N be Borel. If there is a constant selection for S on every compact subset of N^N, then there is a Borel selection for S.

We show that Theorem III cannot be proved using only countably many iterations of the power set operation.

For this purpose, it is convenient to use ZFC\P as the base theory.

We make the following assumptions, sitting within ZFC\P, until the proof of Lemma 4.7 is complete.

a) V = L;
b) □ is a countable ordinal;
c) For all countable □, there is a set of rank □ □ lying in L(□+1)\L(□).

Recall that the rank of a set is the strict sup of the ranks of its elements.

Our aim is to refute Theorem III.

We let KP be the usual Kripke/Platek set theory (see [Ba75], p. 11, Definition 2.5). Let KP’ be KP together with the axiom of infinity.

Let W be the class of all (□,R) such that

i) (□,R) satisfies KP’ + V = L;
ii) there is an internal ordinal of (□,R) whose set of predecessors is of order type □;
iii) (□,R) satisfies “for all □, there is a set of rank □ □ lying in L(□+1)\L(□)”.

Obviously W is Borel. Note that W includes all the well founded (□,R) satisfying KP’ + V = L.

Let (□,R) □ W. For each internal ordinal b of (□,R), we let b* be the set of all sets of rank □ □ that are internal to the L(b) of (□,R). Finally, define (□,R)’ as the set of all b* such that b is an internal ordinal of (A,R). Then (A,R)′ consists of transitive sets of rank □ □+1 which are linearly ordered under inclusion. By clause iii), there are no repetitions among the b*. Hence the mapping that sends b to b* is an order preserving isomorphism from the internal ordinals of (□,R) under R, onto (□,R)′ under inclusion.

LEMMA 4.1. The mapping that sends (□,R) □ W to (□,R)′ is Borel.

Proof: This Lemma is stated precisely in terms of standard codes for countable sets of rank □ □+2. Specifically, we mean that there is a Borel function which sends each (□,R) □ W to some standard code for (□,R)′. This is left to the reader. QED

We say that (□,R) is special if and only if it lies in W and every element of its domain is definable without parameters.

We say that L(□) is special if and only if (□,R) is special for the (□,R) isomorphic to (L(□),□). We write L(□)′ for (□,R)′, where (□,R) is isomorphic to (L(□),□).

LEMMA 4.2. Let L(□) satisfy KP′, □ < $□_1$, and L(□+1)\L(□) meet N^N. Then L(□) is special. There are arbitrarily large countable □ such that L(□) is special.

Proof: Let L(□) be isomorphic to (□,R). Then (□,R) □ W by assumption c). Also a well known Skolem hull and transitive collapse argument establishes that (L(□),□) has every element definable without parameters.

For the second claim, let □ < $□_1$ be so large that there is no countable □ > □ such that L(□) satisfies KP′ and has every element definable without parameters. The least such □ is $□_2$ definable over (L($□_1$),□). Using ZFC\P, let A be the least $□_2$ elementary substructure of ($L_{□1}$,□). Then there exists □ < $□_1$ such that A = L(□). Also □ < □, L(□) satisfies KP′, and (L(□),□) has every element definable without parameters. This is the desired contradiction. QED

Let (□,R) □ W be special. The special code for (□,R), written (□,R)*, is constructed as follows. First let T be the set of all first order sentences that hold in (□,R), coded as T □ N via Gödel numbers. We take (□,R)* to be the first element of

N^N that is recursive in the double jump of T (in some fixed standard indexing) such that

i) (□,R)* eventually strictly dominates every element of N^N that is recursive in T;
ii) for each n, (□,R)*(n) is even if and only if n □ T.

A special code is an element of N^N that is the special code of some special (□,R). Note that the "underlying" (□,R) is unique up to isomorphism.

We also define L(□)* for special L(□), in exactly the same way, using (L(□),□).

Note that the set of special codes is a Borel subset of N^N. In addition, observe that we can recover an underlying (□,R) from the special code in a uniformly effective way.

We now define S □ N^N x N^N as follows. Let x,y □ N^N. Then (x,y) □ S if and only if either x is not a special code, or x,y are both special codes and the following holds:

#) Let x = (□,R)* and y = (□,Y)*, where R,Y are chosen by the uniformly effective procedure from x,y. Let U be the maximum common initial segment of (□,R)′ and (□,Y)′. Then U is the set of strict predecessors of some element of (□,Y)′.

Note that S is Borel.

LEMMA 4.3. There is no Borel selection for S.

Proof: Let F:N^N □ N^N be a Borel selection for S. Let u be a Borel code for F.

According to Lemma 4.2, let □ < $□_1$ be such that L(□) is special and u □ L(□). Then u is recursive in L(□)*, and so F(L(□)*) is hyperarithmetic in L(□)*.

Let x = L(□)*. Then x,F(x) are special codes, and condition #) holds for x, and y = F(x). Let x = (□,R)* and F(x) = (□,Y)*. Here R,Y are chosen by the uniformly effective procedure from x,y. Let U be the maximum common initial segment of (□,R)′ and (□,Y)′. Let U be the set of strict predecessors of some element u of (□,Y)′.

Suppose U is a proper initial segment of (□,R)′. Since U is well founded, u must lie in the well founded part of (□,Y)′. Hence U can be extended to a longer common initial segment of (□,R)′ and (□,Y)′, which is a contradiction.

Hence U = (□,R)′ and U has order type □. Therefore (□,Y)′ has an initial segment of order type □+1. Hence internal ordinals of (□,Y) have an initial segment of order type □+1. So we have an internal copy of L(□) in (□,Y).

Now L(□) has every element definable without parameters. Hence L(□)* is internal to (□,Y). Since (□,Y) satisfies KP′, we see that every element of N^N hyperarithmetic in L(□)* is recursive in F(x). But F(x) = F(L(□)*) is hyperarithmetic in x = L(□)*. This is the desired contradiction. QED

LEMMA 4.4. Let V □ N^N be compact. There is a countable ordinal bound to the lengths of the maximum well ordered initial segments of the internal ordinals of every (□,R) □ W such that (□,R)* lies in V.

Proof: Let f □ N^N be such that every element of V is everywhere strictly dominated by f. Let f □ L(□), □ < $\square_1$. Suppose (□,R)* □ V has maximum well ordered initial segment of order type > □. Then f is internal to (□,R), and so f is recursive in the set of first order sentences that hold in (□,R). Therefore f is eventually strictly dominated by (□,R)*. Since (□,R)* □ V, (□,R)* is everywhere strictly dominated by f. This is a contradiction. QED

LEMMA 4.5. There is a constant Borel selection for S on every compact subset of N^N.

Proof: Let V □ N^N be compact. By Lemma 4.4, let □ < $\square_1$ be such that for every (□,R)* □ V, the maximum well ordered initial segment of (□,R) has type < □.

By Lemma 4.2, let L(□) be special, where □ < □ < $\square_1$. Let (L(□),□) be isomorphic to (□,Y). Let (□,R)* □ V.

We claim that ((□,R)*,(□,Y)*) □ S. To see this, let U be the maximum common initial segment of (□,R)′ and (□,Y)′. Then U is well ordered and must be of order type < □. Hence U is a proper initial segment of (□,Y)′ determined by a point (since (□,Y) is well founded).

For f □ V that is not a special code, (f,(□,Y)*) □ S by default. QED

LEMMA 4.6. Theorem III is false.

Proof: By Lemmas 4.3 and 4.5. QED

We now release the assumptions a) - c) above. We have shown the following.

LEMMA 4.7. ZFC\P + V = L + Theorem III proves (□□ < $□_1$)(□□ < $□_1$)(L(□+1)\L(□) contains no set of rank □ □).

Proof: By Lemma 4.6, and the assumptions a)-c) we used to prove it. QED

LEMMA 4.8. The following is provable in ZFC\P. Let □ □ □ < □ and L(□+1)\L(□) have an element of rank □ □. Then L(□+2)\L(□+1) has an element of rank □ □.

Proof: Let □,□ be as given. Let rk(x) □ □, x □ L(□+1)\L(□). Using the standard definable well ordering of L(□) over (L(□),□), we can assume that x is definable without parameters over (L(□),□). Note that TC(x) □ L(□+1)\L(□). Fix x = {y □ L(□): (L(□),□) satisfies □(y)}, where □ has only the free variable y.

Let A be the set of all elements of L(□) that are definable over (L(□),□) with parameters from TC(x). Then (A,□) is an elementary substructure of (L(□),□), and TC(x) □ A. Also, every element of A is definable over (A,□) with parameters from TC(x), and x = {y □ A: (A,□) satisfies □(y)}.

Let j be the unique isomomorphism from A onto a transitive set B. Then j is the identity on TC(x), x = {y □ B: (B,□) satisfies □(y)}, and every element of B is definable over (B,□) with parameters from TC(x). Since A,B are elementarily equivalent, we see that B = L(□) for some □ □ □. Since x □ L(□), we see that B = L(□).

We now prove that j is the identity. Let x □ A be defined over (L(□),□) using parameters from TC(x). Then x is defined over (A,□) using the same formula and paramters (by elementary substructure). Hence j(x) is defined over (L(□),□) using the same formula and paramters (by isomorphism). Hence x = j(x).

We have thus shown that (L(□),□) has every element definable with parameters from TC(x), where x is definable without parameters. The partial truth predicates for all formulas of bounded complexity with parameters from TC(x), over (L(□),□), all lie in L(□+1), where each is encoded as a set of finite sequences from TC(x). The finite sequences from TC(x) are each encoded as sets of rank < □, and the partial truth predicates as sets of rank □ □.

We claim that the truth predicate for all formulas at parameters from TC(x), over (L(□),□), encoded as a set of rank □ □, does not lie in L(□+1). If it does lie in L(□+1), then it would be definable by a formula with parameters from TC(x), over (L(□),□). This is impossible by diagonalization.

However, this truth predicate does lie in L(□+2) because we can obtain it by piecing together the partial truth predicates that each lie in L(□+1). This yields an element of L(□+2)\L(□+1) of rank □ □. QED

LEMMA 4.9. The following is provable in ZFC\P. Suppose □,□ are ordinals such that □ ≥ □ and □ is the least ordinal with the property that there is no element of L(□+1)\L(□) of rank □ □+3. Then (L(□),□) satisfies ZFC\P + V(□) exists.

Proof: Here "V(□) exists" means that there is a function from □+1 into sets which satisfies the usual inductive definition for the cumulative hierarchy. V(□) itself is taken to be the value of this function at □.

We can easily prove by transfinite induction on □ □ □ that (L(□),□) satisfies V(□) exists, because the relevant functions are of rank □ □+3.

It remains to show that (L(□),□) satisfies ZFC\P. By Lemma 4.6, we see that □ is a limit ordinal.

To verify ZFC\P, it suffices to verify replacement. Let f:□ □ □ be unbounded in □, where □ < □ and f is definable over (L(□),□). For each □ < □, let g(□) be the first constructed element of L(□+1)\L(□) of rank □ □. Using any element of L(□+1)\L(□) of rank □ □, we can piece these sets of rank □ □ together to form a singe set of rank □ □ in L(□+1)\L(□). This is the desired contradiction. QED

THEOREM 4.10. The following are provably equivalent in ZFC\P + V = L.
i) Theorem II;
ii) Theorem III;
iii) Theorem V;
iv) Theorem VII;
v) Borel determinacy;
vi) ($\square\square < \square_1$)($\square\square < \square_1$)(L($\square$) satisfies ZFC\P + V($\square$) exists).

Proof: By Lemma 4.9, the conclusion of Lemma 4.7 is eqvuivalent to vi). The proofs given in section 2 of Theorems II and VII are within ZFC\P + Borel determinacy, and III immediately follows from II, and V immediately follows from VII. Each of Theorems II,V,VII immediately imply Theorem III. Also, it follows easily from [Ma85] that vi) implies v) over ZFC\P + V = L. QED

We remark that by [Fr71], v) $\square$ vi) over ZFC\P. In fact, by [Fr71] and [Ma85], Borel determinacy is provably equivalent to

($\square\square < \square_1$)($\square$x $\square$ $\square$)($\square\square < \square_1$)(L($\square$,x) satisfies ZFC\P + V($\square$) exists).

THEOREM 4.11. Let $\square$(x) be a $\square_1$ formula of set theory with only the free variable shown, where ZFC\P proves ($\square$x)($\square$(x) $\square$ x is an ordinal). Then Theorem III cannot be proved in ZFC\P + ($\square$x)(V(x) exists). However, Theorems II,III,V,VII can all be proved in ZFC\P + ($\square\square < \square_1$)(V($\square$) exists).

Proof: The last claim is obvious since these Theorems were proved from Borel determinacy without use of V = L, and [Ma85] also proves Borel determinacy in ZFC\P + ($\square\square < \square_1$)(V($\square$) exists), without the use of V = L.

We will prove a sharper form of the first claim with both occcurrences of ZFC\P replaced by ZFC\P + V = L.

First note that by $\square_1$ absoluteness, there exists a constructibly countable ordinal $\square$ such that $\square$($\square$) holds in L. Fix the least such $\square$.

There exists $\square > \square$ such that (L($\square$),$\square$) satisfies ZFC\P + V($\square$) exists, since we can take $\square$ to be a sufficiently large successor cardinal.

Let □ be the least ordinal such that (L(□),□) satisfies ZFC\P + V(□) exists.

By hyptheses, (L(□),□) satisfies that □(x) has an ordinal solution. Hence (L(□),□) satisfies that □(x) has a countable ordinal solution. Any such internal solution must be an external solution, and hence must be at least as large as □. Therefore □ is countable in (L(□),□).

Suppose Theorem III holds in (L(□),□). By Theorem 4.10, vi) holds in (L(□),□). By applying vi) to □, we see that (L(□),□) satisfies "there exists □ such that (L(□),□) satisfies ZFC\P + V(□) exists". But this contradicts the choice of □. QED

In light of Theorem 4.11, we say that Theorem III cannot be proved using only countably many iterations of the power set operation, but can be proved using uncountably many iterations of the power set operation.

5. Obtaining DOM.

In this section we focus on the following.

PROPOSITION IV. Let S □ N^N x N^N be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

In this section we show that Proposition IV implies DOM over ZFC.

Until the proof of Lemma 5.5 is complete, we work in ZFC\P + □DOM. Accordingly, fix f □ N^N such that every g □ N^N is eventually strictly dominated by some h □ L(f) □ N^N.

Note that L[f] □ N^N must be uncountable. Hence □$_1$(f) = □$_1$.

We seek to refute Proposition IV. We modify the machinery introduced in section 4.

We use the constructible hierarchy relative to f. Let L(□,f) be the □-th stage of this hierarchy, where we start with L(0,f) = TC(f). Let L[f] be the class of constructible sets relative to f.

Let X be the class of all (□,R,c) such that

i) (□,R) is a model of KP’ with standard integers;
ii) c □ □ represents f in (□,R);
iii) (□,R) satisfies V = L[c].

Note that X is Borel.

Let (□,R,c) □ X. The well founded part of (□,R,c) is the set of all n □ □ such that there is no infinite backwards R-chain starting with n.

We say that (□,R,c) is well founded if and only if its well founded part is the entire domain.

We say that (□,R,c) is unusual if and only if every element of its domain is definable in (□,R,c) with only the parameter c.

We say that L(□,f) is unusual if and only if L(□,f) satisfies KP’, and every element of L(□,f) is definable in (L(□,f),□) with only the parameter f.

Note that the (□,R,c) □ X that are well founded and unusual are exactly the (□,R,c) □ X where (□,R) is isomorphic to some unusual (L(□,f),□).

Let (□,R,c) □ X be unusual. The unusual code for (□,R,c), written (□,R,c)*, is constructed as follows. First let T be the set of all first order sentences that hold in (□,R,c), coded as T □ N via Gödel numbers. Here c is treated as a constant symbol representing the function f.

We take (□,R,c)* to be the first element of N^N that is recursive in the double jump of (T,f), (in some fixed standard indexing), such that

i) (□,R,c)* eventually strictly dominates every element of N^N that is recursive in T,f;
ii) for each n, (□,R,c)*(n) is even if and only if n □ T.

An unusual code is an element of N^N that is the unusual code of some unusual element of X. Note that the “underlying” (□,R,c) is unique up to isomorphism.

Also note that the set of unusual codes is a subset of N^N that is arithmetic in f. In addition, observe that we can recover an underlying (□,R,c) from the unusual code in a uniformly effective way using f.

We now define S′ □ N^N x N^N. Let x,y □ N^N. Then (x,y) □ S′ if and only if

i) x is not an unusual code; or
ii) x is an unusual code and y is an infinite backward R chain. Here x = (□,R,c)*, where (□,R,c) is obtained by the uniformly effective procedure from x,f; or
iii) x,y are unusual codes and x is internal to (□,Y). Here y = (□,Y,d)*, where Y is obtained by the uniformly effective procedure from y,f.

Note that S′ is a subset of N^N x N^N that is arithmetic in f.

LEMMA 5.1. Let L(□,f) satisfy KP′, □ < $□_1$, and L(□+1,f)\L(□,f) meet N^N. Then L(□,f) is unusual. There are arbitrarily large countable □ such that L(□,f) is unusual.

Proof: Argue as in the proof of Lemma 4.2, but this time with the paramter f. Use the fact that $□_1$(f) = $□_1$. QED

LEMMA 5.2. There is no Borel selection for S′.

Proof: Note that being a Borel code for a Borel selection for S′ is a $□^1_1$ property with parameter f. By Shoenfield absoluteness, let F:N^N □ N^N be a Borel section for S′ and let u be a Borel code for F, where u □ L[f].

According to Lemma 5.1, let □ < $□_1$ be such that L(□,f) is unusual and u □ L(□,f). Let x = (□,R,c)*, where (□,R,c) is isomorphic to (L(□,f),□). Let y = F(x). Then y is an unusual code. Let y = (□,Y,d)*. Here (□,R,c) and (□,Y,d) are obtained from x,y by the uniformly effective procedure.

Since (x,y) □ S′, we see that clause iii) applies in the definition of S′. Since (□,R) is well founded, we see that x is internal to (□,Y). Hence every set hyperarithmetic in x is recursive in y. But u is recursive in x and y = F(x) is hyperarithmetic in (x,u). Hence y is hyperarithmetic in x. This is the desired contradiction. QED

LEMMA 5.3. Let V □ N^N be compact. There is a countable ordinal bound to the lengths of the maximum well ordered initial segments of the internal ordinals of every (□,R,c) □ X such that (□,R,c)* that lies in V.

Proof: Let g □ N^N be such that every element of V is everywhere strictly dominated by g. Suppose there are unusual (□,R,c) such that (□,R,c) □ V with arbitrarily long countable well ordered initial segments of the internal ordinals. Then every h □ L[f] will be internal to some unusual (□,R,c) such that (□,R,c) □ V. Hence every h □ L[f] will be eventually strictly dominated by g. This contradicts our assumption. QED

LEMMA 5.4. There is a Borel selection for S on every compact subset of N^N.

Proof: Let V □ N^N be compact. Let □ be the strict bound given by Lemma 5.3. Recall that □ is countable in L[f]. By Lemma 5.1, let L(□,f) be unusual, where every unusual code of every unusual L(□,f), □ < □, is internal to L(□,f). Let y = L(□,f)*.

For x that are not unusual codes, and x □ V, set F(x) = x. For x □ V that are unusual codes, let x = (□,R,c)*, where (□,R,c) is chosen by the uniformly effective procedure in x,f. Because of the bound, □, we can tell in a Borelian way whether (□,R,c) is well founded. If (□,R) is not well founded, then we can construct its well founded part in a Borelian way, and construct an infinite backwards R chain in a Borelian way. So (x,y) □ S’ by clause ii). If (□,R) is well founded, then we can set F(x) = y. By the construction of y, (x,y) □ S’ by clause iii). QED

LEMMA 5.5. Proposition IV is false.

Proof: By Lemmas 5.2 and 5.4. QED

We now release our assumptions.

THEOREM 5.6. Theorem IV □ DOM is provable in ZFC\P. Theorem IV is provably equivalent to (Theorem V □ DOM) over ZFC\P. Theorem IV is provably equivalent to DOM over ZFC. Theorem IV cannot be proved or refuted in ZFC.

Proof: The first claim is by Lemma 5.5 and the assumptions used to prove it. The second claim follows from the first claim and Theorem 3.2. The third claim follows from the second claim. The fourth claim follows from the third claim and the fact that DOM cannot be proved or refuted in ZFC. The consistency of DOM is by Hechler forcing as in the proof of Lemma 3.1. The consistency of ¬DOM is clear since it demonstrably fails in L. QED

6. On the reals.

Recall the following.

THEOREM II. Let S ⊆ N^N x N^N be Borel and E ⊆ N^N be Borel. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

THEOREM VIII. Let S ⊆ ¬ x ¬ be Borel and E ⊆ ¬ be Borel with empty interior. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

THEOREM 6.1. Theorems II and VIII are provably equivalent over ATR_0.

Proof: Assume Theorem II. Let S,E be as given in Theorem VIII, and assume there is a continuous selection for S on every compact subset of E. Let A be a countable dense subset of ¬ such that E ⊆ ¬\A. It is well known that ¬\A is homoeomorphic to ¬\Q, which is in turn homeomorphic to N^N.

Let h:¬\A → N^N be a homeomorphism. Then hS ⊆ N^N x N^N and hE ⊆ N^N are Borel. We claim that there is a continuous selection for hS on every compact subset of hE. To see this, let V ⊆ h^E be compact. Write h^W = V, W ⊆ E. Then W is compact. Hence there is a continuous selection for S on W. This conjugates to a continuous selection for h^S on h^W = V, establishing the claim.

By Theorem II, there is a continuous selection for hS on hE. This conjugates to a continuous selection for S on E.

Assume Theorem VIII. Let S,E be as given in Theorem II, and assume there is a continuous selection for S on every compact subset of E. Let h’:N^N → ¬\Q be a homeomorphism. Then h’S ⊆ ¬\Q x ¬\Q and h’E ⊆ ¬\Q are Borel and hE has empty interior in ¬. We claim that there is a continuous selection for h’S on every compact subset of h’E. To see this, let V ⊆ h’E be compact. Write h’W = V, W ⊆ E. Then W is compact. Hence there is a continuous selection for S on W. This conjugates to a continuous selection for h’S on h’W = V, establishing the claim.

By Theorem VIII, there is a continuous selection for h’S on h’E. This conjugates to a continuous selection for S on E. QED

Recall the following.

PROPOSITION VI. Let S ⊆ N^N x N^N be Borel and E ⊆ N^N be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

PROPOSITION IX. Let S ⊆ ¬ x ¬ be Borel and E ⊆ ¬ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

THEOREM 6.2. Propositions VI and IX are provably equivalent over ATR_0.

Proof: Let Proposition IX’ be the same as Propoistion IX, but with the added hypothesis that E has empty interior. The proof of the equivalence of Proposition VI and IX’ is the same as the proof of the equivalence of Theorem II and VIII, except that continuous replaced everywhere by Borel.

It remains to show that Proposition IX’ implies Proposition IX. Let S,E be as given, where there is a Borel selection for S on every compact subset of E. Then E\Q is Borel, and so by Proposition IX’, there is a Borel selection for S on E\Q. Hence there is a Borel selection for S on E. QED

Recall the following.

THEOREM III. Let S ⊆ N^N x N^N be Borel. If there is a constant selection for S on every compact set, then there is a Borel selection for S.

THEOREM X. Let S ⊆ ¬ x ¬ be Borel. If there is a constant selection for S on every compact set of irrationals, then there is a Borel selection for S on the irrationals.

THEOREM 6.3. Theorems III and X are provably equivalent over ATR_0.

Proof: This is clear using a homeomorphism from N^N onto the irrationals. QED

By Theorem 4.11, Theorems VIII and X can be proved using uncountably many iterations of the power set operation but not with only countably many iterations of the power set operation.

By Theorem 5.6, Proposition IX is independent of ZFC, and implies DOM over ZFC. In fact, Proposition IX follows from COUNT over ZFC, since Proposition VI follows from COUNT over ZFC, by [DSR01X].

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