CONCEPT CALCULUS

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1. INTRODUCTION.

Concept Calculus seeks to isolate fundamental principles about informal concepts that are used in everyday reasoning outside mathematics, science, and engineering. This process creates formal systems associated with various groups of informal concepts.

We have discovered that many of the formal systems that naturally arise have surprisingly great interpretation power - sometimes as much as the most powerful systems of abstract set theory currently under investigation.

If a system interprets the usual ZFC axioms for mathematics, then we say that the system provides a consistency proof for mathematics. This is because any purported inconsistency in ZFC is converted to an inconsistency in the system via the interpretation.

Here we discuss a particular corner of Concept Calculus that was presented in the extended abstract

http://www.cs.nyu.edu/pipermail/fom/2013-January/016898.html 515: Eight Supernatural Consistency Proofs for Mathematics, FOM, January 19, 2013.

The following are our writings on Concept Calculus. All of these can be downloaded from http://www.math.osu.edu/~friedman.8/manuscripts.html

There is one published paper, one (essentially) accepted paper, one submitted paper, and several abstracts, in addition to the above from FOM. We omit earlier versions of 515.

Concept Calculus: Much Better Than, in: New Frontiers in Research on Infinity, ed. Michael Heller and W. Hugh Woodin, Cambridge University Press, 130-164, 2010.

A Divine Consistency Proof for Mathematics, to appear.

**Concept Calculus: universes. October 2, 2012, 33 pages, submitted for publication.
'Concept Calculus', October 25, 2006, 42 pages, abstract.**

'Concept Calculus', Mathematical Methods in Philosophy, Banff, Canada, February 21, 2007, 9 pages.

'Concept Calculus', APA Panel on Logic in Philosophy, APA Eastern Division Annual Meeting, Baltimore Maryland, January 2, 2008, 17 pages.

Concept Calculus, Carnegie Mellon University, March 26, 2009, Pure and Applied Logic Colloquium.

Concept Calculus, Department of Philosophy, MIT, November 4, 2009, 22 pages.

These 8 Consistency Proofs show that, in various ways, we can very naturally interpret set theory in *flat* systems. I.e., in a very natural way, we do not have to rely on anything like the cumulative hierarchy, or an iterative notion of set. Objects and classes of objects suffice, and in this very limited two sorted context, membership is vividly understandable and universally familiar.

Two general principles:

i. The Supernatural World is more extensive than the Real World.
ii. The Real World and the Supernatural World are similar in various respects.

NOTE: Obviously many of these systems have the flavor of very strong forms of nonstandard analysis. Gödel wrote a couple of paragraphs about Nonstandard Analysis. See

K. Gödel, Remark on non-standard analysis, 1974, in: Gödel's Collected Works, volume 2, p. 307-310 (J.E. Fenstad) and p. 311 (K. Gödel).

2. CORE SYSTEM, CORE.

All systems presented here extend CORE, which is a two sorted system.

L(CORE)

1. variables v_i over objects.
2. variables A_i over classes of objects.
3. = between objects.
4. = between classes.
5. binary function symbol P on objects (ordered pairing).
6. binary relation symbol $\in$ between objects and classes (membership).

Note how flat L(CORE) is, with only objects and classes. We obtain great logical strength using only basic principles of a logical nature.

Here are the axioms of CORE.

CORE

1. Logic. The usual axioms and rules of logic for L(CORE).
2. Pairing. $P(v_1,v_2) = P(v_3,v_4) \rightarrow v_1 = v_3 \wedge v_2 = v_4$.
3. Extensionality. $(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow A_1 = A_2$.
4. Comprehension. $(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, **where** φ **is a formula of the language of the system in which** A_1 **is not free.**

Note that CORE has a trivial model where there is one object and two classes.

THEOREM 2.1. The system CORE + $(\exists v_1,v_2)(v_1 \neq v_2)$ is mutually interpretable with Z_2.

3. EQUIVALENCE SYSTEM, EQ.

L(EQ)

1. L(CORE).
2. Unary function symbol CH from classes to objects (choice operator).
3. Class constant symbol RO (class of all real objects).

Thus in EQ, we have a real world and a supernatural world.

For any formula φ in L(EQ), we let φ/RO be the result of replacing quantifiers (Qv), (QA) in φ, by $(Qv \in RO)$, $(QA \subseteq RO)$.

We can expand φ/RO into primitive notation in the usual way.

EQ

1. CORE, expanded to allow all formulas in L(EQ).
2. Choice. $v_1 \in A_1 \rightarrow CH(A_1) \in A_1$.
3. Supernatural Existence. $(\exists v_1)(v_1 \notin RO)$.
4. Real/Supernatural Equivalence. $v_1 \in RO \rightarrow (\varphi \leftrightarrow \varphi^*)$**, where φ is a formula of L(EQ) without RO, where every free variable is** v_1.

4 asserts that every real object has the same properties in the real world as it does in the supernatural world.

EQ gets well past ZFC.

Recall λ is Π^1_n indescribable: $(V(\lambda),\in,R) \models \varphi \rightarrow (\exists\kappa < \lambda)((V(\kappa),\in,R \cap V(\alpha)) \models \varphi)$, where φ is a Π^1_n sentence.

Let T_1 = ZFC + $(\exists\kappa)(\kappa$ is Π^1_n indescribable) as a scheme indexed by n.

Let T_2 = ZFC + $(\exists\kappa < \lambda)(\exists<)(<$ is a w.o. of $V(\lambda)$ where $V(\kappa)$ is an initial segment of $<$, and $((V(\kappa),\in)$ is a Π^1_n elementary substructure of $(V(\lambda),\in))$, as a scheme indexed by n.

THEOREM 3.1. The following are mutually interpretable.

1. EQ.

2. T_1.

3. T_2.

T_1 proves T_2. Assume T_1, and let λ be $\Pi^1{}_{n+1}$ indescribable. Let < be a w.o. of $V(\lambda)$, in order of increasing rank. For R, use the n quantifier diagram for $(V(\lambda),\in,<)$.

To interpret T_1 in T_2, relativize to L. Let n be given, and let $\kappa,\lambda,<$ be a witness for T_2 with n+8, where < is $<_L$. If λ is not $\Pi^1{}_n$ indescribable, take the L least counterexample. Now apply T_2 with $\kappa,\lambda,<$, to show that this is not a counterexample.

EQ is interpretable in T_2 in the obvious way. The objects are elements of $V(\lambda)$, the classes are elements of $V(\lambda+1)$, and RO = $V(\kappa)$.

It remains to interpret T_2 in EQ. First one does some preliminary work concerning CORE + Choice. You get a well ordering of the objects definable from ∈,P,=, and a single real object. Then build the cumulative hierarchy as far as you can along the well ordering. The cumulative hierarchy on a point is coded by a bounded class.

Prove that the construction works at every point. If false, the construction fails at a real point x. But then compare the construction up to that real point, both in the full world and in the real world. Equivalence shows that the construction works at x, contradiction.

In this way, we get the cumulative hierarchy on all points, and we can compare it with the cumulative hierarchy just on the real points. Now apply equivalence.

4. FIRST EXTENSION SYSTEM, EX1.

L(EX1)

1. L(CORE).
2. Class constant symbol RO.
3. Unary function symbol F from objects to classes.

Note that we do not have the choice operator CH.

EX1

1. CORE, expanded to allow all formulas in L(EX1).
2. Supernatural Plenitude.
3. Supernatural Extension.

SUPERNATURAL PLENITUDE

$A_1 \subseteq RO \rightarrow (\exists v_1)(F(v_1) = A1)$.

SUPERNATURAL EXTENSION

$\varphi/RO \wedge A_1 \subseteq RO \rightarrow (\exists A_2 \supseteq\neq A_1)(\varphi[A_1/A_2])$

where φ is a formula in $=,\in,P$ in which all free variables are A_1, and A_2 is not bound in φ, and $\varphi[A_1/A_2]$ is the result of replacing all free occurrences of A_1 in φ by A_2.

This asserts that "any true statement in the real world about a real class lifts to a true statement in the supernatural world about some extension of the real class".

I claimed in the FOM abstract that EX1 and EQ1 are mutually interpretable, but now I doubt this. In any case, I see that $\kappa \rightarrow \omega$ is enough to give a model of EX1.

That T_2 is interpretable in EX1 seems correct. The issue is that instead of using choice to create a well ordering of the objects, we use Supernatural Plenitude, which is enough to develop L in the appropriate way.

Thus $\kappa \to \omega$ is an upper bound, and second order indescribability is a lower bound.

5. SECOND EXTENSION SYSTEM, EX2.

L(EX2) = L(EQ1).

EX2

1. CORE, expanded to allow all formulas in L(EX2).
2. Choice. v1 $\in$ A1 $\to$ CH(A1) $\in$ A1.
3. Supernatural Extension.

Thus EX2 uses Choice, and EX1 uses Supernatural Plenitude.

Again, $\kappa \to \omega$ is an upper bound, second order indescribability is a lower bound.

6. THIRD EXTENSION SYSTEM, EX3.

EX3 is the combining of EX1 and EX2. Again, $\kappa \to \omega$ is an upper bound, and second order indescribability is a lower bound.

7. STRONG EQUIVALENCE SYSTEM, STEQ.

L(STEQ)

1. L(CORE).
2. CH.
3. RO.
4. C, unary function symbol from objects to objects.

Thus L(STEQ) is L(EQ) augmented with C.

STEQ

1. CORE, expanded to allow all formulas in L(STEQ).
2. Choice.
3. Supernatural Existence.
4. Real/Supernatural Equivalence. C not allowed.
5. Correspondence. $C(v_1) \in RO) \wedge (C(v_1) = C(v_2) \rightarrow v_1 = v_2)$.

Thus STEQ is EQ with the addition of Correspondence, asserting a one-one map from objects to real objects (where C allowed in CORE).

We have got past $(\forall x \subseteq \omega)$(x# exists), but STEQ may be far stronger.

8. STRONG EXTENSION SYSTEM, STEX.

L(STEX) = L(STEQ)

1. L(CORE).
2. CH.
3. RO.
4. C.

STEX

1. CORE, expanded to allow all formulas in L(STEX).
2. Choice.
3. Supernatural Extension. C not allowed.
4. Correspondence.

Thus STEX is EX1 with the addition of Correspondence, asserting a one-one map from objects to real objects (where C allowed in CORE).

We got past $(\forall x \subseteq \omega)$(x# exists), but STEX may be far stronger.

9. DIVINE SYSTEM, DIV.

L(DIV).

1. **L(CORE).**
2. **CH.**
3. **POS. Unary predicate on classes.**
4. **DEF. Unary predicate on classes.**

DIV

1. CORE, for L(DIV).
2. Choice.
3. Positive Classes.
$(\forall v_1)(v_1 \in A_1 \vee v_1 \in A_2) \rightarrow POS(A_1) \vee POS(A_2)$. $POS(A_1) \wedge POS(A_2) \rightarrow (\exists v_1 \neq v_2)(v_1, v_2 \in A_1 \wedge v_1, v_2 \in A_2)$.
4. 0-Definable Classes.
$(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge \ldots \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ is a formula of L(DIV) without DEF, with free variables among $v_1, A_2, \ldots, A_n$.
5. Divine Object.
$(\exists v_1)(\forall A_1)(DEF(A_1) \wedge POS(A_1) \rightarrow v_1 \in A_1)$.

Between measurable cardinals and arbitrarily large Ramsey cardinals with stationary homogenous set.

Note that DIV does not use any notion of real object. All objects are real.

10. EXTREME EXTENSION SYSTEM, EXTEX.

L(EXTEX)

1. L(CORE).
2. CH.
3. Class constant symbol RO.
5. Unary function symbol * from classes to classes (extension operator)
6. Unary function symbol H from objects to classes.

EXTEX

1. CORE, expanded to allow all formulas in L(EXTEX).
2. Global Extension. $v_1,...,v_n \in RO \wedge A_1,...,A_m \subseteq RO \rightarrow (\varphi/RO \leftrightarrow \varphi[A_1/A_1^*,...,A_m/A_m^*])$, **where all free variables of** φ **are among** $v_1,...,v_n,A_1,...,A_m$, $n,m \geq 0$.
3. Bijection. H is a bijection from the objects onto the A contained in RO such that $A^* = A$.

NOTE: Under this formalization, the * operator is relevant only applied to subclasses of RO.

NOTE: An alternative axiomatization that is equivalent, and perhaps closer to intuition, is to

have a symbol for a one-one map from objects, to objects in RO, and also a symbol for a one-one map from the A contained in RO with A* = A, into objects.

THEOREM. The Extreme Extension System, EXTEX, interprets ZFC + "there exists a nontrivial elementary embedding from some V(λ) into V(λ), and even I2 (and somewhat more).