COMBINATORIAL SET THEORETIC PRINCIPLES OF GREAT LOGICAL STRENGTH

PRELIMINARY REPORT

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(for mathematicians with some interest in set theory)

(Results obtainedin 1995).

We present a surprisingly simple statement of combinatorial set theory of tremendous logical strength.

We follow the set theorist's convention that an ordinal α is identified with its set of predecessors.

Let $j:\beta \to \beta$, where β is an ordinal. Let $R \subseteq \alpha \times \alpha$, where $\beta \leq \alpha$. We define $j[R] = \{(j(c),j(d)): R(c,d)\}$. We say that j is a nonidentity function if and only if j is not the identity function on β.

Consider the following property of an ordinal α:

$P_1(\alpha)$. For all $R \subseteq \alpha \times \alpha$ there exists a nonidentity $j:\beta \to \beta$, $\beta < \alpha$, such that $j[R] \subseteq R$.

Observe that this properties are preserved upwards; i.e., if $\alpha < \beta$ then $P_1(\alpha)$ implies $P_1(\beta)$.

We introduce the combinatorial statement A_1:

A1. There exists α such that $P_1(\alpha)$ holds.

We now relate these statements to some standard large cardinal axioms that have been extensively studied by set theorists.

LCA_1. There is a nontrivial elementary embedding of $V \to M$ such that $V(\alpha) \subseteq M$, where α is the first fixed point above the critical point.

LCA_2. There is a nontrivial elementary embedding of a rank into itself.

Here LCA stands for "large cardinal axiom."

THEOREM 1. The following is provable in ZFC: A_1 follows from LCA_1 and implies LCA_2.*

We now consider the following very strong property:

$P_2(\alpha)$. For every $R \subseteq \alpha \times \alpha$ there exists a nonidentity $j:\alpha \to \alpha$ such that $j[R] \subseteq R$.

Let A_2 assert that there exists a such that $P_2(a)$.

THEOREM 2. A_2 can be refuted in ZFC.

However, A_2 is consistent with ZFDC:

THEOREM 3. The following is provable in ZF. Suppose every subset of ω_1 is constructible from a set of integers, and every set of integers has a sharp. Then $P_2(\omega_1)$ holds. Hence AD (axiom of determinacy) implies $P_2(\omega_1)$, and hence A_2.

In connection with our research into statements in the integers which require extremely large cardinals to prove, we discuss strengthenings of A_1 which we know are extremely strong in the context of ZF (without any axiom of choice).

Let $F:\alpha \times \alpha \to \alpha$. For $\beta < \alpha$, we write F_β for the restriction of the cross section of F to β; i.e., $F_\beta(c) = d$ if and only if $c < b$ and $F(b,c) = d$.

$P_3(a)$. There exists an $F:\alpha \times \alpha \to \alpha$ such that for every $R \subseteq \alpha \times \alpha$, there is a nonidentity $F_\beta:\beta \to \beta$, $\beta < \alpha$, such that $F_\beta[R] \subseteq R$.

Let A_3 assert that there exists a such that $P_3(\alpha)$. Obviously A_3 implies A_1.

THEOREM 4. Assume LCA_1. Then $P_3(a)$ holds in a generic extension.

We now consider the following weakening of P_3:

$P_4(\alpha)$. There exists $F:\alpha \times \alpha \to \alpha$ such that for every $R \subseteq \alpha \times \alpha$ that is constructible from F, there is a nonidentity $F_\beta:\beta \to \beta$, $\beta < \alpha$, such that $F_\beta[R] \subseteq R$.

Consider the following proposition:

A_4. $P_4(\alpha)$ holds for some countable α.

THEOREM 5. ZFC + LCA_1 implies A_4. ZF + A_4 implies the existence of a transitive class containing all ordinals which satisfies ZFC + LCA_2.

Notice that A_4 is a Π-1-3 sentence.

Continuing on, we can consider the following weakening of A_4:

A_5. There is a countable function of the form $F:\alpha \times \alpha \to \alpha$ such that for every unbounded $R \subseteq \alpha \times \alpha$ that is first order definable in $(\alpha,<,F)$, some F_β is a nontrivial injection (embedding) of a proper initial segment of R.

THEOREM 6. ZFC + LCA_1 implies A_5. ZF + A_5 implies the existence of a transitive model of ZFC + LCA_2.

Notice that A_5 is a Σ-1-2 sentence. So A_5 is a consequence of LCA_1 in the constructible universe that implies the existence of transitive models of LCA_2.

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*We are indebted to Donald Martin for suggesting the possible use of an absoluteness/tree argument in connection with the proof of A_1 from LCA_1. Such arguments have been previously used in the theory of extremely large cardinals.