ORDER EMULATION THEORY STATUS 3/27/18

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ABSTRACT. Presently, CMI = Concrete Mathematical Incompleteness consists of Boolean Relation Theory (BRT), Emulation Theory, and Inductive Equation Theory (and, of course, a great deal of earlier work preceding BRT, as detailed in the Introduction to the BRT book on my website). Many highlights across CMI are documented in [2], including explicitly Π^0_1 forms and formulations corresponding to the HUGE cardinal hierarchy. Here we are concerned only with the implicitly finite part of Emulation Theory, which promises to be the most immediately thematically resonating part of CMI for the general mathematical community. We first present a General Emulation Theory conception. To date, we have only considered the special very concrete case of Order Emulation Theory. Within Order Emulation Theory, we have so far concentrated almost entirely on what we now call Combinatorial Order Emulation Theory. We will follow this status report up with a later status report which focuses on Geometric Emulation Theory. The Combinatorial and the Geometric resonate very differently across the mathematical community, and the separation between Combinatorial Order Emulation Theory and Geometric Order Emulation Theory appears to be rather important for the development of Order Emulation Theory.

1. GENERAL EMULATION THEORY

The most general conception that we have of Emulation Theory at present is as follows.
Let W be a class of mathematical structures. Let E be a particular mathematical structure in W. An emulator S of E in W is a mathematical structure S in W such that any pattern (of prescribed kind) that arises in S already arises in E. A Maximal Emulator S of E in W is an Emulator S of E in W where no proper extension (of prescribed kind) is an Emulator of E in W.

We refer to the above as a General Emulation Setting. More specifically, the class W, the patterns considered, and the notion of extension used.

So far we have only considered Order Emulation Theory where the mathematical structures are sets of tuples of rational numbers, and extensions are supersets. The patterns only involve the usual ordering of rational numbers.

In General Emulation Theory, we expect that by using Zorn's Lemma,

1) Every structure in W has a maximal emulator

where the maximality refers to extensions.

In Order Emulation Theory, we will see that there is no need for Zorn's Lemma. In fact there is an effective construction of a maximal emulator of every structure from W.

Let P be a significant or interesting or desirable property of structures in W. I.e., P may or may not hold of a given structure in W. We seek to determine whether the following holds.

2) For structures in W, some maximal emulator has property P

This depends critically on P and the General Emulation Setting.

EXPOSITIONAL POINT: We avoid writing the apparently more attractive

3) Every structure in W has a maximal emulator with property P

because of a subtle ambiguity. This might mean that for every structure in W, there is an emulator with property P that is maximal among emulators with property P. That is of course NOT what we intend by 2).

Now what kind of properties P do we have in mind?

Since we have only focused on a the very specific concrete Order Emulation Theory, our ideas of the kinds of P are at present rather limited. We have been considering invariance under binary relations given in advance. Generally speaking, these amount to what we view as Symmetry Conditions.

We anticipate a much further development of Order Emulation Theory - whose current status is presented here - before we begin to investigate General Emulation Theory.

NOTE: A formulation considerably less general than the above General Emulation Theory, but far more general than our Order Emulation Theory, was briefly considered in section 1.1 of [1]. We haven't even started to seriously investigate even that.

GENERAL CONJECTURE. Just as in Order Emulation Theory, we rather naturally and quickly run out of axioms in ZFC in order to prove perfectly natural theorems of kind 2), as long as the General Emulation Setting is just a little bit rich.

All of [1] resides in Order Emulation Theory, and in fact in Combinatorial Order Emulation Theory. [2] discusses Emulation Theory and Inductive Equation Theory. [2] discusses several aspects of Emulation Theory not reflected in Order Emulation Theory. Also there are things discussed here and in [1] that are not present in [2].

There is some PRELIMINARY INDICATION that Inductive Equation Theory may profitably be reconfigured as part of Order Emulation Theory - where the notion of EXTENSION is not superset as it is here. But this is beyond the scope of this abstract.

2. ORDER EMULATION THEORY

DEFINITION 2.1. A rational interval is an interval of rationals in the usual sense with endpoints from $Q \cup \{-\infty,\infty\}$. We use I for nondegenerate rational intervals.

Since Order Emulation Theory uses only the usual linear ordering on the rationals, there are really only three relevant choices of intervals: Q(0,1), Q(0,1], Q[0,1). Q[0,1]. Other intervals in Q are order isomorphic to one of these four.

DEFINITION 2.2.. S is an emulator of $E \subseteq I^k$ if and only if $S \subseteq I^k$ and every element of $S \times S$ is order equivalent to an element of $E \times E$ (as 2k-tuples). S is a maximal emulator of $E \subseteq I^k$ if and only if S is an emulator of $E \subseteq I^k$, where no proper superset of S is an emulator of $E \subseteq I^k$.

THEOREM 2.1. (RCA_0) Every $E \subseteq I^k$ has the same emulators as some finite subset of E.

Theorem 2.1 tells us that we can formulate our statements using only finite $E \subseteq I^k$, and have the same effect as stating them for all $E \subseteq I^k$. We would rather state then using finite $E \subseteq I^k$, as this is more concrete.

Order Emulation Theory starts with the following easy fundamental result.

THEOREM 2.2. (RCA_0) Every finite subset of I^k has a (algorithmically computable) maximal emulator.

ORDER EMULATION THEORY. What properties can we require of a maximal emulator? I.e., for which properties P is it the case that for finite subsets of I^k, some maximal emulator satisfies P?

The KIND of Order Emulation Theory being pursued is determined by the KIND of properties of subsets of I^k that one is considering. Thus in Combinatorial Order Emulation Theory, combinatorial properties are used. In Geometric Order Emulation Theory, geometric properties are used. In the future, we anticipate perhaps Algebraic Order Emulation Theory, Topological Order Emulation Theory, Number Theoretic Order Emulation Theory, and so forth.

More specifically, we have concentrated on invariance or symmetry conditions of a rather standard kind throughout mathematics.

DEFINITION 2.3. Let $R \subseteq A \times A$ and $S \subseteq A$. S is R closed if and only if for all x,y, $x R y \wedge x \in S \rightarrow y \in S$. S is R invariant if and only if for all x,y, $x R y \rightarrow (x \in S \leftrightarrow y \in S)$.

DEFINITION 2.4. $f::A \rightarrow B$ if and only if f is a function which is a subset of $A \times B$. $f::I \rightarrow I$ is a partial isomorphism from S to S if and only if
i. $S \subseteq I^k$, for some k.
ii. f is one-one.
iii. For all $p_1,...,p_k \in dom(f)$, $S(p_1,...,p_k) \leftrightarrow S(f(p_1),...,f(p_k))$.

The present development of Order Emulation Theory is based on the following three notions.

DEFINITION 2.5. Let $R \subseteq I^k \times I^k$. R is ME usable if and only if the following holds. For finite subsets of I^k, some maximal emulator is R closed. R is ME invariantly usable if and only if the following holds. For finite subsets of I^k, some maximal emulator is R invariant.

DEFINITION 2.6. Let $f::I \rightarrow I$. f is ME iso usable if and only if the following holds. For all k and finite subsets of I^k, some maximal emulator carries the partial isomorphism f. Let V be a set of $f::I \rightarrow I$. V is ME iso usable if and only if the following holds. For all k and finite subsets of I^k, some maximal emulator carries all of the partial isomorphisms in V.

In Order Emulation Theory, we seek necessary and/or sufficient conditions for ME usability, ME invariant usability, and ME iso usability. In Combinatorial Order Emulation Theory we seek combinatorial conditions on R,f, and in Geometric Order Emulation Theory we seek geometric conditions on R,f.

There is a fundamental necessary condition established in [1].

DEFINITION 2.7. $R \subseteq I^k \times I^k$ is order preserving if and only if for all $x,y \in I^k$, $x R y \rightarrow$ x,y are order equivalent.

Obviously f::I → I is order preserving if and only if f is strictly increasing.

THEOREM 2.3. (RCA_0) Order preservation is a necessary condition for ME usability, ME invariant usability, and ME iso usability. Strictly increasing is a necessary condition for ME iso usability.

3. COMBINATORIAL ORDER EMULATION THEORY

In the present development of Combinatorial Order Emulation Theory, we seek necessary and/or sufficient conditions for ME usability, ME invariant usability, and ME iso usability of a combinatorial nature.

In sections 3.1, 3.2 we consider some general relations and functions. In section 3.3, we consider some special relations and functions. Some readers may prefer to read section 3.3 first, especially Maximal Emulation Shift = MES.

3.1. FINITE R,f

DEFINITION 3.1.1. Let $R \subseteq I^k \times I^k$. p divides R if and only if p is in the interior of I, and for all x R y and i, we have $x_i,y_i < p \vee x_i,y_i > p$. R is divisible if and only if some p divides R.

COMBINATORIAL MAXIMAL EMULATION/FINITE RELATIONS. CME/FR. Every divisible order preserving finite $R \subseteq I^k \times I^k$ is ME invariantly usable.

COMBINATORIAL MAXIMAL EMULATION/FINITE FUNCTIONS. CME/FF. Every divisible strictly increasing finite f::I → I is ME iso usable.

COMBINATORIAL MAXIMAL EMULATION/FINITE FUNCTION SETS. CME/FFS. Every finite set of divisible strictly increasing finite f::I → I not altering two distinct endpoints is ME iso usable.

In case I does not contain two distinct endpoints, finite R,f are automatically divisible. For I containing no endpoints, these statements are immediate consequences of the usual finite Ramsey theorem of 1930, and are provable in RCA_0. For I containing exactly one endpoint, the statements are much more difficult to prove, and we prove them in ACA'. We conjecture that they are provably equivalent to Con(PA) over WKL_0.

Now suppose I contains two distinct endpoints. Then divisibility cannot be eliminated in any of the three statements (see Theorem 3.1.1). Again, we use ACA' for the proof. In fact, we prove the above three statements in ACA', and conjecture that they are provably equivalent to Con(PA) over WKL_0.

THEOREM 3.1.1. (RCA_0) Let $f::Q[0,1] \rightarrow Q[0,1]$ be given by $f(0) = 1/2 \wedge f(1/2) = 1$, and $g::Q[0,1] \rightarrow Q[0,1]$ be given by $g(0) = 1/2 \wedge g(1/3) = 1$. Then f,g are not divisible, strictly increasing, and not ME iso usable.

We obviously have a determination of the ME usability, ME invariant usability, and ME iso usability of finite R,f in case I does not contain two distinct endpoints. Namely, order preservation (and strictly increasing). However, we do not have a determination of any of these ME usabilities for I = Q[0,1].

There are some weak partial results from [1].

THEOREM 3.1.2. Every order preserving $R \subseteq I^k \times I^k$ of cardinality 1 is ME usable.

Also in [1], there is a complete determination of the ME invariant usability of finite $R \subseteq Q[0,1]^2 \times Q[0,1]^2$.

3.2. LOWER PARAMETERIZATIONS AND EXTENSIONS

We now consider infinite R,f. But we like to continue to use only very concrete objects.

DEFINITION 3.2.1. $V \subseteq Q^k$ is order theoretic if and only if V is (quantifier free) definable over (Q,<) with parameters.

Order preservation and strictly increasing are not sufficient no matter what I is.

THEOREM 3.2.1. (RCA_0) Let p be in the interior of I and $R \subseteq I \times I$ be given by $x \; R \; y \leftrightarrow x < p \wedge y \geq p$. Then R is order theoretic, order preserving, and not ME invariantly usable.

THEOREM 3.2.2. (RCA_0) Let I = [(a,b)] and f::I → I be given by f(p) = q if and only if a < p < (a+b)/2 ∧ q = p+((a+b)/2). Then f is strictly increasing and not ME iso usable.

Note that f is not order theoretic. We conjecture that there is an order theoretic f for Theorem 3.2.2.

DEFINITION 3.2.2. Let $R \subseteq I^k \times I^k$. The lower parameterizations of R are the relations $R' \subseteq I^{k+m} \times I^{k+m}$ given by (x,z) R' (y,z) ↔ x R y ∧ max(z) < min(x,y).

DEFINITION 3.2.3. Let f::I → I. The lower extension of f is the extension of f below dom(f) ∪ rng(f) by the identity.

Note that for finite R,f, the lower parameterizations and lower extensions are order theoretic.

COMBINATORIAL MAXIMAL EMULATION/LOWER RELATIONS. CME/LR. Every lower parameterization of every finite order preserving $R \subseteq I^k \times I^k$ is ME invariantly usable.

COMBINATORIAL MAXIMAL EMULATION/LOWER FUNCTIONS. CME/LF. The lower extension of every divisible strictly increasing finite f::I → I is ME iso usable.

COMBINATORIAL MAXIMAL EMULATION/LOWER FUNCTIONS*. CME/LF*. The set of lower extensions of every finite set of divisible strictly increasing finite f::I → I is ME iso usable.

Our proof of CME/LF over Q(0,1) and Q[0,1) is carried out in ZC, where we use the $V(\omega+n)$. Our proof of CME/LF* and CME/LR over Q(0,1) and Q[0,1) is carried out in ZC, where we use that for every $V(\alpha)$, the ordinal $|V(\alpha)|$ and the set $V(|V(\alpha)|)$ exist. So this is a serious use of Replacement far beyond say ω_1 iterations of the power set operation.

THEOREM 3.2.3. CME/LR, CME/LF, CME/LF* are provably equivalent to Con(SRP) over WKL_0. This is true over Q(0,1] and over Q[0,1].

CONJECTURE. The ME usability and ME invariant usability of any given order theoretic $R \subseteq I^k \times I^k$ is either provable in SRP or refutable in RCA_0.

CONJECTURE. The ME iso usability of any given finite set of order theoretic f::I → I is provable in SRP^+ or refutable in RCA^0.

We know from the above results that SRP is definitely involved in these Conjectures.

3.3. SPECIFICS

Some readers may wish to read this section before sections 3.1, 3.2, especially MES below.

Here we discuss some very natural specific R,f. It is particularly convenient to use intervals Q[0,n] with distinguished points 0,1,...,n, rather than the order isomorphic interval Q[0,1] with corresponding distinguished points 0,1/n,...,n/n, or 0,1/n,1/(n-1),...,1/1.

MAXIMAL EMULATION SHIFT. MES. For finite subsets of $Q(0,k]^k$, some maximal emulation has for all p < 1, S(p,1,...,k-1) ↔ S(p,2,...,k).

MES is proved in [1] in WKL_0 + Con(SRP).

We can present MES in terms of ME invariant usability.

MES. For all k ≤ n, the relation $R \subseteq Q[0,k]^k \times Q[0,k]^k$ is ME invariantly usable, where R(x,y) if and only if $x_1 = y_1 = 1$, each $x_{i+1} = i$, and each $y_{i+1} = i+1$.

Note that these R are lower parameterizations of divisible finite order preserving $R \subseteq Q[0,k]^k \times Q[0,k]^k$ up to an unimportant permutation of coordinates. So we can apply CME/LR.

MAXIMAL EMULATION ISO SHIFT. MEIS. The function f::Q[0,k] → Q[0,k] is ME iso usable, where f(p) = p if p < 1; p+1 if p = 1,...,k-1; undefined otherwise.

Note that these f are lower extensions of divisible strictly increasing finite f::Q[0,k] → Q[0,k]. So we can apply CME/LF.

In [2] we go on to define a rather combinatorially natural equivalence relation on $Q[0,n]^k$ called OE↓, involving the distinguished points 1,2,...,n in Q[0,n]. We then stated, in our notation here,

GENERAL MAXIMAL EMULATION. GME. Each OE↓ $\subseteq$ $Q[0,n]^k \times Q[0,n]^k$ is ME invariantly usable.

OE↓ is also a lower parameterization of a divisible finite order preserving R $\subseteq$ $Q[0,n]^k \times Q[0,n]^k$, again falling under CME/LR.

We also give a multiple form of MEIS.

MAXIMAL EMULATION ISO SHIFT*. MEIS*. The set of functions f_i::Q[0,n] → Q[0,n], $1 \le i \le n-1$, is ME iso usable, where $f_i(p) = p$ if $p < i$; $p+1$ if $p = i,...,n-1$.

Again, this falls under CME/LF*.

THEOREM 3.3.1. MES, MEIS, GME, MEIS* are provably equivalent to Con(SRP) over WKL_0.

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