CONCRETE INCOMPLETENESS FROM EFA THROUGH LARGE CARDINALS

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THE ABSTRACT/CONCRETE DIVIDE

There is a deep and growing conceptual divide between abstract set theory and normal mathematical culture.

Normal mathematical culture is overwhelmingly concerned with finite structures, finitely generated structures, discrete structures (countably infinite), continuous and piecewise continuous functions between complete separable metric spaces, with lesser consideration of pointwise limits of sequences of such functions, and Borel measurable functions between complete separable metric spaces.

More abstract mathematical objects are normally considered for two reasons:

1. They simplify presentations of material by avoiding the need for extra hypotheses.
2. They are used in proofs of normal statements.

For 1, if great generality causes technical difficulties unrelated to the material, they are avoided in favor of less abstract formulations.

This leaves 2 for a substantial role of abstract set theory in normal mathematical culture.

THE ABSTRACT/CONCRETE DIVIDE

We will take the working definition of the Concrete as the universe of

Borel measurable functions between complete separable metric spaces.
Borel measurable functions of finite Borel rank between complete separable metric spaces.

We have mentioned both as we like to keep the divide somewhat flexible.

Obviously, this includes mathematical objects of far greater concreteness - particularly, countable structures.

There are examples of uses of the Abstract for proving theorems in the Concrete - by normal mathematicians.

USES OF ABSTRACT FOR THE CONCRETE IN NORMAL MATHEMATICS

The examples of normal mathematicians using the Abstract for the Concrete fall into these categories:

1. Convenience. It simplifies matters, and does not cause any special irrelevant difficulties. But it can obviously be removed, although actually removing it is judged not to be worth the effort.

2. Apparent necessity. There is no obvious way to remove it. Some ideas are needed to remove it. Interest varies concerning the issue. Subsequently, the Abstract is removed.

3. Necessity. It is known or believed that there is no way to remove it.

Using “every field has an algebraic closure” is typical of category 1.

We mention two particularly interesting cases of Apparent Necessity.

REMOVAL OF THE ABSTRACT FOR THE CONCRETE: TWO PARTICULARLY INTERESTING CASES FROM NORMAL MATHEMATICS

We mention two cases. Both situations are fluid and ongoing.

1. Laver proved a number of new results about the free left distributive algebra on one generator, using spectacularly abstract principles going far far beyond the usual axioms of mathematics - ZFC. Many but not all of these results were reproved using expectedly normal principles by Dehornoy. There are some remaining results where the removal has not been achieved. See

R. Dougherty, T. Jech, Left-Distributive Algebras, Electronic Research Announcements of the AMS, Vol 3, pp. 28–37 (April 9, 1997) S 1079-6762(97)00020-6.

2. Wiles proved Fermat’s Last Theorem using a lot of sophisticated machinery from algebraic geometry, including Grothendieck Topoi (in the strong sense), that is roughly equivalent to using a strongly inaccessible cardinal. It is clear to experts that this is a removable convenience. However, lower forms of the Abstract remain. There have been substantial efforts to prove FLT in the Concrete - particularly in PA = Peano Arithmetic. See the careful discussion in http://www.cwru.edu/artsci/phil/Proving_FLT.pdf by Colin McLarty. I conjectured that FLT is provable in EFA.

NATURE OF CONCRETE INCOMPLETENESS

By Concrete Incompleteness, we mean that a particular concrete mathematical statement

is neither provable nor refutable in one of the standard formal systems from mathematical that are used to organize the logical structure of mathematics.

We distinguish two sources of Concrete mathematical statements for present purposes.

1. Concrete mathematical statements that appear to arise only from the investigation of formal systems for mathematical reasoning.

2. Concrete mathematical statements that can arise from considerations present in normal mathematical culture.

NATURE OF CONCRETE INCOMPLETENESS

1. Concrete mathematical statements that appear to arise only from the investigation of formal systems for mathematical reasoning.

2. Concrete mathematical statements that have arisen, or can arise, through considerations present in standard mathematical culture.

There is a huge divide between 1 and items that are arguably in 2. Within 2, there is quite a range from the arguably 2 and the obviously 2. The latter includes, of course, celebrated theorems from the most respected Journals concentrating on mainstream mathematics.

I am firmly convinced that this is not a matter of sociology, but of substance. However, at this time, we know of no formal way to make the relevant distinctions.

These matters are currently being put to the test with actual presentations to and contacts with working mathematicians from a variety of standard mathematical areas.

VARIETIES OF CONCRETE MATHEMATICAL INCOMPLETENESS

We have a largely completed draft of the Introduction to my forthcoming BRT book. Its title is Concrete Mathematical Incompleteness. It is, itself, a 100+ page book.

By Concrete Mathematical Incompleteness, we mean the independence from various standard formal systems of

2. Concrete mathematical statements that have arisen, or can arise, through considerations present in standard mathematical culture.

But in the early sections, we also have a discussion of general incompleteness, and some important completeness.

The section headings of Concrete Mathematical Incompleteness are as follows:

0.1. General Incompleteness.
0.2. Some Completeness.
0.3. Abstract and Concrete Mathematical Incompleteness.
0.4. Reverse Mathematics.
0.5. Incompleteness in Exponential Function Arithmetic.
0.6. Incompleteness in Primitive Recursive Arithmetic, Single Quantifier Arithmetic, RCA_0, and WKL_0.
0.7. Incompleteness in Nested Multiply Recursive Arithmetic, and Two Quantifier Arithmetic.
0.8. Incompleteness in Peano Arithmetic and ACA_0.
0.9. Incompleteness in Predicative Analysis and ATR_0.
0.10. Incompleteness in Iterated Inductive Definitions and ${}^1{}_1$-CA_0.
0.11. Incompleteness in Second Order Arithmetic and ZFC\P.
0.12. Incompleteness in Russell Type Theory and Zermelo Set Theory.
0.13. Incompleteness in ZFC using Borel Functions.
0.14. Incompleteness in ZFC using Discrete Structures.
0.15. Templates and the Template Conjecture.
0.16. Overview of book contents.
0.17. Open problems.

VARIETIES OF CONCRETE MATHEMATICAL INCOMPLETENESS

In 0.5 through 0.14, we give concrete mathematical examples of Theorems that are proved in a bit stronger systems than indicated by the section heading, yet cannot be proved using the system indicated by the section heading.

We will present some highlights from each of these 10 sections.

INCOMPLETENESS IN EXPONENTIAL FUNCTION ARITHMETIC

EFA = Exponential Function Arithmetic is the weakest system in use for which the coding of finite objects by nonnegative integers is worry free. It is also called $I\Delta_0$(exp) = $I\Sigma_0$(exp) in the literature.

Its primitives are 0,S,+,$\cdot$,<,^. The axioms are the usual successor axioms, recursion equations for +,$\cdot$,^, definition of <, and induction for Δ0 formulas. It is finitely axiomatizable. It is very robust.

We have conjectured that it proves FLT. We consider a system T to "have logical strength" if and only if EFA is interpretable in T.

The "next" stronger system normally considered is SEFA = Superexponential Function Arithmetic based on the superexponential

$n^{[m]}$ = iterate base n exponentiation m times.

INCOMPLETENESS IN EXPONENTIAL FUNCTION ARITHMETIC

FINITE RAMSEY THEOREM. For all $k,p,r \geq 1$ there exists n so large that the following holds. In any coloring of the unordered k tuples from $\{1,...,n\}$ using p colors, there is an r element subset of $\{1,...,n\}$ whose unordered k tuples have the same color.

THEOREM. The Finite Ramsey Theorem is provable in SEFA, but not in EFA, even for 2 colors only. It is provable in EFA for fixed k.

ADJACENT RAMSEY THEOREM. For all $k,p \geq 1$ there exists t so large that the following holds. For all $f:\{1,...,t\}^k \to \{1,...,p\}$, there exist $1 \leq x_1 < ... < x_{k+1} \leq t$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

THEOREM. The Adjacent Ramsey Theorem is provable in SEFA, but not in EFA.

INCOMPLETENESS IN PRIMITIVE RECURSIVE ARITHMETIC, SINGLE QUANTIFIER ARITHMETIC, RCA_0, AND WKL_0

PRA = primitive recursive arithmetic, is based on 0,S, and all primitive recursive function symbols. The axioms are the successor axioms, the defining equations, and induction for all quantifier free formulas.

PA = Peano arithmetic, is based on 0,S,+,•. The axioms are the successor axioms, defining equations for +,•, and induction for all formulas.

$I\Sigma_n$ is the fragment of PA, where induction is only for all Σ_n formulas. We identify single quantifier arithmetic with $I\Sigma_1$. $I\Sigma_1$ is a conservative extension of PRA for Π02 sentences.

RCA_0, WKL_0 are my two systems from Reverse Mathematics. RCA_0 is a conservative extension of $I\Sigma_1$ that is based on recursive comprehension. WKL_0 is a conservative extension of RCA_0 for arithmetical sentences (and more), that is based on the 0,1 finite tree theorem of Konig.

Thus PRA, $I\Sigma_1$, RCA0, WKL_0 are at the same level.

INCOMPLETENESS IN PRIMITIVE RECURSIVE ARITHMETIC, SINGLE QUANTIFIER ARITHMETIC, RCA_0, AND WKL_0

For x,y in N^k, write $x \leq_c y$ if and only if each $x_i \leq y_i$.

THEOREM. Every infinite sequence from N^k has a finite initial segment such that every term is $\leq_c$ some term in that finite initial segment.

THEOREM. For all $k \geq 1$ and $p \geq 0$, there exists n such that the following holds. For all $x_1,...,x_n$ from N^k such that each $\max(x_b) \leq b+p$, there exists $1 \leq i < j \leq n$ such that $x_i \leq_c x_j$.

These Theorems are not provable in WKL_0. The first Theorem is provable in ACA_0, and the second Theorem is provable in $I\Sigma_0$, or even in Ackermann Function Arithmetic.

The first Theorem is provable in RCA_0 for fixed k. The second Theorem is provable in PRA for for fixed k.

INCOMPLETENESS IN PRIMITIVE RECURSIVE ARITHMETIC, SINGLE QUANTIFIER ARITHMETIC, RCA_0, AND WKL_0

HBT (Hilbert's Basis Theorem). Let $P_1,P_2,\ldots$ be an infinite sequence of polynomials from the polynomial ring in k variables over a countable field. There exists n such that all P's are ideal generated by $P_1,P_2,\ldots,P_n$.

THEOREM. HBT is not provable in WKL_0, but is provable in ACA_0. It is provable in RCA_0 for fixed k.

INCOMPLETENESS IN NESTED MULTIPLY RECURSIVE ARITHMETIC, AND TWO QUANTIFIER ARITHMETIC

There is an obviously extension of primitive recursion to nested multiple recursion, where the Ackermann function is a nested double recursion. Thus we obtain NMRA = nested multiply recursive arithmetic.

We identify two quantifier arithmetic with $I\Sigma_2$, which is a conservative extension of NMRA for Π02 sentences.

BLOCK SEQUENCE THEOREM. There is a longest finite sequence $x_1,x_2,\ldots,x_n$ from {1,...,k} in which no consecutive block $x_i,\ldots,x_{2i}$ is a subsequence of any later consecutive block $x_j,\ldots,x_{2j}$.

THEOREM. The Block Sequence Theorem is not provable in $I\Sigma_2$, or NMRA. It is provable in $I\Sigma_3$.

INCOMPLETENESS IN PEANO ARITHMETIC AND ACA_0

ACA_0 is my main system of Reverse Mathematics that is conservative over PA. It is based on arithmetic comprehension.

The usual next higher system is ACA′ which is based on "for all x $\subseteq$ N, for all n, the n-th Turing jump of x exists". This can be stated more mathematically as the scheme of recursion on N for arithmetic formulas.

Most well known are the historically earliest Goodstein Sequence Theorem, and Paris Harrington Ramsey Theorem. Both are provable in ACA′ but not in PA or ACA_0.

There are substantially better examples now from various points of view. Here is one.

INCOMPLETENESS IN PEANO ARITHMETIC AND ACA_0

THEOREM. For all $f:N^k$ into N, there exists $0 \leq n_1 < ... < n_{k+2}$ such that $f(n_1,...,n_k) \leq f(n_2,...,n_{k+1}) \leq f(n_3,...,n_{k+2})$.

THEOREM. For all k there exists t such that the following holds. For all $f:\{1,...,t\}^k \to N$, there exists $0 \leq n_1 < ... < n_{k+2} \leq t$ such that $f(n_1,...,n_k) \leq f(n_2,...,n_{k+1}) \leq f(n_3,...,n_{k+2})$.

These are provable in ACA′ but not in ACA_0. The second is not provable in PA.

INCOMPLETENESS IN PREDICATIVE ANALYSIS AND ATR_0

The philosophy of mathematics called predicativity emphasizes a similarity between $\{x: x \notin x\}$ and $\{x \in N: P(x)\}$, where P has quantifiers over all subsets of N. Platonists/realists emphasize the difference between them.

The most commonly accepted analysis of predicativity is through systems of S. Feferman surrounding the proof theoretic ordinal Γ_0. The most common system is based on the hyperarithmetic hierarchy of subsets of N of each length $< \Gamma_0$.

This analysis has been criticized from various points of view, but no alternative has emerged with greater following.

We proved that the one of the main systems ATR_0 of Reverse Mathematics corresponds to Γ_0.

INCOMPLETENESS IN PREDICATIVE ANALYSIS AND ATR_0

We proved the following simple assertion.

THEOREM. Between any two countable sets of reals, there is a pointwise continuous one-one map from one into the other.

This innocent looking result needs transfinite induction on all countable ordinals. One aspect of this is

THEOREM. The above Theorem is provably equivalent to ATR_0 over RCA_0.

INCOMPLETENESS IN PREDICATIVE ANALYSIS AND ATR_0

KT (Kruskal's tree theorem). In any infinite sequence of finite trees, one tree is inf preserving embeddable into a later one.

FKT (a finite form). For all $c \geq 0$ there exists n such that the following holds. In any sequence of finite trees $T_1,...,T_n$, where each Ti has at most c+i vertices, some tree is inf preserving embeddable into a later tree.

KT and FKT are provable in Π12-TI but not in $\Pi12\text{-}TI_0$. In particular, they are not provable in predicative analysis, or ATR_0.

There are many finite forms, and there are many restrictions, on the trees considered, some of which hit ATR_0 and Γ_0 on the button. E.g., binary trees with two labels.

INCOMPLETENESS IN ITERATED INDUCTIVE DEFINITIONS AND PIΠ^1_1-CA_0

We extended Kruskal's Theorem to EKT = extended Kruskal's theorem, by using finitely many labels, and introducing the gap condition:

if y is an immediate successor of x, then for all z in the gap (hx,hy), l(z) ≥ l(hy).

EKT. In any infinite sequence of finite trees with finitely many labels, one tree is gap condition preserving embeddable into a later tree.

We proved that EKT is provable in Π11-CA but not in Π11-CA_0.

We also gave finite forms and showed the above result for the finite forms.

INCOMPLETENESS IN ITERATED INDUCTIVE DEFINITIONS AND $PI^1{}_1$-CA_0.

With N. Robertson and P. Seymour, we made a connection between my EKT and their Graph Minor Theorem = GMT.

We showed that GMT implies EKT, and that bounded GMT (GMT for bounded tree width) is equivalent to EKT.

Thus even the bounded GMT is not provable in $\Pi 11$-CA_0.

Bounded GMT is provable in Π11-CA. However, it is not clear where GMT is provable. The discussions have been too vague. Most likely is that iteration of the hyperjump along a small proof theoretic ordinal should be sufficient.

INCOMPLETENESS IN ITERATED INDUCTIVE DEFINITIONS AND Π^1_1-CA_0

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The Borel Ramsey theorem, also known as the Galvin/Prikry theorem, asserts the following. Every Borel subset of P(N) contains or is disjoint from all infinite subsets of some fixed infinite subset of N.

INCOMPLETENESS IN ITERATED INDUCTIVE DEFINITIONS AND Π^1_1-CA_0

The Borel Ramsey theorem corresponds to Pi11 transfinite recursion, which is the iteration of the hyperjump along any countable well ordering. Thus it is not provable in Π11-CA.

There has also been work on my EKT extended to allow well ordered labels. This again leads to Pi11 transfinite recursion.

INCOMPLETENESS IN SECOND ORDER ARITHMETIC AND ZFC\P

Cantor proved that in any infinite sequence of reals, some real number is missing. The missing real can be constructed by a low level Borel function from $\Re^\infty$ into $\Re$. But can we find the missing real in terms of only the given reals, and not their order of presentation?

An invariant Borel function $F: \Re^\infty \to \Re$ is a Borel function where if x,y have the same set of terms, then $F(x) = F(y)$.

THEOREM. (Borel diagonalization). If $F: \Re^\infty \to \Re$ is an invariant Borel function, some $F(x)$ is a coordinate of x.

THEOREM. Borel Diagonalization cannot be proved in Z2, ZFC\P. It can be proved in Z3, ZFC\P + P(N) exists.

INCOMPLETENESS IN SECOND ORDER ARITHMETIC AND ZFC\P

We also consider Borel $F:\Re^\infty \to \Re^\infty$. We say that F is invariant if and only if for all x,y,

if x,y have the same set of coordinates then F(x),F(y) have the same set of coordinates.

THEOREM. Let $F:\Re^\infty \to \Re^\infty$ be Borel and invariant. Some F(x) is a subsequence of x.

THEOREM. The above Theorem cannot be proved in Z2, ZFC\P. It can be proved in Z3, ZFC\P + P(N) exists.

INCOMPLETENESS IN RUSSELL TYPE THEORY AND ZERMELO SET THEORY

THEOREM. Every Borel set $Y \subseteq \Re^2$ symmetric about the line $y = x$, contains or is disjoint from the graph of a Borel function from $\Re$ into $\Re$.

THEOREM. This Theorem is provable using uncountably many iterations of the power set operation, but not with any definite countably number of iterations of the power set operation.

In particular, the Theorem is not provable in Z.

The above is closely tied to Borel determinacy, which had previously been shown by Donald Martin and me to have this metamathematical status.

INCOMPLETENESS IN ZFC USING BOREL STRUCTURES

Let GRP be the Borel space of countable groups, and FGG be the Borel space of finitely generated groups.

PROPOSITION. For all invariant (finitely) Borel F:FGG$^\infty$ → GRP, there exists towered x in FGG$^\infty$ such that for all infinite subsequences y of x, F(y) is embeddable in $\cup_n x_n$.

The above Proposition is provable from a measurable cardinal but not from a Ramsey cardinal.

INCOMPLETENESS IN ZFC USING DISCRETE STRUCTURES

This title refers to Boolean Relation Theory.

Let $f:N^k \to N$ and $A \subseteq N$. We write $fA = f[A^k]$.

Let MF be the family of functions of the form $f:N^k \to N$. Let SD be the $f \in MF$ which are strictly dominating. Let ELG be the $f \in MF$ which are of expansive linear growth. Let INF be the family of infinite subsets of N.

THIN SET THEOREM. For all f in MF there exists $A \in INF$ such that $fA \neq N$.

COMPLEMENTATION THEOREM. For all f in SD there exists (unique) A in INF such that $fA = N\setminus A$.

INCOMPLETENESS IN ZFC USING DISCRETE STRUCTURES

A **∪**. B means A **∪** B if A,B are disjoint; undefined otherwise.

TEMPLATE. For all f,g ∈ ELG, there exists A,B,C ∈ INF, such that

$$P \cup. fQ \subseteq R \cup. gS$$
$$T \cup. fU \subseteq V \cup. gW.$$

Of the 6561 cases, all but 12 are provable or refutable in RCA_0. The remaining 12 are all symmetric to

EXOTIC CASE. For all f,g ∈ ELG, there exists A,B,C ∈ INF, such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

The Exotic Case is provable using Mahlo cardinals of finite order, but not without. V = L will not help.