ASPECTS OF CONSTRUCTIVE SET THEORY AND BEYOND

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Set Theory, Classical and Constructive

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WHAT ARE THESE THREE ASPECTS?

1. FORMALIZING CONSTRUCTIVE ANALYSIS.

Provide a formal system that is a conservative extension of PA for Π^0_2 sentences, and even a conservative extension of HA, that supports the worry free smooth development of constructive analysis in the style of Errett Bishop.

2. FORMALIZING CLASSICAL ANALYSIS.

Prove a formal system that is a conservative extension of PA, that supports the worry free smooth development of classical analysis.

3. STRONG INTUITIONISTIC ZF.

Understand strong intuitionistic versions of ZF.

4. INTUITIONISTIC LARGE CARDINAL THEORY.

Understand intuitionistic versions of large cardinal hypotheses.

1. FORMALIZING CONSTRUCTIVE ANALYSIS

We gave a foundation for Errett Bishop style constructive analysis in terms of intuitionistic set theory, in

H. Friedman, Foundations for Constructive Analysis, Annals of Mathematics, 105 (1977), 1-28.

In this paper, we show that any arithmetic sentence provable in the system B is provable in PA, and that B and PA prove the same Π^0_2 sentences.

In another paper with a different purpose (discussed below), I wrote

"In Friedman (1977), we presented a fragment B of Zermelo set theory with intuitionistic logic, and proved that any arithmetic sentence provable in B is provable in PA. (It is now known that B is a conservative extension of HA.)"

1. FORMALIZING CONSTRUCTIVE ANALYSIS

Unfortunately, I didn't give a reference, and I don't quite remember what that reference should be, or whether it was an unpublished observation of ours. In any case, I'm sure that this (sort of thing) must be embedded in the current literature.

I think this was done using the formalized realizability method that Michael Beeson presented in his thesis to reprove a theorem of Goodman in his thesis, that HA is conservative over HA.

The primitives of B are "being a natural number", "being a set", "y is the successor of x", and identity. The axioms of B are as follows.

A. (Ontological axioms.) Every object is either a set or a natural number, but not both. 0 is a natural number. If x is the successor of y, then x,y are natural numbers. If x y then y is a set. If x = y then x is a set if and only if y is a set.

B. (Equality axioms).

1. FORMALIZING CONSTRUCTIVE ANALYSIS

C. (Extensionality.) If two sets have the same elements then they are equal.

D. (Successor axioms.) Any two successors of a number are equal. Any two numbers with the same successor are equal. 0 is not the successor of any number.

E. (Infinity.) The set of all natural numbers exists.

F. (Induction.) If a set contains 0 and is closed under successor, then it contains all natural numbers.

G. (Pairing.) There is a set which consists of just the objects x,y.

H. (Union.) There is a set consisting of all the elements of elements of x.

I. (Exponentiation.) There is the set of all total functions (in the sense of univalent sets of ordered pairs) from one set into another.

J. (Limited dependent choice.) Dependent choice with respect to a Δ_0 relation.

K. (Limited separation.) Separation for Δ_0 formulas.

L. (Abstraction.) Let $A(y_1,\ldots,y_k,u)$ be a Δ_0-formula, x a set. Then $\{\{u \in x: A(y_1,\ldots,y_k,u): y_1,\ldots,y_k \in x\}$ exists.

2. FORMALIZING CLASSICAL ANALYSIS

In a later paper, we attacked the problem of giving a strong conservative extension of PA which is sufficient to do a great deal of classical analysis in a smooth and direct manner. This is

H. Friedman, A Strong Conservative Extension of Peano Arithmetic, J. Barwise, H.J. Keisler and K. Kunen, eds., The Kleene Symposium, North-Holland, (1980), 113-122.

This is also the paper where I wrote

"In Friedman (1977), we presented a fragment B of Zermelo set theory with intuitionistic logic, and proved that any arithmetic sentence provable in B is provable in PA. (It is now known that B is a conservative extension of HA.)"

2. FORMALIZING CLASSICAL ANALYSIS

We exploited the idea that although mathematics is heavily classical in and around the integers, it is normally constructive higher up. E.g., there is a huge practical distinction in mathematics between say, a real number or function from ω into ω, and a subset of ω. From this point of view, the "good" subsets of ω are those that have a characteristic function.

ALPO = analysis with the limited principle of omniscience, has the same primitives as B. I.e., "being a natural number", "being a set", "y is the successor of x", and identity.

A. Ontological Axioms. As in B.

B. Extensionality. As in B.

C. Successor axioms. As in B.

2. FORMALIZING CLASSICAL ANALYSIS

D. Infinity. As in B.

E. Sequential induction. For functions from ω into ω.

F. Sequential recursion. For functions from ω into ω, as in primitive recursion.

G. Pairing. As in B.

H. Union. As in B.

I. Exponentiation. As in B.

J. Countable choice. Produces a function from ω, and uses any formula.

K. Limited separation. As in B.

L. Strong collection. Collection, using any formula.

M. Limited principle of omniscience. Every function from ω into ω has a zero or is everywhere nonzero.

2. FORMALIZING CLASSICAL ANALYSIS

PROBLEM. Create much stronger natural conservative extensions of PA that support the direct formalization of yet more mathematics.

PROBLEM. At some point in classical analysis, we encounter transfinite inductions. This should be handled by ATR_0. Thus we should also develop the subject "strong conservative extensions of ATR_0".

3. STRONG INTUITIONISTIC ZF

Intuitionistic ZF comes in several flavors.

1. There are systems in which power set is severely weakened. For example, by using the set of all functions from one set into another, as in the systems we have discussed thus far. Michael Rathjen also talked about these kind of systems. These are no stronger than ZF without power set, and often considerably weaker than that.

2. The systems with separation and power set. These are very strong. Either as strong as ZF, or at least as strong as Z.

Of course, we are assuming the Axiom of Infinity.

We now focus on 2.

3. STRONG INTUITIONISTIC ZF

An early divide was the systems of

J. Myhill, Some properties of intuitionistic Zermelo-Fraenkel set theory, in "Lecture Notes in Mathematics Vol. 337," pp. 206-231, Springer-Verlag, Berlin, 1973.

H. Friedman, The consistency of classical set theory relative to a set theory with intuitionistic logic, JSL, Vol. 38, No. 2, June 1973, 315-319.

where the Myhill system uses Replacement instead of my system, which uses Collection.

To avoid confusion, we will write Myhill's system as ZFI(R), and my system as ZFI(C). Both systems use only $\in$.

3. STRONG INTUITIONISTIC ZF

I formulated the axioms of ZFI(C) as follows.

1. Pairing. $(\exists x)(a \in x \wedge b \in x)$.
2. Union. $(\exists x)(\forall y)((\exists z \in a)(y \in z) \rightarrow y \in x)$.
3. Infinity. $(\exists x)((\exists y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z))$.
4. Separation. $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in a \wedge \varphi))$, where x is not free in φ.
5. Transfinite Induction. $(\forall x)((\forall y \in x)(\varphi[x/y]) \rightarrow \varphi) \rightarrow \varphi$, where y is not in φ.
6. Collection. $(\forall x \in a)(\exists y)(\varphi) \rightarrow (\exists z)(\forall x \in a)(\exists y \in z)(\varphi)$, where z is not free in φ.
7. Extensionality. $(\forall x)(x \in a \leftrightarrow x \in b) \rightarrow (a \in c \leftrightarrow b \in c)$.
8. Power set. $(\exists x)(\forall y)((\forall z \in y)(z \in a) \rightarrow y \in x)$.

3. STRONG INTUITIONISTIC ZF

I needed to consider S-, where Extensionality is dropped, and 8 (power set) is weakened to

8'. Weak power set. $(\exists x)(\forall y)(\exists z \in x)(\forall w)(w \in z \leftrightarrow (w \in y \wedge w \in a))$.

8' says "given a, there exists x such that the intersection of any set with a is extensionality equivalent to some element of x".

First I interpreted ZF into S. These are both classical systems.

Then I gave a syntactic translation of S into S-. It is an adaptation of Gödel's negative interpretation.

THEOREM. EFA proves the following. Con(ZF) $\leftrightarrow$ Con(ZFI(C)). ZF and ZFI(C) prove the same Π^0_1 sentences. The same is true with S- instead of ZFI(C).

Here EFA = exponential function arithmetic.

PROBLEM. Explore fragments of ZFI(C) that are strong enough for the above Theorem.

3. STRONG INTUITIONISTIC ZF

In the following paper, I developed an unexpectedly simple method for obtaining Π^0_2 conservative extension results in contexts where we have the negative interpretation. I applied it to HA and PA, replacing the Gödel Dialectica proof with a new proof of a few lines.

H. Friedman, Classically and intuitionistically provably recursive functions, in "Lecture Notes in Mathematics Vol. 669", pp. 21-27, Springer-Verlag, Berlin, 1978.

The method works generally enough to cover ZF, ZFI(C), and S-.

THEOREM. EFA proves the following. ZF and ZFI(C) prove the same Π^0_2 sentences. The same is true with S- instead of ZFI(C).

3. STRONG INTUITIONISTIC ZF

Myhill's system, which we write as ZFI(R), is the same as our ZFI(C), but with Replacement instead of Collection:

6'. Replacement. $(\forall x \in a)(\exists! y)(\varphi) \rightarrow (\exists z)(\forall x \in a)(\exists y \in z)(\varphi)$, where z is not free in φ. Here ! is formulated using extensional equality.

Joint work with Myhill appears in his 1973 paper:

THEOREM. ZFI(R) has the set existence property. I.e., if ZFI(R) proves the sentence $(\exists x)(\varphi)$, then there is a formula ψ with at most the free variable y, such that ZFI(R) proves the sentence $(\exists x)((\forall y)(y \in x \leftrightarrow \psi) \wedge \varphi)$.

The above uses general techniques developed in my 1973 paper in the same volume, which adapts the Kleene slash | :

H. Friedman, Some applications of Kleene's methods for intuitionistic systems, in "Lecture Notes in Mathematics Vol. 337", pp. 113-170, Springer-Verlag, Berlin, 1973.

3. STRONG INTUITIONISTIC ZF

In that paper, I proved

THEOREM. ZFI(C) has the disjunction property and the numerical existence property. I.e., if ZFI(C) proves the sentence $(\forall n)(\varphi)$, then there is a nonnegative n such that ZFI(C) proves the sentence $\varphi[n/n^*]$, where n^* is a standard name for n.

Much, but not nearly all, is known about the relationship between ZFI(R) and ZFI(C). Obviously ZFI(R) $\subseteq$ ZFI(C).

In the Introduction of the following paper, credit is explicitly divided:

H. Friedman, Andrej Scedrov, The Lack of Definable Witnesses and Provably Recursive Functions in Intuitionistic Set Theories, Advances in Mathematics, vol. 57, No. 1, July 1985, 1-13.

3. STRONG INTUITIONISTIC ZF

THEOREM. ZFI(C) does not have the existence property. Hence ZFI(R) is properly contained in ZFI(C).

THEOREM. There are Π^0_2 sentences provable in ZFI(C) that are not provable in ZFI(R). The provably recursive functions of ZFI(R) are eventually bounded by a provably recursive function of ZFI(C).

PROBLEM. Is every Π^0_1 sentence provable in ZFI(C) provable in ZFI(R)? Does EFA prove Con(ZFI(R)) $\leftrightarrow$ Con(ZFI(C))?

3. STRONG INTUITIONISTIC ZF

In the paper

H. Friedman, A. Scedrov, Set Existence Property for Intuitionistic Theories with Dependent Choice, Annals of Pure and Applied Logic 25 (1983) 129-140

RDC = relativized dependent choice, is added to ZFI(R), and the resulting system is proved to have the set existence property. This uses our paper extending Kleene's slash.

Incidentally, there is my shocking paper

H. Friedman, The disjunction property implies the numerical existence property, Proc. Natl. Acad. Sci. USA, Vol. 72, No. 8, pp. 2877-2878, August 1975

which shows that any r.e. extension of intuitionistic arithmetic which obeys the disjunction property obeys the numerical existence property. And any r.e. extension of intuitionistic arithmetic proves its own disjunction property if and only if it proves its own inconsistency.

PROBLEM. Make more out of this shock.

4. INTUITIONISTIC THEORY OF LARGE CARDINALS

I took up the matter of intuitionistic large cardinal theory in my joint paper

H. Friedman, A. Scedrov, Large Sets in Intuitionistic Set Theory, Annals of Pure and Applied Logic 27 (1984) 1-24.

We give intuitionistically sensible definitions of

1. inaccessible set.
2. Mahlo set.
3. k-Mahlo set.
4. $j:V \to M$, as in elementary embeddings in classical set theory.

We show that ZFI(C) together with the existence principles associated with 1-4, forms a theory with the usual properties of sensible intuitionistic theories.

4. INTUITIONISTIC THEORY OF LARGE CARDINALS

They are equiconsistent with their classical counterparts, and also have the same provable $\Pi 01$ sentences. The idea is that the negative interpretation goes through, as well as various forms of Kleene's realizability. In particular, we get compatibility with Church's Thesis.

In the last section of this paper, we extend the work to appropriate intuitionistic formulations of supercompact, huge, and Reinhardt's axiom (inconsistent with ZFC).

PROBLEM. What can we say about those formulations that work well under intuitionistic logic, versus those that do not? What are the relevant syntactic conditions for a successful formulation in the intutionistic framework?