CONCEPT CALCULUS: universes

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Abstract. This is the second paper on Concept Calculus, following the first paper that treats Better Than and Much Better Than. Concept Calculus investigates basic formal systems arising from informal commonsense thinking, establishing interpretability relations with a range of formal systems from mathematical logic. Here we focus on commonsense notions of universe, plenitude, and explosion. Interpretations with PA, Z_2, Z_3, ..., ZF, and various large cardinal axioms are presented.

1. Concept calculus.

The initial publication on concept calculus has appeared in [Fr11]. This is the second publication on concept calculus.

Concept calculus is an open ended investigation that seeks to connect commonsense thinking and mathematical thinking in a rigorous way. In concept calculus, we focus on informally described notions of general familiarity. We isolate simple fundamental principles that generate simple formal systems which are then shown to be mutually interpretable with familiar formal systems of mathematical logic. For a discussion of interpretations, following Alfred Tarski, see [Fr07].

In [Fr11], we investigated a number of systems based on the two commonsense partial orderings of better than and much better than, capitalization on subtle interactions. We showed that these systems are mutually interpretable with the systems Z (Zermelo set theory) and ZF (Zermelo Frankel set theory).

Here we consider the informal idea of a physical universe. We use two driving commonsense ideas. One is a principal of plenitude, asserting that if you go out far enough, any situation will be found. The second is that the universe can explode, suddenly creating more space, not only at the outer end, but possibly in the middle or in the front.

We shall see that the simple idea of plenitude generates comprehension axioms, whereas the simple idea of an exploding universe of various kinds generates power sets, replacement, and elementary embeddings.

The axiom of choice turns out to be foreign in this context, as it was in [Fr11]. Therefore, the corresponding formal set theories are missing AxC. This creates some unresolved difficulties with the systems involving elementary embeddings. It is not clear how strong there are, although it is well known that they are considerably stronger than the existence of measurable cardinals. See section 7.

We have based the investigation thus far on the particularly naïve but particularly manageable structure of a linear ordering.

Thus an SOU (second order universe) is a triple U = (S,<,F), where

i. (S,<) is a linear ordering, F:S → S, and S has at least 3 elements.
ii. every S bounded subset of S is F[x,y], for some x,y ∈ S.

Condition ii is called second order plenitude.

An FOU (first order universe) is a triple U = (S,<,F), where

i. (S,<) is a linear ordering, F:S → S, and S has at least 3 elements.
ii. every U definable S bounded subset of S is F[x,y], for some x,y ∈ S.

Condition ii is called first order plenitude.

We show that every FOU is infinite. The condition $|S| \geq 3$ is needed to avoid the trivialities $(\emptyset,\emptyset,\emptyset)$ and $(\{1,2\},<,\text{identity})$.

The triples (S,<,F) are a particularly simple and convenient representation of universes. We can view F as responsive to the following informal idea. We use the points of S as points in space, but also as magnitudes in two senses. One corresponds to the "distance" from the beginning (although we do not assume that there is an actual first point). The other corresponds to a measurement of a physical quantity at any point in space. Thus F(x) is the, say, "temperature" at $x \in S$.

As in [Fr11], we emphasize a minimalist approach. Generally, we seek the weakest and simplest axiomatizations that generate the greatest logical strength. Accordingly, we use only one plenitude axiom for universes (in both first order and second order formulations). It addresses only one basic aspect of plenitude that is particularly easy to formulate.

Specifically, any upper bounded subset of S can be realized as the set of values of F on some closed interval with endpoints from S. In the first order formulation, we use definable subsets of S, and in the second order formulation, we use all subsets of S.

In this way, the first order formulation are approximations to the second order formulation. However, the second order formulation does not directly constitute a formalism for reasoning. For that, second order formulations need to have an additional, ultimately first order apparatus, addressing "arbitrary set" or "arbitrary property", which again constitutes an approximation to the general notion.

Obviously, there is more structure to F than just its range of values on intervals. There is of course the order in which these values appear, and even various interactions. These kind of features are not readily formalizable without bringing in, for example, translation structure. This is beyond the scope of the paper.

What if we use the unrestricted second order plenitude principle "every subset of S is of the form F[x,y], $x,y \in$

S"? Let use call these unrestricted SOU's, and unrestricted SOU's.

Note that (∅,∅,∅) does not satisfy the unrestricted second order plenitude principle. But {{1,2],<,identity) does.

Unrestricted SOU's can be dispensed with by the following very easy argument.

THEOREM 1.1. (ZFC). There is no unrestricted SOU.

Proof: Let (S,<,F) be an unrestricted SOU. First assume that S is finite. We are assuming that S has at least three elements. Since S = F[x,y], for some x,y ∈ S, we see that F is a bijection. Let a < b < c be the first three elements. We see that {a,c} cannot be so represented.

Now suppose S is infinite. There is a surjection from S^2 onto ℘(S). Using the axiom of choice, we have $|S^2| = |S|$, and so we have a surjection from S onto ℘(S). This is impossible. QED

But can the above be proved in ZF? Yes. In fact, in Lemma 3.4 vi, we prove that there are no unrestricted FOU, without using the axiom of choice.

Theorem 1.1 suggests a purely set theoretic question which is not specifically about universes. The following is left to the reader.

THEOREM 1.2. (ZFC). Let S be a set. The following are equivalent.
i. There is a surjection from S^2 onto ℘(S).
ii. S has exactly 2,3, or 4 elements.

But can Theorem 1.2 be proved in ZF?

We will use my base theory RCA_0 for Reverse Mathematics, extensively as a base theory here. See [Si99], p. 23.

We will also use EFA = exponential function arithmetic = $I\Sigma_0(\exp)$, as a base theory for all claims of interpretability between theories. This is convenient, since the theories are all axiomatized by schemes. See [HP98], p. 272.

The system ZFWO\P is a very useful base theory for set theoretic statements. This is ZF without the power set axiom, plus "every set can be well ordered'. We use it for almost all results concerning the second order formulations.

We use Z for Zermelo set theory and ZF for Zermelo Frankel set theory.

We use the systems Z_n, $n \geq 2$, of n-th order arithmetic. By convention we set Z_1 to be PA (Peano arithmetic).

Zn, $n \geq 2$, has n sorts, 1,...,n. We use = on each sort. We use 0,SC,+,• on sort 1 (here SC is read "successor"). We use epsilon relations $\in_1,\ldots,\in_{n-1}$, where $\in_i$ is between sort i and sort i+1.

The nonlogical axioms of Z_n are as follows.

i. The usual quantifier free axioms for 0,SC,+,•.
ii. $(\forall x(x \in_i y \leftrightarrow x \in_i z) \rightarrow y = z$, where x is of sort i and y,z are of sort i+1.
ii. Induction in the form $0 \in_1 x \wedge (\forall n)(n \in_1 x \rightarrow SC(n) \in_1 x) \rightarrow n \in_1 x$, where n is of sort 1 and x is of sort 2.
iii. $(\exists x)(\forall y)(y \in_i x \leftrightarrow \varphi)$, where x is of sort i+1, y is of sort i, and φ is a formula in the language of Z_n in which x is not free.

For standard definitions of strongly inaccessible, indescribable, measurable, extendible, supercompact, cardinals, and critical points of elementary embeddings, see [Ka94].

2. Universes and explosions.

We model a universe as follows. We take space to be a nonempty linear ordering. At each point $x \in S$, we have a "physical" quantity $F(x) \in S$. Thus the "physical" scale used for the quantities is the same as the scale used for space.

DEFINITIOIN 2.1. Let (S,<) be a linear ordering. For $A \subseteq S^n$, we write $x > A$ ($x \geq A$, $x < A$, $x \leq A$) if and only if for all $y \in A$, $x > \max(y)$ ($x \geq \max(y)$, $x < \max(y)$, $x \leq \max(y)$). For $x \in S^n$, $y \in S^m$, we write $x > y$ ($x \geq y$, $x < y$, $x \leq y$) for $\max(x) > \max(y)$ ($\max(x) \geq \max(y)$, $\max(x) < \max(y)$, $\max(x) \leq \max(y)$). For $A,B \subseteq S^n$, we write $A > B$ ($A \geq B$, $A < B$, $A \leq B$) if and only if for all $x \in A$, $y \in B$, $\max(x) > \max(y)$ ($\max(x) \geq \max(y)$, $\max(x) < \max(y)$, $\max(x) \leq \max(y)$). We say that $A \subseteq S^n$ is S bounded if and only if $(\exists x \in S)(x \geq A)$. For

$A \subseteq S^n$, we define fld(A) to be the set of all coordinates of elements of A. We define $[x,y] = [z: x \le z \le y\}$, $(-\infty,y] = \{z: z \le y\}$. For $F:S \to S$ and $x,y \in S$, we write $F[x,y] = \{F(z): x \le z \le y\}$.

DEFINITION 2.2. An SOU (second order universe) is a triple U = (S,<,F), where
i. (S,<) is a linear ordering, $F:S \to S$, and S has at least 3 elements.
ii. every S bounded subset of S is F[x,y], for some $x,y \in S$.
Condition ii is called second order plenitude.

Note that second order plenitude only partially reflects the idea that if we go far enough out, any imaginable local pattern emerges. Consideration of stronger principles of plenitude are beyond the scope of this paper.

We impose the condition $|S| \ge 3$ in order to avoid the trivial triples $(\emptyset,\emptyset,\emptyset)$ and ({1,2},<,identity), which obey second order plenitude.

This limited second order formulation does have a suitably powerful first order formulation, which we present now. Throughout the paper, we follow the usual convention that definability always allows parameters, and 0-definability does not.

DEFINITIONI 2.3. An FOU (first order universe) is a triple U = (S,<,F), where
i. (S,<) is a linear ordering, $F:S \to S$, and S has at least 3 elements.
ii. every U definable S bounded subset of S is F[x,y], for some $x,y \in S$.
Condition ii is called first order plenitude.

We also use FOU (SOU) to refer to the corresponding first (second) order theory in <,F.

The focus of this investigation is on SOU and FOU explosions, as well as SOU and FOU explosion sequences. These "explosions" are loosely motivated by informal accounts of the big bang and cosmological inflation, whereby additional space is created.

DEFINITION 2.4. Let triples U = (S,<,F), U' = (S',<',F') be given. We define $U \subseteq U'$ if and only if $S \subseteq S'$, $< \subseteq <'$, $F \subseteq$

F'. We say that U,U' is an SOU explosion if and only if U,U' are SOU's with U ⊆≠ U' and (∃x ∈ S')(x > S).

FOU explosions are required to obey an additional first order plenitude condition.

DEFINITION 2.5. We say that U,U' is an FOU explosion if and only if U,U' are FOU's with U ⊆ U', (∃x ∈ S')(x > S), and where two strengthened plenitude conditions hold.
i. every (S',<',F',S) definable S bounded subset of S is F[x,y], for some x,y ∈ S.
ii. every (S',<',F',S) definable S' bounded subset of S' is F'[x,y], for some x,y ∈ S'.

DEFINITION 2.6. We say that U,U' is an outer SOU (FOU) explosion if and only if U,U' is an SOU (FOU) explosion, where S < S'\S.

Thus in an explosion, new space is added at the end. In an outer explosion, new space is added at the end only.

We write FOU[exp], FOU[out-exp] for the first order theories corresponding to an FOU explosion, outer FOU explosion, respectively.

How do things change for points in U, after U explodes to U'?

DEFINITION 2.7. We say that U,U' is a conservative SOU (FOU) explosion if and only if U,U' is an SOU (FOU) explosion such that the following holds. Let φ be a second order (first order) formula in the language of universes, with free variables among $v_1,...,v_k$. For all $x_1,...,x_k$ ∈ S, U |= φ[$x_1,...,x_k$] ↔ U' |= φ[$x_1,...,x_k$].

The above equivalence is the so called second order (first order) elementary substructure condition from model theory. Thus in a conservative explosion, properties are conserved.

We write FOU[con-exp] for the first order theory corresponding to a conservative FOU explosion. We also consider outer conservative explosions, and write FOU(out-con-exp] for the corresponding first order theory.

We also consider explosion series.

DEFINITION 2.8. An SOU (outer SOU, conservative SOU, outer conservative SOU) explosion series is a sequence of SOU's

$U_1,\ldots,U_n$, $n \geq 1$, such that for all $1 \leq i < n$, U_i,U_{i+1} is an SOU (outer SOU, conservative SOU, outer conservative SOU) explosion. An FOU (outer FOU, conservative FOU, outer conservative FOU) explosion series is a sequence of FOU's $U_1,\ldots,U_n$, $n \geq 1$, such that for all $1 \leq i < n$, U_i,U_{i+1} is an FOU (outer FOU, conservative FOU, outer conservative FOU) explosion.

We write FOU[n-exp], FOU[n-out-exp], FOU[n-con-exp], FOU[n-out-con-exp], for the first order theories corresponding to an FOU explosion of length n, outer FOU explosion of length n, conservative FOU explosion of length n, outer conservative FOU explosion of length n, respectively.

We now introduce a more powerful kind of (two stage) explosion involving three universes. This is the most powerful kind of explosion considered in this paper.

DEFINITION 2.9. We say that U_1,U_2,U_3 is a super SOU (FOU) explosion if and only if
i. U_1,U_2,U_3 is a conservative SOU (FOU) explosion series.
ii. $S_1 < \min(S_3/S_1) < S_2\backslash S_1$.

It follows easily that U_1,U_2, and U_1,U_3 are outer SOU (FOU) explosions, and U_2,U_3 is not an outer SOU (FOU) explosion.

We write FOU[sup-exp] for the first order theory corresponding to a super FOU explosion.

As promised in section 1, we refute unrestricted plenitude in first order form.

THEOREM 2.1. There is no $(S,<,F)$, where $(S,<)$ is a linear ordering with at least 3 elements, $F:S \to S$, and every definable $A \subseteq S$ is of the form $F[x,y]$, $x,y \in S$.

Proof:

3. FOU, SOU.

THEOREM 3.1. (RCA_0). There exists an SOU.

Proof: Let $U = (\omega,<,F)$, where $<$ is the usual linear ordering on ω, and F is defined as follows. Enumerate the finite subsets of ω, and make them, successively, the range of F on successively higher finite intervals in an obvious inductive way. Then second order plenitude is satisfied.
QED

The following set theoretic fact is well known.

LEMMA 3.2. (ZFWO\P). Let (A,<) be a linear ordering with no greatest element. Every cardinality less than that of A is the cardinality of a bounded subset of A.

Proof: Let A be as given, and let B have lesser cardinality. We can assume B $\subseteq$ A. If B is bounded in A, then we are done. Otherwise, A is the union of proper initial segments determined by elements of B, which is a union indexed by elements of B. Hence at least one term has cardinality greater than that of B. Take a subset of cardinality exactly that of B. QED

DEFINITION 3.1. S has property *) if and only if S is infinite, and the power set of every subset of S of smaller cardinality than S exists and has cardinality at most that of S.

Note that ZFWO\P + CH (continuum hypothesis) implies that V(ω+1) and $\mathfrak{R}$ have property *). ZFWI\P + GCH (generalized continuum hypothesis) implies that every infinite set has property *).

THEOREM 3.3. (ZFWO\P). Let S be given. The following are equivalent.
i. S is the domain of an SOU.
ii. S is the domain of a well ordered SOU of type |S|.
iii. S has property *).

Proof: It suffices to show i $\rightarrow$ iii $\rightarrow$ ii. For i $\rightarrow$ iii, let U = (S,<,F) be an SOU. By Lemma 3.4 below, there is no greatest element (this has a more direct proof for SOU's). By second order plenitude, every S bounded set is of the form F[x,y], x,y $\in$ S. Hence for every S bounded set, the cardinality of its power set is at most that of S. By Lemma 3.2, the S bounded set can have any cardinality less than that of S.

For iii $\rightarrow$ ii, let S have property *). Write |S| = κ, and identify S with κ. The power set of every bounded subset of κ exists, and so by Replacement, the set of all bounded subsets of κ exists. Let A_α, $\alpha < \kappa$, enumerate the bounded subsets of κ with type <, so that each $A_\alpha \subseteq \alpha$. We define U = ($\kappa$,<,F), where F:$\kappa \rightarrow \kappa$ is constructed as follows.

We successively arrange that each A_α is some F[x,y], $\alpha \leq x,y < \kappa$, where the interval [x,y] has length α. By transfinite arithmetic, we will eventually handle every A_α, $\alpha < \kappa$. QED

LEMMA 3.4. (EFA).
i. for all n, FOU proves that there exists at least n elements.
ii. there is a 0-definable partial $f:S \to S^2$ such that FOU proves $rng(f) = \{(x,y) \in S^2: x < y\}$.
iii. there is a 0-definable map $g:S \to S^2$ such that FOU proves g is surjective.
iv. Let $n \geq 1$. There is a 0-definable map $g_n:S \to S^n$, where FOU proves that g_s is surjective.
v. FOU proves that there is no greatest element.
vi. There is a 0-definable set $A \subseteq S$ such that FOU proves that A is not of the form F[x,y], $x,y \in S$.

Proof: Suppose $|S| = n$. Recall that $n \geq 3$. The number of nonempty subsets of S is 2^n-1, and the number of pairs $x \leq y$ is $C(n,2)+n = (n^2+n)/2$. We claim $2^n-1 > (n^2+n)/2$. It suffices to show that $2^{n+1} > n^2+n+2$. This is easily shown by external induction on $n \geq 3$.

For ii, define $\alpha(x) = F[x,y]$, where $|F[x,y]| = 2$, if $(\exists y)(|F[x,y]| = 2)$; undefined otherwise. The choice of y may not be unique, but the choice of F[x,y] is unique (if it exists). Define $f(x) = \alpha(x)$ in increasing order.

For iii, let $2 \leq i \leq 7$. Define $\alpha_i:rng(f) \to S$ by $\alpha_i(x,y) = \min(F[x,y])$ if $|F[x,y]| = i$ and i is even; $\max(F[x,y])$ if $|F(x,y)| = i$ and i is odd. For $1 \leq i \leq 3$, define $\beta_i(x,y) = \alpha_{2i} \cup \alpha_{2i+1}$. Then $\beta_1,\beta_2,\beta_3:rng(f) \to S$ are partial functions with disjoint domains and range S. Define the partial functions $\gamma_1,\gamma_2,\gamma_3:S \to S$, by $\gamma_i(x) = \beta_i(f(x))$. Then $\gamma_1,\gamma_2,\gamma_3$ have disjoint domains and range S.

Define $g:S \to S^2$ by $g(x) = f(\gamma_1(x))$ if $x \in dom(\gamma_1)$; reverse $f(\gamma_2(x))$ if $x \in dom(\gamma_2)$; $(\gamma_3(x),\gamma_3(x))$ if $x \in dom(\gamma_3)$; (x,x) otherwise. Then g is as required.

For iv, let $n \geq 2$. $g_n:S \to S^n$ is defined as follows. $g_1:S \to S$ is the identity function. $g_2 = g$. $g_{n+1}(x) = (g(x)_1,g_n(g(x)_2))$, where $g(x) = ((g(x))_1,(g(x))_2)$.

For v, suppose there is a greatest element. Then first order plenitude applies to all definable subsets of S. Let $A = \{x: x \notin F[y,z]$, where $g(x) = (y,z)\}$. By first order plenitude, let A = F[b,c]. By iii, let g(d) = (b,c). Hence d ∈ A ↔ d ∉ F[b,c] ↔ d ∉ A, which is a contradiction.

For vi, use the same 0-definable A = {x: x ∉ F[y,z], where g(x) = (y,z)}, and repeat the argument in the previous paragraph. QED

Note that Lemma 3.4 vi shows that there is no unrestricted FOU as discussed in the material surrounding Theorem 1.1.

LEMMA 3.5. (EFA). FOU is interpretable in PA.

Proof: Let M be a model of PA. We build an FOU U = (S,<,F) as follows.

We take S = dom(M) and < to be the less than relation of M. In M, we enumerate the M finite subsets of S along S. We take each successive M finite subset of S and make it the image of F on successively higher intervals with endpoints from dom(M), in an obvious M inductive way. We then extend F to be min(S) where we have not yet defined F. Set U = (S,<,F).

Every U set is M definable, and therefore M finite. Hence by construction, every U set is the image of F on some interval with endpoints from S. This verifies first order plenitude. QED

We now show that PA is interpretable in FOU.

Let U = (S,<,F) be an FOU. We build a model M = (S,≡,0,SC,ADD,MULT) of PA out of U, in a manner suitable for establishing that FOU is interpretable in PA. Here ≡ is the equality relation, which, in interpretations, is not required to be the identity. Also 0 is an equivalence class of ≡, SC (successor) is a binary relation that respects ≡, and ADD,MULT are ternary relations that respect ≡. Here SC,ADD,MULT are functions when we factor out by ≡.

DEFINITION 3.2. Let U = (S,<,F). We say that A ⊆ S is a U set if and only if A is of the form F[x,y], x,y ∈ S, and A is S bounded. We fix the surjective g_n, n ≥ 2, given by Lemma 3.2. Take g_1 to be the identity on S.

Note that we only define the U subsets of S. U subsets of S^n are defined in section 5, where we have enough strength to take advantage of this extended definition. However, we will define the U finite subsets of S^n in Definition 3.4.

LEMMA 3.6. (FOU). Every definable subset of S that is S bounded, is a U set.

Proof: This is formulated as a scheme. It follows immediately from first order plenitude. QED

DEFINITION 3.3. We say that $A \subseteq S$ is U finite if and only if A is a U set, where every nonempty U set contained in A has a least and a greatest element. Let $A \subseteq S$. We define $x <_{A,n} y$ if and only if $x,y \in A^n$ are distinct, and for the least i such that $x_i \neq y_i$, we have $x_i < y_i$.

LEMMA 3.7. (FOU). Let A be a U finite subset of S. Every nonempty definable subset of A^n has a $<_{A,n}$ least and a $<_{A,n}$ greatest element. Let $R \subseteq S^{n+m}$ be U definable. $(\forall x \in A^n)(\exists y \in S^m)(R(x,y)) \rightarrow (\exists z \in S)(\forall x \in A^n)(\exists y < z)(R(x,y))$. $\{x\}$, $x \in S$, is U finite. Any U set contained in a U finite subset of S is U finite. The union of two U finite subsets of S is U finite.

Proof: Let A be as given. The claims are formulated as schemes. Use external recursion on 1,...,n. The first claim supports internal induction arguments along $<_{A,n}$. The second claim is proved using such an internal induction argument. The remaining claims are left to the reader. QED

Note that Lemma 3.7 supports internal lexicographic induction on the elements of A^n, where $A \subseteq S$ is U finite.

DEFINITION 3.4. We say that $A \subseteq S^n$ is U finite if and only if $A = g_n[B]$ for some U finite $B \subseteq S$.

LEMMA 3.8. (FOU). A definable subset of S^n is U finite and only if its field is U finite. $\{x\}$, $x \in S^n$, is U finite.

Proof: This is formulated as a scheme. Let $A \subseteq S^n$ be definable. First suppose A is U finite. Let $A = g_n[B]$, where $B \subseteq S$ is U finite. Show by internal induction on B that fld(A) is U finite. $\{x\}$, $x \in S^n$, is U finite. Any U set contained in a U finite subset of S^n is U finite. The union of two U finite subsets of S^n is U finite.

Now suppose fld(A) is U finite. Show by internal lexicographic induction on A^n that A^n is U finite.

The remaining claims are left to the reader. QED

DEFINITION 3.5. For $A \subseteq S^n$, $x \in S$, write $A|<x = \{y \in A: y < x\}$.

LEMMA 3.9. (FOU). Let $A,B \subseteq S$ be U finite. Exactly one of the following holds.
i. There is a U finite order preserving bijection from A onto B.
ii. There is a U finite order preserving bijection from A onto some $B|<x$, $x \in B$.
iii. There is a U finite order preserving bijection from B onto some $A|<y$, $y \in A$.
In case i, the bijection is unique. In case ii, the x and the bijection is unique. In case iii, the y and the bijection is unique.

Proof: By internal induction arguments. QED

DEFINITION 3.6. $A \sim B$ if and only if $A,B \subseteq S$ are U finite and there exists a U finite order preserving bijection from A onto B. SC(A,B) if and only if there exists $x \in S\backslash A$ such that $A \cup \{x\} \sim B$. ADD(A,B,C) if and only if there is a U finite $D \subseteq S\backslash A$ such that $D \sim B$ and $A \cup D \sim C$. MULT(A,B,C) if and only if there is a finite bijection from $A \times B$ onto C.

LEMMA 3.10. (FOU). $\sim$ is an equivalence relation on the U finite $A \subseteq S$. $SC(A,B) \rightarrow A,B \subseteq S$ are U finite. $ADD(A,B,C) \vee MULT(A,B,C) \rightarrow A,B,C \subseteq S$ are U finite. SC, ADD, MULT respect $\sim$. $SC(A,B) \wedge SC(A,B') \rightarrow B \sim B'$. $ADD(A,B,C) \wedge ADD(A,B,C') \rightarrow C \sim C'$. $MULT(A,B,C) \wedge MULT(A,B,C') \rightarrow C \sim C'$. $(\forall$ U finite $A)(\exists B)(SC(A,B))$. $(\forall$U finite $A,B)(\exists$U finite $C,D)(ADD(A,B,C) \wedge MULT(A,B,D))$.

Proof: Left to the reader. QED

LEMMA 3.11. (FOU). $\neg SC(A,\emptyset)$. $SC(A,B) \wedge SC(C,B) \rightarrow A \sim C$. $ADD(A,\emptyset,B) \leftrightarrow A \sim B$. $MULT(A,\emptyset,B) \leftrightarrow B = \emptyset$. $ADD(A,B,C) \wedge SC(B,B') \wedge SC(C,C') \rightarrow ADD(A,B',C')$. $MULT(A,B,C) \wedge SC(B,B') \wedge ADD(C,A,D) \rightarrow MULT(A,B',D)$.

Proof: Left to the reader. QED

DEFINITIOIN 3.7. For each $x \in S$, let SET(x) be $F[g_2(x)]$. Let $D = \{x \in S$: SET(x) is a U finite subset of $S\}$. $x \equiv y \leftrightarrow x,y \in D \wedge SET(x) \sim SET(y)$. $0 = \{x: SET(x) = \emptyset\}$. SC(x,y) if and only if SC(SET(x),SET(y)). ADD(x,y,z) if and only if ADD(SET(x),SET(y),SET(z)). MULT(x,y,z) if and only if MULT(SET(x),SET(y),SET(z)).

LEMMA 3.12. (FOU). Let $A \subseteq D$ be U definable, and respect ≡. Suppose $(\forall x)(SET(x) = \emptyset \rightarrow x \in A) \wedge (\forall x \in A)(\forall y \in D)(SC(x,y) \rightarrow y \in A)$. Then $A = D$.

Proof: This is formulated as a scheme. Let A be as given. Let $x \in D$. Let $b \in SET(x)$ be greatest such that the $y \in D$ with $SET(y) \sim SET(x) \cap (-\infty,b]$ lie in A. By internal induction, $b = max(x)$. Hence the $y \in D$ with $SET(y) = SET(x)$ lie in A. Therefore $x \in A$, as required. QED

LEMMA 3.13. (FOU). (S,≡,0,SC,ADD,MULT) satisfies each axiom of PA.

Proof: By Lemmas 3.10 - 3.12. QED

THEOREM 3.14. (EFA). FOU and PA are mutually interpretable.

Proof: By Lemmas 3.5 and 3.13. QED

4. FOU, SOU explosions.

THEOREM 4.1. (ZFWO\P). Let $S \subseteq S'$. The following are equivalent.
i. There is an SOU explosion with domains S,S'.
ii. There is an outer SOU explosion with domains S,S'.
iii. S,S' have property *) and $|S| < |S'|$.

Proof: Let $S \subseteq S'$. Let U,U' be an SOU explosion with domains S,S'. By Theorem 3.3, S,S' have property *). Since S is S' bounded, $|\wp(S)| \leq |S'|$. Hence $|S| < |S'|$.

Now suppose S,S' have property *) and $|S| < |S'|$. By Theorem 3.3, let $U = (S,<,F)$ be a well ordered SOU with domain S. Extend (S,<) by a well ordering of S'\S on top, which must have cardinality that of S'. Then proceed as in the proof of Theorem 3.3, by extending F to F'. QED

THEOREM 4.2. (ZFWO\P). There is an (outer) SOU explosion if and only if there is an uncountable set with property *). If $V(\omega+\omega)$ exists then there is an (outer) SOU explosion. If there is an SOU explosion then $V(\omega+1)$ exists. If CH then there is an (outer) SOU explosion if and only if $V(\omega+1)$ exists.

Proof: Let U,U' be an SOU explosion. By Theorem 3.3, S,S' have property *). Since S is infinite and S' bounded, S' must be uncountable. The converse is by Theorem 4.1.

For the second claim, note that $V(\omega+\omega)$ has property *). For the third claim, note that if there is an uncountable set with property *), then $V(\omega+1)$ exists. For the fourth claim, note that if CH then $V(\omega+1)$, if it exists, has property *). QED

DEFINITION 4.1. For $n \geq 1$, let $Z_n\backslash IND$ be Z_n with the induction axioms.

LEMMA 4.3. (EFA). Each Z_n, $n \geq 2$, is interpretable in $Z_n\backslash IND$.

Proof: Let $n \geq 2$, and $(D_1,\ldots,D_n,\in_1,\ldots,\in_{n-1},0,SC,+,\bullet)$ satisfy $Z_n\backslash IND$. We build a model $(D_1',\ldots,D_n',\in_1',\ldots,\in_{n-1}',0',SC',+',\bullet',\equiv_1,\equiv_2,\ldots,\equiv_n)$ of Z_n, where $\equiv_1,\ldots,\equiv_n$ are the equality relations on $D_1',\ldots,D_n'$, respectively.

Define D_1' to be the set of all $x \in D_1$ that are inductive, in the sense that x lies in every set of natural numbers that contains 0 and is closed under successor. Define $\equiv_1$ to be equality on D_1'. Define $D_2',\ldots,D_n' = D_2,\ldots,D_n$. Define $\in_1'$ to be $\in_1$ restricted to $D_1 \times D_2$.

We define $\equiv_i$, $\in_i$, inductively. We have already defined $\equiv_1$, $\in_1$. Suppose $\equiv_i$, $\in_{i-1}$ have been defined, $1 \leq i < n$. (We don't define $\in_0$). Define $x \equiv_{i+1} y \leftrightarrow x,y \in D_{i+1} \wedge (\forall z \in D_i)(z \in_i x \leftrightarrow z \in_i y)$. Define $x \in_i' y \leftrightarrow x \in D_i \wedge y \in D_{i+1} \wedge (\exists z)(x \equiv_i z \wedge z \in_i y)$. Define $0' = 0$.

To complete the interpretation, we need to verify, in $Z_n\backslash IND$, that the inductive natural numbers are closed under $SC,+,\bullet$, so that we can define $SC',+',\bullet'$ by restriction. Closure under SC is immediate. For +, let $A = \{x: x$ is inductive $\wedge$ ($\forall$ inductive y)(x + y is inductive)$\}$. Then A contains 0 and is closed under SC. Hence A consists of the inductive integers. For $\bullet$, let $B = \{x: x$ is inductive $\wedge$ ($\forall$ inductive y)($x \bullet y$ is inductive)$\}$. Then B contains 0 and is closed under SC. Hence B consists of the inductive integers. QED

THEOREM 4.4. (EFA). FOU[exp], FOU[out-exp], Z_2 are mutually interpretable.

Proof: We first interpret FOU[out-exp] in $Z_2 + V = L$. Let $U = (\omega,<,F)$, from the proof of Theorem 3.1. Set $U' = (\omega \cup. \wp(\omega),<_L,F)$, where F as constructed as in the proof of Theorem 3.3 (or Lemma 3.5). Since $Z_2 + V = L$ is interpretable in Z_2, we have that FOU[out-exp] is interpretable in Z_2.

We now interpret Z_2\IND in FOU[exp]. Let U,U' be an FOU explosion. Let the arithmetic part be the (S,≡,0,SC,ADD,MULT) of Definition 3.7. The sets of integers will be the F'[x,y], x,y ∈ S', that respect ≡. Now apply Lemma 4.3. QED

5. Conservative FOU, SOU explosions.

We start with outer conservative FOU explosions, and build on the development in section 3. We show that FOU[out-con-exp] is mutually interpretable with ZF.

We conjecture that FOU[con-exp] is mutually interpretable with ZF. By Theorem 4.4, Z_2 is interpretable in FOU[con-exp].

LEMMA 5.1. (EFA). If every finite fragment of FOU[out-con-exp] is interpretable in ZF, then FOU[out-con-exp] is interpretable in ZF.

Proof: This follows from the Orey compactness theorem [Or61]. To use it here, we cite the well known fact that ZF is reflexive, in the sense that it proves the consistency of each of its finite fragments. QED

LEMMA 5.2. (EFA). FOU[out-con-exp] is interpretable in ZF.

Proof: By Lemma 5.1, it suffices to show that each finite fragment of FOU[out-con-exp] is interpretable in ZF. Let T be a finite fragment of FOU[out-con-exp]. Since ZFC + V = L is interpretable in ZF, it suffices to show that T is interpretable in ZFC + V = L.

Let M be a model of ZFC + V = L. We build U,U' satisfying T as follows. Set U' = $(L,<_L,F')$, where L is the universe of M, $<_L$ is its usual definable well ordering in the sense of M, and F' is defined below.

By a straightforward transfinite recursion on $<_L$, as in the proof of Theorem 3.3, build M definable F':L → L such that $(L,<_L,F')$ is an SOU in the sense of M. Of course, we are not asserting that $(L,<_L,F')$ is an actual SOU, externally. Note that $(L,<_L,F)$ is an FOU.

For any M cardinal λ, the same M definition defines the restriction $F:L(\lambda) \to L(\lambda)$ of $F':L \to L$. such that $(L(\lambda),<_L,F)$

Now T asserts conservativity with respect to formulas $\varphi_1,\ldots,\varphi_n$ in $<,F$, with free variables collectively among $x_1,\ldots,x_k$. Clearly there are formulas $\varphi_1^*,\ldots,\varphi_n^*$ in set theory, with the same free variables, such that for all $1 \leq i \leq k$, we have, from the point of view of M,

i. $(\forall x_1,\ldots,x_k \in L)((L,<L,F') \models \varphi_i[x_1,\ldots,x_k] \leftrightarrow (L,\in) \models \varphi_i[x_1,\ldots,x_k])$.
ii. for all M cardinals λ, $(\forall x_1,\ldots,x_k \in L(\lambda))((L(\lambda),<_{L(\lambda)},F' \cap L(\lambda)) \models \varphi_i[x_1,\ldots,x_k] \leftrightarrow (L(\lambda),<_{L(\lambda)},F' \cap L(\lambda)) \models \varphi_i^*[x_1,\ldots,x_k])$.

To verify the above, we need to observe that $<_{L(\lambda)},F' \cap L(\lambda)$ are defined over $(L(\lambda),\in)$ in the same way that $<_L,F \cap L(\lambda)$ are defined over $(L,\in)$.

By reflection, let λ be any M cardinal such that

iii. $(\forall x_1,\ldots,x_k \in L(\lambda))((L,\in) \models \varphi_i^*[x_1,\ldots,x_k] \leftrightarrow (L(\lambda),\in) \models \varphi_i^*[x_1,\ldots,x_k])$.

Then

iv. $(\forall x_1,\ldots,x_k \in L(\lambda))((L,<_L,F') \models \varphi_i[x_1,\ldots,x_k] \leftrightarrow (L(\lambda),<_{L(\lambda)},F) \models \varphi_i[x_1,\ldots,x_k])$.

QED

We now show that ZF is interpretable in FOU[out-con-exp]. We start with an outer conservative explosion U,U', and define a model of ZF. We have already defined many of the concepts that we need in sections 2 and 3. In particular, Definitions 2.1, 3.2 - 3.7 should be recalled.

DEFINITION 5.1. $A \subseteq S^n$ is a U set if and only if $A = g_n[B]$ for some U set $B \subseteq S$. We use g_n' for the g_n construction applied to U'. This defines the U' sets $A \subseteq S'^n$.

LEMMA 5.3. (FOU[out-con-exp]). Let $x \in S$ and $n \geq 1$. $g_n \subseteq g_n'$. The one dimensional U' sets $A < x$ are the same as the one dimensional U sets $A < x$. Every n-dimensional U set is an n-dimensional U' set.

Proof: Let x,n be as given. $g_n \subseteq g_n'$ is by conservativity.

Let $A \subseteq S'$ be a U' set, where $A < x$. By outer, $A \subseteq S$, $A < x$. By combined first order plenitude for U, there exists $y,z \in S$ such that $A = F[y,z]$. Hence $A \subseteq S$ is a U set.

Let $A \subseteq S$ be a U set, where $A < x$. By combined first order plenitude for U', there exists $y,z \in S'$ such that $A = F'[y,z]$. Hence $A \subseteq S'$ is a U' set.

Now let $A \subseteq S^n$ be a U set, where $A < x$. Let $A = g_n[B]$, where $B \subseteq S$ is a U set. Then $B \subseteq S'$ is a U' set. Also $A = g_n'[B]$, and so $A \subseteq S'^n$ is a U' set. QED

LEMMA 5.4. (FOU[out-con-exp]). Let $R \subseteq S^{n+m}$ be U definable, and $u \in S$. $(\forall x < u)(\exists y)(R(x,y)) \rightarrow (\exists z \in S)(\forall x < u)(\exists y < z)(R(x,y))$. Every U subset of S^n is S bounded.

Proof: Let R,x be as given. Write R as a U definition with parameters from S. Let R' be defined as a U' definition with the same parameters. By outer conservativity, $R = R' \cap S^{n+m}$. We claim that by outer conservativity,

$$(\forall x < u)((\exists y \in S')(R'(x,y)) \rightarrow (\exists z \in S')(\forall x < u)(\exists y < z)(R'(x,y)).$$

To see this, assume $(\forall x < u)((\exists y \in S')(R'(x,y))$, and set $z > S$. By conservativity,

$$(\forall x < u)((\exists y \in S)(R(x,y)) \rightarrow (\exists z \in S)(\forall x < u)(\exists y < z)(R(x,y)).$$

It follows that every U definable function maps S bounded sets into S bounded sets. Then every U subset of S^n is S bounded. QED

LEMMA 5.5. (FOU[out-con-exp]). $(\forall x \in S)(\exists y \in S)$(range of g_n on $(-\infty,y]$ includes $(-\infty,x]^n$. Let $x \in S$ and $n \geq 1$. The n dimensional U' sets $< x$ are the same as the n dimensional U sets $< x$.

Proof: The first claim follows from the surjectivity of g_n and the first claim of Lemma 5.4.

Let x,n be as given. Let $A \subseteq S'^n$, $A < x$, $A = g_n'[B]$, $B \subseteq S'$, B a U' set. By outer, $A \subseteq S^n$, $A < x$. Let $y \in S$ be as given by the first claim. Let $C = \{z < y: g_n(z) \in A\}$. Then C is U,U' definable and S bounded. By combined first order plenitude for U, $C \subseteq S$ is a U set. This establishes the forward direction of the second claim. The reverse direction of the second claim follows from the first claim of Lemma 5.4. QED

DEFINITION 5.2. x U codes $y \in S^n$ if and only if $g_n(x) = y$. x U codes the U set $A \subseteq S$ if and only if $A = F[y,z]$, where $g_2(x) = (y,z)$. x U codes $A \subseteq S^n$ if and only if $A = g_n[B]$, where x U codes the U set $B \subseteq S$. These definitions are also made for U' and S'.

LEMMA 5.6. (FOU[out-con-exp]). Let $A \subseteq S^n$ be a U set and $x \in S$. x U codes A if and only if x U' codes A.

Proof: By outer conservativity. QED

LEMMA 5.7. (FOU[out-con-exp]). Every U' finite subset of S'^n is in U' one-one correspondence with some U finite subset of S^n. There exists $x \in S$ such that every U finite set is in U one-one correspondence with some U finite subset of $(-\infty,x)$ that is U coded by some $y < x$.

Proof: The first claim is proved by an internal lexicographic induction on the U' finite subset of S'^n. For the second claim, note that

$$(\exists x \in S')(\text{every U' finite set is in U' one-one correspondence with some U' finite subset of } (-\infty,x] \text{ with code} < x)$$

by using any $x \in S'\backslash S$. The claim follows by conservativity. QED

LEMMA 5.8. (FOU[out-con-exp]). Fix $n \geq 1$. For all $x \in S$ there exists $y \in S$ such that the following holds. Every element of S^n is U coded by some $z < y$. Every U subset of $(-\infty,x]^n$ is U coded by some $z < y$.

Proof: The first claim is from Lemma 5.4. For the second claim, let $x \in S$. BY Lemma 5.6, $(\exists y \in S')$(every U' subset of $(-\infty,x]^n$ is U' coded by some $z < y$), by using any $y \in S'\backslash S$. The claim follows by conservativity. QED

DEFINITION 5.3. A U system is a quadruple $J = (A,\sim,R,a)$, where

i. A is a U set, $\sim$,R are U subsets of A^2, and $a \in A$.
ii. $\sim$ is an equivalence relation on A.
iii. $x \sim x' \wedge y \sim y' \rightarrow (R(x,y) \leftrightarrow R(x',y'))$.

iv. $x,y \in A \wedge (\forall z)(R(z,x) \leftrightarrow R(z,y)) \rightarrow x \sim y$.
v. if B is a U set containing a, and all R predecessors of every one of its elements, then B = A.
vi. for all nonempty U sets B contained in A, there exists $x \in B$ such that no $y \in B$ has $R(y,x)$.

DEFINITION 5.4. Let $J = (A,\sim,R,a)$, $J' = (A',\sim',R',a')$ be U systems. A U comparison relation is a U relation $T \subseteq A \times A'$ such that for all $x \in A$ and $y \in A'$, we have
$T(x,y) \leftrightarrow (\forall z)(R(z,x) \rightarrow (\exists w)(R'(z,y) \wedge T(z,w))) \wedge (\forall z)(R'(z,y) \rightarrow (\exists w)(R(w,x) \wedge T(w,z)))$. A U comparison relation is said to be a U isomorphism if and only if every element of A is related to an element of A', and every element of A' is related to an element of A.

DEFINITION 5.5. Let $J = (A,\sim,R,a)$ be a U system. For each $b \in A\backslash\{a\}$, define $J|b = (B,\sim \cap B^2,R \cap B^2,b)$, where B is the least U subset of A containing b, and all R predecessors of every one of its elements.

LEMMA 5.9. (FOU[out-con-exp]). Let J,J' be U systems. Each J|b is a U system. Any two U systems have a unique U comparison relation. The U comparison relation respects $\sim,\sim'$. The U comparison relation is an isomorphism if and only if it relates a and a'.

Proof: To prove existence, take the union of all U isomorphisms from some J|x onto some J'|y. They cohere, using U well foundedness. QED

DEFINITION 5.6. $J \equiv J'$ if and only if the U comparison relation is a U isomorphism relation. $J \in^* J'$ if and only if J and J'|b are U isomorphic, for some b with $R(b,a')$.

DEFINITION 5.7. W is the family of all U systems. The prospective interpretation of ZF is $(W,\equiv,\in^*)$.

Officially speaking, for interpretations in FOU[out-con-exp], we are only allowed to use points, or more liberally, tuples of points of fixed length (even more liberally, tuples of points of various bounded lengths) as the elements. So instead of U systems, we officially use U codes for U systems via Definition 5.2, according to the following.

DEFINITION 5.8. A U code for a U system $J = (A,\sim,R,a)$ is a U code for a 4-tuple (x,y,z,a), where x is a U code for A, y is a U code for ~, and z is a U code for R.

Interpretations of set theory roughly like our (W,≡,∈*) have been made for various purposes. E.g., see [Fr73], [Fr13]. So we will leave many details to the reader, and focus on the main ideas.

LEMMA 5.10. (FOU[out-con-exp]). Let J = (A,~,R,a) be a U system. There is a U set B such that
i. every element of B is a U code of some J|b, b ∈ A\{a}.
ii. every J|b, b ∈ A\{a}, has a U code in B.

Proof: Let J be as given. Let u > A. Apply Lemma 5.4. QED

LEMMA 5.11. (FOU[out-con-exp]). Let A be a U set consisting of U codes for U systems. There is a U system J[A] such that (∀J ∈ W)(J ∈* J[A] ↔ J is U isomorphic to a U system U coded by an element of A).

Proof: Let A be as given. We will first give a virtual form of J[A] in which the domain is a U subset of S^2 rather than a U subset of S. We will then fix this so that we have a legitimate U system.

We disjointify the domains of the U systems coded by elements of A, by replacing the U system J coded by each x ∈ A, by the virtual U system obtained by replacing the elements u of dom(U) by the ordered pair (x,u).

Let X be the union of the domains of these virtual U systems. We put an equivalence relation $\sim_X$ on X, where y $\sim_X$ y' if and only if the virtual comparison relation between the two virtual U systems with the ordered pairs y,y' relate y and y'. Define R_X(y,z) if and only if this virtual comparison relation relates the ordered pair y to some ordered pair w, where w is related to z in the virtual U system containing w. Choose any new (u,u) and make its predecessors exactly the distinguished points in the virtual U systems.

This results in the desired U system J[A], except that J[A] is made up of ordered pairs from S instead of elements of S. Choose v ∈ S such that that all of the ordered pairs used in J[A] have a U code below v. Replace each ordered pair in the virtual J[A] by its inverse image under g_2 below v, and adjust the virtual J[A] accordingly, thickening the equivalence relations, and resulting in the desired J[A}. QED

We view Lemma 5.11 as a kind of second order separation axiom for (W,≡,∈*).

LEMMA 5.12. (FOU[out-con-exp]). (W,≡,∈*) satisfies extensionality.

Proof: Let J = (A,~,R,a), J' = (A',~',R',a'). Assume that each J|x, R(x,a), is isomorphic to some J'|y, R'(y,a'), and each J'|z, R'(z,a'), is isomorphic to some J|w, R(w,a). It is easy to see that the union of all of these isomorphisms is an isomorphism from J to J'. QED

LEMMA 5.13. (FOU[out-con-exp]). (W,≡,∈*) satisfies pairing.

Proof: This follows from Lemma 5.11. QED

LEMMA 5.14. (FOU[out-con-exp]). (W,≡,∈*) satisfies union.

Proof: Let J = (A,~,R,a). We can form a U set of U codes for the J|b, where there exists c such that R(b,c) ∧ R(c,a), using 5.11. QED

LEMMA 5.15. (FOU[out-con-exp]). (W,≡,∈*) satisfies foundation.

Proof: Let J = (A,~,R,a), where (∃x)(R(x,a)). Let x be R minimal such that R(x,a). Use J|x. QED

LEMMA 5.16. (FOU[out-con-exp]). (W,≡,∈*) satisfies separation.

Proof: Let J = (A,~,R,a), Use Lemmas 5.10, 5.11. QED

LEMMA 5.17. (FOU[out-con-exp]). (W,≡,∈*) satisfies power set.

Proof: Let J = (A,~,R,a). Let b > A. For each U set B ⊆ {x: R(x,a)}, form the U system B* = (B ∪ {b},~',R',b), where ~' = $(A \cap B^2) \cup \{(b,b)\}$, and R' = $(R \cap B^2) \cup (B \times \{b\})$. These B* comprise a complete set of representatives for the "subsets" of J, in the sense of (W,≡,∈*). Now use Lemmas 5.8, 5.11. QED

We next verify the collection axiom scheme from set theory. This follows easily from the following U collection lemma.

LEMMA 5.18. (FOU[out-con-exp]). (W,≡,∈*) satisfies collection.

Proof: Let J = (A,~,R,a) be the base for the collection. We have a U definable relation relating R predecessors of a to at least one U system. By Lemma 5.4, there exists u such that each R predecessor of a is related to some U system with code < u. Now apply Lemma 5.11 to the resulting U set of codes U < u. QED

We now come to Infinity, which we take to be $(\exists x \neq \emptyset)(\forall y \in x)(\exists z \in x)(y \in z)$.

LEMMA 5.20. (FOU[out-con-exp]). (W,≡,∈*) satisfies infinity.

Proof: Let x ∈ S be as given by Lemma 5.7. We say that y is an x-code if and only if y < x is a U code for some U finite set. Define J = (A,~,R,x) as follows. A is the set of all x-codes. y ~ z if and only if the U sets U coded by y,z are in U one-one correspondence. R(y,z) if and only if the U set U coded by y is in U one-one correspondence with a proper U subset of the U set U coded by z. Then J is the required U system. QED

LEMMA 5.21. (FOU[out-con-exp]). (W,≡,∈*) satisfies ZF.

Proof: From Lemmas 5.12 - 2.20. QED

THEOREM 5.22. (EFA). FOU[out-con-exp] is mutually interpretable with ZF and ZFC.

Proof: By Lemmas 5.2, 5.21, and that ZF, ZFC are mutually interpretable. QED

We now shift attention to outer conservative SOU explosions U,U'. We use (W,≡,∈*), (W',≡',∈*') constructed from U, U' by Definition 5.7. Here we will not take W (W') to be the set of all U (U') codes for U (U') systems, as we did, officially, right after Definition 5.7.

LEMMA 5.23. (ZFWO\P). $W \subseteq W'$, ≡ is $\equiv' \cap W^2$, $\in^* = \in^{*\prime} \cap W^2$. (W,≡,∈*) is a second order elementary substructure of (W',≡',∈*'). If x ∈* y, where y ∈ W, then x ≡' z for some z ∈ W.

Proof: The first two claims are immediate from the second (in fact first) order construction of (W,≡,∈*),(W',≡',∈*') out of W,W', conservativity, and that U is a second order elementary substructure of U'. Now let x ∈* y, y ∈ W. Then x is U' isomorphic to some y|b, b ∈ S. QED

LEMMA 5.24. (ZFWO\P). The U (U') subsets of S^n (S'^n) consist of all S (S') bounded subsets of S^n (S'^n).

Proof: By second order plenitude, the U subsets of S are the S bounded subsets of S. Let $A \subseteq S^n$, u > A. By Lemma 5.4, let $(-\infty,u]^n \subseteq g_n(-\infty,v]$. Then $A = g_n[B]$ for some $B \subseteq (-\infty,v]$. Let B = F[x,hy], x,y ∈ S. Then $A = g_n[F[x,y]]$, x,y ∈ S. Hence A is a U subset of S^n. QED

LEMMA 5.25. (ZFWO\P). (W',≡',∈*',W) is isomorphic to (V(λ),=,∈,V(κ)), for some unique strong limit cardinals κ < λ (after factoring out by ≡'). The isomorphism is unique. (V(κ),∈), (V(λ),∈) both satisfy ZF. (V(κ),∈) is a second order elementary substructure of (V(λ),∈).

Proof: (W',≡',∈*') and (W,≡,∈*) both satisfy ZF with second order separation. Hence they must be well founded, and so the internal cumulative hierarchy must match the external cumulative hierarchy as far as it goes. QED

LEMMA 5.26. (ZFWO\P). If there is a conservative SOU explosion, then there exists strong limit cardinals κ < λ such that

i. (V(κ),∈) is a second order elementary substructure of (V(λ),∈).

ii. λ is a second order indescribable cardinal in the sense of L.

Proof: Claim i is by Lemma 5.26. We show that claim ii follows from claim i. Suppose ii is false in L. Let (V(λ),∈,A) |= φ, but for all α < λ, (V(α),∈,A ∩ V(α)) |= ¬φ, where φ is a second order sentence and A ⊆ V(λ). We can assume that A is the constructibly least subset of V(λ) with this property.

Then (V(λ),∈) satisfies the second order sentence that asserts that (V(λ),∈,A) |= φ, for this specific A ⊆ V(λ). Hence by i, (V(κ),∈) satisfies that (V(κ),∈,A') |= φ, for the specific A' ⊆ V(κ) defined in the same way. Also by i, A' = A ∩ V(κ). Hence (V(κ),∈,A ∩ V(κ)) |= φ, contradicting the choice of A. QED

LEMMA 5.27. (ZFWO\P). If there is a Π^3_1 indescribable cardinal, then there is an outer conservative SOU explosion.

Proof: Let κ be a Π^3_1 indescribable cardinal. Then κ is strongly inaccessible, and so we can build a well ordered SOU $(V(\kappa),<,F)$ of type κ, according to Theorem 3.3. Let A be the set of all finite sequences $(x_1,\ldots,x_n,r)$, where $x_1,\ldots,x_n \in V(\kappa)$, r is the Gödel number of some second order formula φ of set theory with free variables among $v_1,\ldots,v_n$, and $(V(\kappa),\in) \models \varphi[x_1,\ldots,x_n]$.

Clearly $(V(\kappa),<,F,A)) \models (V(\kappa),<,F \cap V(\kappa))$ is an SOU $\wedge$ $A \cap V(\kappa)$ is as constructed $\wedge$ κ is a limit ordinal. This conjunction can be expressed as a Π^3_1 sentence. By Π^3_1 indescribability, let $\alpha < \kappa$ be such that $(V(\alpha),<,F \cap V(\alpha),A \cap V(\alpha)) \models (V(\alpha),<,F \cap V(\alpha))$ is an SOU $\wedge$ $A \cap V(\alpha)$ is as constructed $\wedge$ α is a limit ordinal. Set $U = (V(\alpha),<,F \cap V(\alpha))$ and $U' = (V(\kappa),<,F)$. Then U,U' is a conservative SOU explosion. QED

THEOREM 5.28. (ZFWO\P). If there is a Π^3_1 indescribable cardinal then there is an outer conservative SOU explosion. If there is an outer conservative SOU explosion then there are strongly inaccessible cardinals κ,λ such that $V(\kappa),\in)$ is a second order elementary substructure of $V(\lambda,\in)$, and λ is a second order indescribable cardinal in the sense of L. Assume V = L. The existence of a Π^3_1 indescribable cardinal implies the existence of an outer conservative SOU explosion implies the existence of a second order indescribable cardinal.

Proof: By Lemmas 5.26, 5.27. QED

6. FOU, SOU explosion series.

THEOREM 6.1. (ZFWO\P). Let $n \geq 1$. There is an SOU explosion series of length n if and only if there exists an outer SOU explosion series of length n if and only if there exists $S_1,\ldots,S_n$ with property *) of strictly increasing cardinality. If $V(\omega \bullet n)$ exists then there is an outer SOU explosion series of length n. If there is an SOU explosion series of length n then $V(\omega+n-1)$ exists. If GCH then there is an SOU explosion series of length n if and only if there exists an outer SOU explosion series of length n if and only if $V(\omega+n-1)$ exists.

Proof: For the first claim, argue as in the proof of Theorem 4.2 using Theorem 4.1. For the second claim, note that the $V(\omega \bullet n)$ have property *). and strictly increasing cardinality. For the third claim, use the first claim. For the fourth claim, use the first claim. QED

LEMMA 6.2. (EFA). Each FOU[n-out-exp], $n \geq 1$, is interpretable in Z_n.

Proof: For the case $n = 1$, FOU[n-out-exp] is the same as FOU[out-exp]. So we use Theorem 4.4. Now let $n \geq 2$. The theory $Z_n + V = L$ makes perfectly good sense, and is interpretable in Z_n. Now adapt the proof of Theorem 6.1 with GCH. QED

DEFINITION 6.1. A strong (outer) FOU explosion series of length n is a (outer) FOU explosion series $U_1,...,U_n$, where first order plenitude is strengthened as follows. For all $1 \leq i \leq n$, every $(S_n,<,F_n,S_1,...,S_{n-1})$ definable S_i bounded subset of S_i is some $F_i[x,y]$, $x,y \in S_i$. Here $S_1,...,S_{n-1}$ are used as unary predicates on S_n. We use strong FOU[n-exp], strong FOU[n-out-exp] for the corresponding first order theoriess.

LEMMA 6.3. Every FOU explosion series of length $n \geq 1$ is strong. FOU[n-exp] and strong FOU[n-exp] are logically equivalent.

Proof: By induction on $n \geq 1$. Obviously this is true for the basis case $n = 1$. Assume true for $n \geq 1$. Let $U_1,...,U_{n+1}$ be an FOU explosion series of length n+1.

By the induction hypothesis, $U_2,...,U_{n+1}$ is a strong FOU explosion series of length n. Since U_1,U_2 is an FOU explosion, S_1 is U_2 definable. Hence S_1 is $(S_{n+1},<,F_{n+1},S_2,...,S_{n+1})$ definable. Therefore for all $2 \leq i \leq n+1$, every $(S_{n+1},<,F_{n+1},S_1,...,S_{n+1})$ definable S_i bounded subset of S_i is some $F_i[x,y]$, $x,y \in S_i$.

Let $A \subseteq S_1$ be S_1 bounded and $(S_{n+1},<,F_{n+1},S_1,...,S_{n+1})$ definable. By the previous paragraph, let $A = F_2[x,y]$, $x,y \in S_2$. Then A is $(S_2,<,F_2,S_1)$ definable (even without S_1). Since U_1,U_2 is an FOU explosion, A is some $F_1[x,y]$, $x,y \in S_1$. QED

THEOREM 6.4. (EFA). Let $n \geq 1$. FOU[n-exp], FOU[n-out-exp], Z_n are mutually interpretable. FOU[n-exp], FOU[n-out-exp] are logically equivalent to strong FOU[n-exp], strong FOU[n-out-exp], respectively.

Proof: By Lemmas 4.4, 6.2, 6.3, it suffices to show that $Z_n \backslash IND$ is interpretable in strong FOU[n-exp].

Let $U_1,...,U_n$ be a strong FOU[n-exp]. We build a model of $Z_n\backslash IND$ with sorts $D_1,...,D_n$, epsilon relations $\in_1,...,\in_{n-1}$, equality relations $\equiv_1,...,\equiv_n$, and arithmetic. For $D_1,\equiv_1$, and arithmetic we use the interpretation of PA in U_1 constructed in section 3, $(S_1,\equiv,0,SC,ADD,MULT)$.

For $1 \leq i \leq n$, we inductively define $(D_i,\equiv_i)$, $D_i \subseteq S_i$, as follows. We have already defined $D_1,\equiv_1$, and the arithmetic, by $(S_1,\equiv,0,SC,ADD,MULT)$.

Suppose $(D_i,\equiv_i)$ have been defined. Define $D_{i+1} = \{x \in S^{i+1}$: $SET(x) \subseteq D_i$ respects $\equiv_i$, where $SET(x)$ is calculated in $U_{i+1}\}$. Define $x \equiv_{i+1} y$ if and only if $x,y \in D_{i+1}$ and $SET(x) = SET(y)$, calculated in U_{i+1}.

For $1 \leq i \leq n-1$, define $x \in_i y$ if and only if $x \in D_i$, $y \in D_{i+1}$, $x \in SET(y)$ calculated in U_{i+1}. QED

LEMMA 6.5. (ZFWO\P). Let κ be a Π^3_1 indescribable cardinal, and φ be a Π^3_1 sentence. Let $A \subseteq V(\kappa)$. Suppose $(V(\kappa),\in,A) \models \varphi$. Then $\{\alpha < \kappa: (V(\alpha),\in,A \cap V(\alpha)) \models \varphi\}$ is unbounded in κ. In particular, it is infinite.

Proof: Let κ be as given. We can assume $A \neq \emptyset$. Let $\gamma < \kappa$. Use $A \times \{\gamma\}$ instead of A, and adjust φ to assert φ for A. Then α must be greater than γ. QED

THEOREM 6.6. (ZFWO\P). If there is a Π^3_1 indescribable cardinal then there is a conservative SOU explosion series of every finite length. If there is a conservative SOU explosion then there is a second order indescribable cardinal in L.

Proof: The first claim has the same proof as Lemma 5.28, obtaining an SOU explosion series of any given finite length using the last claim of Lemma 6.5. The second claim is from Theorem 5.29. QED

THEOREM 6.7. (EFA). Let $n \geq 1$. FOU, FOU[n-out-con-exp], ZF, ZFC are mutually interpretable.

Proof: By Theorem 5.22, it suffices to interpret FOU[n-out-con-exp] in ZF. We follow the proof of Lemma 5.2 until we introduce an M cardinal λ with property iii. Instead, we introduce n cardinals $\lambda_1 < ...< \lambda_n$ with property iii. QED

7. Super FOU, SOU explosions.

We follow the usual convention that "elementary embedding" always means "first order elementary embedding". We also use "second order elementary embeddings".

Let U_1,U_2,U_3 be a super FOU explosion. It is easy to see that U_1,U_2 and U_1,U_3 are outer conservative FOU explosions, and U_2,U_3 is a conservative SOU explosion that is not outer.

We extensively rely on the development in section 5.

DEFINITION 7.1. Let h be the following map from U_2 systems into U_3 systems. Let J be a U_2 system. Let x be a U_2 code for J. By conservativity, x is also a U_3 code. Define h(J) as the U_3 system coded by x. By conservativity, h is well defined.

DEFINITION 7.2. We say that J is special if and only if J is a U_3 system which is, according to U_3, a pre well ordering of least length greater than all pre well orderings defined on a bounded initial segment of S1.

LEMMA 7.1. No special J is a value of h.

Proof: Let h(J') = J, where J' is a U_2 system, and J is special. Let u be any U_2 code for J'. Then u is a U_3 code for J.

Now $\min(S_3\backslash S_1)$ can be defined in U_3 as the least $x \in S_3$ such that there is a pre well ordering on $(-\infty,x)$ of the same length as J (using u). This same definition, with only the parameter $u \in S_2$, must also define some $x' \in S_2$ when interpreted in U_2. By conservativity, $x' = x \in S_2$, which contradicts Definition 2.9. QED

From section 5, we have an isomorphism relation and an epsilon relation on the U_i systems defined in U_i. Because of the joint first order plenitude, these notions have the same meaning when we increase i.

From section 5, the U_i systems each form a model of ZF with equality as isomorphism, where the separation and replacement schemes are formulated in the joint language.

We now factor out by the isomorphism relation. The U_i systems form a model M_i of ZF, and h becomes a map $h^*:M_2 \rightarrow M_3$.

LEMMA 7.2. M_1, M_2, M_3 forms an elementary chain of models of ZF, with separation holding for the joint language. M_2, M_3 are end extensions of M_1. M_1 forms a set in M_2, and M_2 forms a set in M_3. The least ordinal in $M_3\backslash M_1$ is greater than the least ordinal in $M_2\backslash M_1$. $h^*:M_2 \rightarrow M_3$ is an elementary embedding. h^* fixes every element of M_1. The critical point of h^* is the least ordinal in $M_3\backslash M_1$.

Proof: Mostly left to the reader. By conservativity, it is clear that $h^*:M_2 \rightarrow M_3$ is elementary. h^* fixes the elements of M_1 by Lemma 5.6, since U_1,U_2, and U_1,U_3 are outer conservative FOU explosions. M_1 forms a set in M_2, and M_2 forms a set in M_3, since U_1,U_2,U_s is an explosion series.

Let α be the least ordinal in $M_3\backslash M_1$. Suppose $h^*(\alpha) = \alpha$. Note that the special J are of type α. By Lemma 7.1, they are not values of h. Hence α is not a value of h^*, and so $h^*(\alpha) \neq \alpha$. Now $h^*(\alpha) < \alpha$ is impossible since h^* is one-one and fixed all $\beta < \alpha$. QED

DEFINITION 7.3. Let T_1 be in the language $\in,=,\kappa_1,\kappa_2,j$, where κ_1,κ_2 are constant symbols and j is a unary function symbol. The axioms of T_1 are ZF + $\kappa_1 < \kappa_2$ are limit cardinals + $(V(\kappa_1),\in)$ is an elementary substructure of $(V(\kappa_2),\in)$ + $j:V(\kappa_2) \rightarrow V$ is an elementary embedding with critical point κ_1. Here the schemes of ZF use j, and the elementarity is formulated as schemes in $\in,=$.

DEFINITION 7.4. Let T_2 be in the language $\in,=,\kappa_1,\kappa_2,j,<_2,<_3$, where κ_1,κ_2 are constant symbols, $<_2,<_3$ are binary relation symbols, and j is a unary function symbol. The axioms of T_2 are ZFC + $\kappa_1 < \kappa_2$ are cardinals + $<_2$ is a well ordering of $V(\kappa_2)$ of type κ_2 + $<_3$ is a well ordering of V + $(V(\kappa_1),<_2,\in)$ is an elementary substructure of $(V(\kappa_2),<_2,\in)$ + $j:(V(\kappa_2),<_2,\in) \rightarrow (V,<_3,\in)$ is an elementary embedding with critical point κ_1. Here the schemes of ZF use $j,<_2,<_3$, and the elementarity is formulated as schemes in $\in,=$.

LEMMA 7.3. (EFA). T_1 is interpretable in FOU[sup-exp].

Proof: By Lemma 7.2, from the point of view of M_3, we see that M_1 and M_2 are models of ZF with second order separation. Now M_3 is an end extension of M_1. Therefore we know, from the point of view of M_3, that M_1 is some $V(\kappa_1)$ satisfying ZF, where κ is a limit cardinal. Also M_2 is a model of ZF with second order separation, from the point of view of M_3.

Now perform the transitive collapse of M_2 within M_3. From the point of view of M_3, M_2 is collapsed to some $V(\kappa_2)$, where κ_2 is a limit cardinal. Convert h* to the corresponding elementary embedding $j:V(\kappa_2) \to V(\kappa_3)$, by composition. By Lemma 2, κ_1 is the critical point of j. QED

LEMMA 7.4. (EFA). FOU(sup-exp) is interpretable in T_2.

Proof: Assume T_2, and let $\kappa_1,\kappa_2,\kappa_3,<_2,<_3,j$ be as given. Build $F:V(\kappa_3) \to V(\kappa_3)$ by a standard transfinite recursion on $<_3$ so that $(V(\kappa_3),<_3,F)$ is an FOU. Take $U_1 = (V(\kappa_1),<_2,F)$, $U_2 = (rng(j),<_3,F)$, $U_3 = (V(\kappa_3),<_3,F)$, by restrictions. Note that $\kappa_1 \notin rng(j)$. QED

DEFINITION 7.5. Let H_1 assert the existence of strongly inaccessible cardinals $\kappa_1 < \kappa_2 < \kappa_3$ and j, where $(V(\kappa_1),\in)$ is a second order elementary substructure of $(V(\kappa_2),\in)$, and $j:V(\kappa_2) \to V(\kappa_3)$ is a second order elementary embedding with critical point κ_1.

DEFINITION 7.6. Let H_2 assert the existence of strongly inaccessible cardinals $\kappa_1 < \kappa_2 < \kappa_3$, well orderings $<_2,<_3$ of $V(\kappa_2),V(\kappa_3)$ of types κ_2,κ_3, and j, where $(V(\kappa_1),<_2,\in)$ is a second order elementary substructure of $(V(\kappa_2),<_2,\in)$, and $j:(V(\kappa_2),<_2,\in) \to (V(\kappa_3),<_3,\in)$ is a second order elementary embedding with critical point κ_1.

LEMMA 7.5. (ZFC). If there is a super SOU explosion then H_1.

Proof: Analogous to Lemma 7.3. At first, we only get limit cardinals $\kappa_1,\kappa_2,\kappa_3$. However, the critical point κ_1 must be strongly inaccessible (and much more) by a standard argument. Second order elementary substructure implies that κ_2 is strongly inaccessible, and the second order elementary embedding implies that κ_3 is strongly inaccessible. QED

LEMMA 7.6. (ZFC). H_2 implies the existence of a super SOU explosion.

Proof: From Lemma 7.5, and argue as in Lemma 7.4. QED

We now discuss the status of the theories ZFC + H_1, ZFC + H_2, T_1, and T_2.

LEMMA 7.7. (ZFC). If there is an extendible cardinal then H_2.

Proof: Let κ be an extendible cardinal. By extendability, let $j:V(\kappa+1) \to V(\lambda+1)$ be an elementary embedding with critical point κ. Then $\kappa < \lambda$ are strongly inaccessible cardinals. Let <' be a well ordering of $V(\kappa)$ of order type κ. Then $(V(\kappa),<',\in)$ is a second order elementary substructure of $(V(\lambda),j(<'),\in)$, and $<' = j(<') \cap V(\kappa)$.

By extendability, let $j':V(\lambda+1) \to V(\lambda'+1)$ be an elementary embedding with critical point κ. Then $j':V(\lambda) \to V(\lambda')$ is a second order elementary embedding with critical point κ. QED

LEMMA 7.8. (ZFC). H_2 implies the existence of strongly inaccessible cardinals $\kappa < \lambda$ such that $(V(\lambda),\in) \models \kappa$ is a supercompact cardinal.

Proof: Assume H_1, and let $\kappa < \lambda$ and j be as given by H_2. Let μ be the critical point of j.

We can assume that $j(\mu) \geq \kappa$. To see this, suppose $j(\mu) < \kappa$. We can restrict j to $j:V(j(\mu)) \to V(j(j(\mu)))$, which will still be an elementary embedding with critical point μ, and obviously $V(j(j(\mu)))$ satisfies ZFC, since $\mu,j(\mu)$ are strongly inaccessible cardinals.

We use the characterization of supercompact cardinals from [Ma71], [Ka94], p. 302, stated in section 1. We show that $j(\mu)$ is supercompact in $(V(\lambda),\in)$.

Suppose $j(\mu)$ is not supercompact in $(V(\lambda),\in)$. Let $\alpha < \lambda$ be least such that $\alpha > j(\mu)$, and there is no β,j' such that $\beta < j(\mu)$ and $j':V(\beta) \to V(\alpha)$ is an elementary embedding with critical point γ, where $j'(\gamma) = j(\mu)$.

Since α is definable in $V(\lambda)$ from $j(\mu)$, we see that $j^{-1}(\alpha)$ exists. Use $\beta = j^{-1}(\alpha)$, $j' = j$, $\gamma = \kappa_1$. Note that $j^{-1}(\alpha) < \kappa \leq j(\mu)$. This contradicts the second sentence of the previous paragraph. Hence $j(\mu)$ is supercompact in $(V(\lambda),\in)$. QED

It is not clear just how strong the choiceless theory T_1 is. The standard construction within T_1 of a measure on the critical point κ_1 and the original inner model $L(\mu)$ construction easily gives an interpretation of ZFC + "there exists a measurable cardinal" in T_1. This can be pushed much further, using inner model theory, probably to as far as the inner theory has so far been developed. That is, to roughly Woodin cardinals. We will be content here to stop at "there are arbitrarily large measurable cardinals".

We now summarize what we have established about super explosions.

THEOREM 7.9. (EFA). In ZFC, "there exists an extendible cardinal" proves the existence of an SOU super explosion, which proves the existence of strongly inaccessible cardinals $\kappa < \lambda$ such that $(V(\lambda),\in) \models \kappa$ is a supercompact cardinal. ZFC + "there are arbitrarily large measurable cardinals" is interpretable in FOU[sup-exp], which is interpretable in ZFC + "there exists an extendible cardinal".

Proof: The first claim is from Lemmas 7.6, 7.7, 7.8. For the second claim, we use Lemmas 7.3, 7.4, 7.7. QED

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