The book is expected to have three major parts:

CONCRETE MATHEMATICAL INCOMPLETENESS

PART 1. BOOLEAN RELATION THEORY.
PART 2. EMULATION THEORY.
PART 3. INDUCTIVE EQUATION THEORY.

1. Emulation Theory/infinite/SRP.
2. Inductive Equation Theory/infinite/SRP.
3. Inductive Equation Theory/infinite/HUGE.
4. Inductive Equation Theory/finite/SRP.
5. Inductive Equation Theory/finite/HUGE.

I have recently been determined to move to functions rather than stick with sets. However, I have now realized how I can do more with sets than I had previously thought, and certain things are more naturally transparent with sets.

So I am now moving back to sets throughout.

1. EMULATION THEORY/INFINITE/SRP

LARGE CARDINAL PROPERTY. For every $R \subseteq \lambda^k$ there exists $0 < \alpha_1 < \ldots < \alpha_k$ such that for all $\beta < \alpha_1$, $R(\beta, \alpha_1, \ldots, \alpha_{k-1}) \leftrightarrow R(\beta, \alpha_2, \ldots, \alpha_k)$.

This large cardinal property corresponds exactly to the SRP hierarchy of large cardinals as $k \rightarrow \infty$. This kind of thing is very familiar in combinatorial set theory - although there are serious simplifying details here.
MAXIMAL EMULATION SHIFT. MES. For (finite) subsets of $Q[0,k]^k$, some maximal emulation has for all $p < 1$, $S(p,1,...,k-1) \leftrightarrow S(p,2,...,k)$.

It is easily seen that both forms of MES are equivalent (provably in RCA$_0$). It is easily seen, via Gödel's Completeness Theorem, that MES is provably equivalent, over WKL$_0$, to a $\Pi^0_1$ sentence, and therefore is PROVABLY FALSIFIABLE.

It is easy to see that every subset of $Q^k$ has a maximal emulation. This is provable in RCA$_0$.

*supporting definitions*

$Q(a,b]$ is the set of all rationals in the interval $[a,b]$.

$S$ is an emulation of $E \subseteq Q[0,k]^k$ if and only if $S \subseteq Q[0,k]^k$, and every $x \in S^2$ is order equivalent to some $y \in E^2$.

Maximal refers to maximality under inclusion.

THEOREM 1.1. MES is provably equivalent to Con(SRP) over WKL$_0$. This result hold even if we require that $S$ be recursive in 0'.

2. INDUCTIVE EQUATION THEORY/INFINITE/SRP

INDUCTIVE UPPER SHIFT. IUS. Every order invariant $R \subseteq Q^{2k}$ has some $S = S\#\setminus R[S] \supseteq \text{ush}(S)$.

*supporting definitions*

Fix $R \subseteq Q^{2k}$. $R$ is order invariant if and only if for all order equivalent $x,y \in Q^{2k}$, $x \in R \iff y \in R$. $R[S]$ is the upper image of $R$ on $S$, which is $\{y \in Q^k: (\exists x \in S)(\max(x) < \max(y) \land x \in R y)\}$. The notation $R[S]$ requires that $S \subseteq Q^k$.

Let $S \subseteq Q^k$, $S\#$ is the least set $E^k \supseteq S \cup \{(0,...,0)\}$. The upper shift of $S$, $\text{ush}(S)$, results from adding 1 to all nonnegative coordinates of $S$.

THEOREM 3.1. IUS is provably equivalent to Con(SRP) over WKL$_0$. This holds for fixed small dimension $k$. These results hold even if we require that $S,S\#$ be recursive in 0'.
3. SET EQUATION THEORY/FINITE/SRP

FINITE INDUCTIVE UPPER SHIFT. FIUS. Every order invariant \( R \subseteq Q^{2k} \) has some nonempty finite \( S_1 \subseteq S_2 \subseteq \ldots \subseteq S_k \subseteq Q^k \), where each \( S_{i+1} = S_{i+1} \setminus R \subseteq S_{i+2} \supseteq \text{ush}(S_i) \).

FIUS is explicitly \( \Pi^0_2 \). There is an obvious double exponential bound on the numerators and denominators used in \( S_k \) as a function of \( k \). This puts FIUS in explicitly \( \Pi^0_1 \) form.

THEOREM 3.1. FIUS is provably equivalent to Con(SRP) over EFA.

4. SET EQUATION THEORY/INFINITE/HUGE

INTERNAL INDUCTIVE UPPER SHIFT. IIUS. Every order invariant \( R \subseteq Q^{2k} \) has some \( S = S \subseteq S \# \setminus R \subseteq S \) that strongly contains its upper shift.

*supporting definitions*

Let \( A, B \subseteq Q^k \). \( A =_\leq B \) if and only if \( A, B \) have the same elements with increasing (\( \leq \)) coordinates. The lower sections of \( A \) are the sets \( A^p = \{ x \in Q^{k-1} : \max(x) < p \land (p, x) \in A \} \).

\( A \) strongly contains \( B \) if and only if \( A \supset B \) and every lower section of \( B \) is a lower section of \( A \).

THEOREM 4.1. IIUS is provably equivalent to Con(HUGE) over WKL_0. This holds even for fixed small dimension \( k \). These results hold even if we require that \( S \) be recursive in \( 0' \).

5. SET EQUATION THEORY/FINITE/HUGE

FINITE INTERNAL INDUCTIVE UPPER SHIFT. FIIUS. Every order invariant \( R \subseteq Q^{2k} \) has some nonempty finite \( S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k \), where for \( 1 \leq i \leq j \leq k \), \( S_i =_\leq S_i \# \setminus R \subseteq S_{i+1} \) and \( \text{ush}(S_j) * i = S_j * (i + (1/2)) \).

FIIUS is explicitly \( \Pi^0_2 \). There is an obvious double exponential bound on the numerators and denominators used in \( S_k \) as a function of \( k \). This puts FIIUS in explicitly \( \Pi^0_1 \) form.
THEOREM 5.1. FIIUS is provably equivalent to Con(HUGE) over EFA.