Note: This is an interim version that corrects the axioms in section 1.1. There was a problem with interpreting the axiom of infinity in the earlier version. As we are now rushing to finish the book “BRT and Incompleteness”, we will not make a major anticipated revision of this extended abstract at this time.

CONCEPT CALCULUS

by

Harvey M. Friedman

Ohio State University

July 27, 2006

August 7, 2006

October 25, 2006

PREFACE. We present a variety of basic theories involving fundamental concepts of naive thinking, of the sort that were common in "natural philosophy" before the dawn of physical science. The most extreme forms of infinity ever formulated are embodied in the branch of mathematics known as abstract set theory, which forms the accepted foundation for all of mathematics. Each of these theories embodies the most extreme forms of infinity ever formulated, in the following sense. ZFC, and even extensions of ZFC with the so called large cardinal axioms, are mutually interpretable with these theories. This is an extended abstract. Proofs of the claims will appear elsewhere.

INTRODUCTION

We present a variety of basic theories involving fundamental concepts of naive thinking, of the sort that were common in "natural philosophy" before the dawn of physical science.

The most extreme forms of infinity ever formulated are embodied in the branch of mathematics known as abstract set theory, which forms the accepted foundation for all of mathematics.

Each of these theories embodies the most extreme forms of infinity ever formulated, in the following sense. ZFC, and even extensions of ZFC with the so called large cardinal axioms, are mutually interpretable with these theories.

Physical science, as we know it today, is based on measuring quantities via the usual real number system, and counting objects via the usual natural number system.

Here we instead use abstract linear orderings (sometimes without linearity), and formulate principles that can be seen to be incompatible with the identification of these linear orderings with the real numbers, or segments of the real numbers. The principles have a clear conceptual meaning to any naive thinker.

At this point, we are not suggesting the overthrow of the real number orthodoxy in physical science. That would be grossly premature, and may never be appropriate. Our aims are quite different and more philosophical.

i. We show how a complex of very simple intuitive ideas that can be absorbed and reasoned with by naive thinkers, leads surprisingly quickly and naturally to a consistency proof for mathematics (ZFC, even with large cardinals).

Thus these naive ideas are surprisingly powerful in light of the fact that mathematics itself is not sufficient to prove its own consistency (in the sense of Gödel's second incompleteness theorem).

ii. Armed with this array of very powerful naive principles, we can search for entirely new contexts in which principles of a similar nature arise with similar results. We expect to find more richly philosophical, or even theological, contexts in which these emerging naive principles are particularly compelling, and relevant to ordinary thinking and life experiences.

iii. In particular, there are contexts in which there are orderings but where the very idea of quantitative measurement (as presently construed) is inappropriate or even absurd, and so there will not be or cannot be any associated real number orthodoxy. For example, "x is more beautiful than y".
Or "idea x is more interesting than idea y". Or "act x is morally preferable to act y". Or "agent x is morally superior to agent y". Or "outcome x is more just than outcome y". Or "act x is more pleasurable than act y." Or "activity x is preferable to activity y" or "state of affairs x is preferable to sate of affairs y".

iv. What is emerging is a true calculus of conceptual principles, whose "logical strengths" is being "calculated".

There is already an entirely solid mathematical basis for the comparison of theories formulated in first order predicate calculus. This is through the fundamental notion of *interpretation* or *interpretability* due to Alfred Tarski.

See http://en.wikipedia.org/wiki/Interpretability and A.Tarski, A.Mostovski and R.M.Robinson, **Undecidable Theories**. North-Holland, Amsterdam, 1953.

v. We call this prospective calculus of conceptual principles, the CONCEPT CALCULUS.

vi. What is the methodology of the Concept Calculus? We identify fundamental informal notions from ordinary thinking, particularly from philosophical subjects where there is a rich literature of discussion going back for long periods of time.

We then consider various combinations of fundamental principles capable of clear and concise formulation.

We then experiment with appropriate combinations of such principles. Some of these first order theories may of course be incompatible with others.

We then "calculate" the logical strengths of these theories.

Here logical strength has come to mean "interpretation power". Calculating logical strength has come to mean

a) an identification of a theory among the robust linearly ordered hierarchy of set theories, arising out of the foundations of mathematics.

b) the verification that the theory in question is mutually interpretable with the identified set theory.

Thus the "measuring tool" is the robust hierarchy of set theories already developed in the foundations of mathematics.

vii. This measuring tool is the appropriate tool needed to compare the logical strengths of two interesting theories arising in Concept Calculus. For, let S and T be two theories arising in Concept Calculus. To compare S and T, we first calculate the logical strengths of S,T in the sense above - through identifying the appropriate two levels of set theory. Then we can merely note the comparison of these two levels of set theory.

viii. This is a good analogy with the measurement of, say, the height of buildings. First they are measured in, say, meters - as, say, a base 10 rational. Then the two base 10 rationals are compared. Generally, one may only commit to an interval and not a single number, where the intervals are both small enough that the comparison can be made.

ix. We are at the very beginning of the development of Concept Calculus, and here great experience with set theories and various techniques developed from mathematical logic are needed to obtain the first significant results. In particular, one has to be facile with how to interpret set theories and interpret into set theories in a wide variety of contexts.

x. However, at some point, basic tools should arise that will make the development of Concept Calculus more amenable to more scholars. In particular, what I want to do now is to develop an array of most basic theories for which I can calculate - or approximately calculate - their logical strength. Then, scholars can work with these basic theories – which should be more 'friendly' than set theories - in order to make calculations, or at least the interpretations needed for approximate calculations.

We will make a number of calculations in Concept Calculus. For a while, we will not try to operate fully systematically, as our main interest at the moment is in showing how naturally one obtains theories that are at least as strong, logically, as mathematics - as identified with ZFC.

What is emerging, over and over again, is two fundamental principles, each of which take somewhat different forms depending on the context in which they are applied.

A. Completeness/Randomness/Creativity (anything that can happen will).

B. Symmetry/Indiscernibility/Horizons (any two horizons are indiscernible to observers on the basis of their extent).

The first principle is well established in the Philosophical literature, as the Aristotelian Principle of Plenitude.

http://en.wikipedia.org/wiki/Principle_of_plenitude

http://poznanstudies.swps.edu.pl/vols/ps51abs.html

http://www.philosophypages.com/dy/p5.htm

http://plato.stanford.edu/entries/aristotle-natphil/notes.html footnote 30 http://64.233.167.104/search?q=cache:GYFG3KOAiJwJ:ethesis.helsinki.fi/julkaisut/teo/syste/vk/kukkonen/studiesi.pdf+Aristotle+Plenitude&hl=en&gl=us&ct=clnk&cd=4

In

http://www.philosophyprofessor.com/philosophies/plenitude-principle.php

some different forms of the Principle of Plentitude are discussed, including the relevant one of Aristotle.

The second principle is probably as least implicit in the Philosophical literature. But since I don't yet have references, let me try to say something intelligible about this general principle.

Suppose you look up to the sky with different strong powers of vision. You will then experience different horizons, some much farther out than others. Ideally speaking, you should not be able to distinguish what you see under the different strong powers of vision, since what you see is, in each case, unimaginably vast.

The idea is that this principle is meant to apply to contexts far more general than cosmology. There should be much more clarifying things to say about it, including possibly a way to combine it with the Principle of Plenitude. However, here we will be content, here, to mold it to some specific contexts of naive thinking.

From looking at the specific contexts in which we apply this Principles, we see the emergence of RIGOROUS theories of

naive probability.
naive statistics.
naive geometry.
naive physics.
naive theory of agents (minds).

More speculatively,

naive differential equations?
naive biology?
naive psychology?

...
naive anything?

For instance, we could try to exploit the naïve idea of the instantaneous rate of change of a quantity varying according to time.

For instance, naive probability and statistics already suggested by the Principle of Plentitude:

time is so vast, that any possibility will eventually occur

This basic principle corresponds very well with what we know from standard mathematically formalized probability theory.

Specifically, in
http://en.wikipedia.org/wiki/Law_of_large_numbers

we see the following formulation of the law of large numbers (although this is not the only formulation):

"The phrase "law of large numbers" is also sometimes used to refer to the principle that the probability of any possible event (even an unlikely one) occurring at least once in a series increases with the number of events in the series. For example, the odds that you will win the lottery are very
low; however, the odds that someone will win the lottery are quite good, provided that a large enough number of people purchased lottery tickets."

In fact, the usual comprehension axiom scheme or separation axiom scheme in set theory can be viewed as a kind of informal, intuitive, naive probability theory in the following sense. A more explicit form is

time is so vast, that any given possible behavior over time intervals will be realized over some time interval

We can also think of a binary relation on two separate scales as an ensemble of data. I.e., we can plot a diagram of pairs (height, weight) of persons. We can assert that the two parts - height and weight - are completely independent (which is of course not actually the case). More abstractly, we can speak of naive independence.

I.e., that the heights and weights have nothing to do with each other. Of course, in reality they are correlated. But when we handle binary relations, we use naïve independence. An important challenge is to incorporate some dependence (correlations) in this program.

In reading the numerous theories presented below in diverse contexts, note that the issue is not (at least yet) one of truth. Various principles can be objected to on various grounds - even if the underlying concepts are taken to be idealized conceptions residing in the mind.

We believe that the various theories can be adjusted in order to meet many kinds of objections. The resulting adjusted theories would also have very high logical strengths which can be calculated. In particular, they can also be used to give consistency proofs of mathematics (as formalized by ZFC).

Examples pursued here of adjusting theories according to reasonable objections, appear in sections 1.3, 2.2, 2.4, the use of discrete point masses in chapter 5, and further developments anticipated in chapter 6.

1. BETTER THAN

1.1. Better Than, Much Better Than.

We use a one sorted predicate calculus with equality, with a binary relation symbol > for "better than", and a binary relation symbol >> for "much better than".

BASIC. Nothing is better than itself. x >> y □ x > y. x >> y □ y > z □ x >> z. x > y □ y >> z □ x >> z. Given any two things, there is something much better than both.

MINIMALITY. There is nothing that is better than all minimal things. Anything that is much better than a given thing is also much better than something minimally better than that given thing.

EXISTENCE. Let x be a thing better than a given range of things. There is something that is better than the given range of things and the things that they are better than, and nothing else. Here we use L(>,>>) to present the range of things.

HORIZON. Let y > x be given, as well as a true statement about x, using “better than”, and “much better than x”. The corresponding statement about x, using “better than”, and “much better than y” is also true. Here we use formulas in L(>) with at most x free to present the true statement.

THEROEM 1.1.1. Basic + Minimality + Existence + Horizon is mutually interpretable with ZFC. This is provable in EFA.

There is a sharper version of Horizon.

HORIZON (binary). Let y > x be given, as well as a true statement about x, using the binary relations > and z >> w >> x. The corresponding statement about x, using the binary relations > and z >> w >> y is also true.

HORIZON LIMIT. There is something that is better than something, and also much better than everything it is better than.

THEOREM 1.1.2. Basic + Minimal + Existence + Horizon (binary) interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

THEOREM 1.1.3. Basic + Minimal + Existence + Horizon (binary) + Horizon Limit interprets ZFC + "there exists a measurable cardinal □ with kappa many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

1.2. Better Than, Real.

We use a one sorted predicate calculus with equality, with a binary relation symbol > for "better than", and a unary predicate R for "being real".

The idea is that we are dividing the objects up into those that are real and those that are imaginary.

We define

x is minimal if and only if x is not better than anything.

BASIC. Nothing is better than itself. (x > y □ y > z) □ x > z. Something is real.

IMAGINARY. There is something that is better than all real things, and nothing else.

MINIMALITY. There is nothing that is better than all minimal things.

"Even the lowest level things, collectively, have something to offer." This has economic, political, and social ramifications.

REAL EXISTENCE. Let something real be better than a given range of things. There is something real that is better than the given range of things and the things they are better than, and nothing else. Here we use L(>) to present the range of things.

REAL EXAMPLES. If two real things bear a certain relation to something, then they bear that relation to something real. Here we use L(>) to present the relation.

THEOREM 1.2.1. Basic + Imaginary + Minimal + Real Existence + Real Examples is mutually interpretable with ZFC. This is provable in EFA.

In Real Existence, we can put a strong restriction on the formula: Let x be a real thing that is better than a given range of things that is defined where all quantifiers are bounded to x (i.e., to the y < x), but with arbitrary parameters. There is something real that is better than the given range of things and the things they are better than, and nothing else. Here we use L(>) to present the range of things.

If we use this restricted form of Real Existence, then Theorem 1.2.1 remains unchanged.

1.3. Better Than, Real, Conceivable.

From some points of view, one can criticize Real Existence of section 1.2 on the grounds that one is asserting the existence of a real object using a formula that involves things that are not real. By the claim after Theorem 1.2.1, we have met this objection to a considerable extent. However, we still use arbitrary parameters.

We meet this criticism by adding the new notion "conceivable". We replace Real Existence by Conceivable Existence, which has heavy restrictions.

We use a one sorted predicate calculus with equality, with a binary relation symbol > for "better than", and two unary predicates R, C. Here R(x) means "x is real". C(x) means "x is conceivable".

We define

x is minimal iff x is not better than anything.

BASIC. Nothing is better than itself. If a first thing is better than a second thing, and the second thing is better than a third thing, then the first thing is better than the third thing. Everything that is real is conceivable. The only things that a real thing can be better than are real.

REAL MINIMAL. There is nothing real that is better than all real minimal things.

CONCEIVABLE EXISTENCE. Let a range of real things be given, defined with reference to real things only. There is a conceivable thing that is better than the given range of things and the things that they are better than, and nothing else.

TRANSFER. Let a true statement be given involving two specific real things, "better than", and "being real". The corresponding statement is true involving the two real things, "better than", and "being conceivable". Here we use L(>,R) to present the true statement.

In Conceivable Existence, the range is given by a formula □ all of those parameters are real things, and whose quantifiers all range over real things.

THEOREM 1.3.1. Basic + Real Minimal + Conceivable Existence + Transfer interprets ZFC + "there exists a totally indescribable cardinal" and is interpretable in ZFC + "there exists a subtle cardinal". This is provable in EFA.

2. VARYING QUANTITIES

2.1. Single Varying Quantity.

The language L(<,F,T) is in first order predicate calculus with equality, the binary relation symbol <, the unary function symbol F, and the unary relation symbol T.

We think of < as the linear ordering of time. F(x) is the value of the varying quantity at time x.

Note that the values of the varying quantity are from the same scale as time. This can rather easily be objected to, and section 2.2 is in response to this objection.

T(x) is read "x is a transition". Thus if T(x) then x represents the beginning of a new epoch. Thus the transitions are spaced unimaginably apart. One can also think of x as representing an "horizon". If y < x and T(x), then x is unimaginably far beyond y.

Bounded intervals are intervals of the form

(x,y), [x,y], (x,y], [x,y)

where x,y are points. We allow x ≥ y when using interval notation.

LINEARITY. < is a linear ordering.

TRANSITIONS. Every time is < some transition.

ARBITRARY BOUNDED RANGES. Every bounded range of values is the range of values over some bounded interval. Here we use L(<,F,T) to present the bounded range of values.

TRANSITION SIMILARITY. Any true statement involving x,y, and a given transition z > x,y, remains true if z is replaced by any transition w > z. Here we use L(<,F) to present the true statement.

THEOREM 2.1.1. Linearity + Transitions + Arbitrary Bounded Ranges + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

TAIL TRANSITION SIMILARITY. Let x,y,z be specific times, and let a true statement involving the times x,y, the relation <, and "being a transition > x,y" be given. The statement remains true about the times x,y, the relation <, and "being a transition > x,y,z". Here we use L(<,F) to present the true statement.

THEOREM 2.1.2. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

TRANSITION ACCUMULATION. There is a point which is the limit of earlier transitions.

THEOREM 2.1.3. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with kappa many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

THEOREM 2.1.4. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □", and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinals". This is provable in EFA.

2.2. Two Varying Quantities, Three Separate Scales.

In most contexts from naive thinking, a quantity varies over time, where the scale for that quantity is not appropriately identified with the time scale. In mathematical science, there is usually just one scale, as the real numbers are normally used for all scales.

We can carry out a development that is analogous to that of section 2.1, except that

i. We use two varying quantities rather than one.
ii. We use (and need) a deeper formulation of Arbitrary Bounded Ranges.

We introduce axioms formulated in the three sorted predicate calculus with equality on each sort.

The three sorts are: the time scale, the first quantity scale, and the second quantity scale.

We use $t,t_1,t_2,\ldots$ for variables over the time scale.
We use $x,x_1,x_2,\ldots$ for variables over the first quantity scale.
We use $y,y_1,y_2,\ldots$ for variables over the second quantity scale.

We use the binary relation symbol $<_1$ on the time scale.
We use the binary relation symbol $<_2$ on the first quantity scale.
We use the binary relation symbol $<_3$ on the second quantity scale.

We use the unary function symbol F from the time scale to the first quantity scale.

We use the unary function symbol G from the time scale to the second quantity scale.

We also use the unary relation symbol T on the time scale for "transitions" (as in section 2.1).

We refer to this language as L[3,T]. If we do not use T, then we write L[3].

It will also be convenient to refer to the time scale as "the first scale", the first quantity scale as "the second scale", and the second quantity scale as "the third scale".

LINEARITY. $<_1,<_2,<_3$ are strict linear orderings.

BOUNDEDNESS. The values of F on any bounded interval of time are included in some bounded interval in the second scale. The values of G on any bounded interval of time are included in some bounded interval in the third scale.

TRANSITIONS. Every time is earlier than some transition.

By a "suitable pair" we mean a pair x,y, where x is from the first scale and y is from the second scale.

ARBITRARY BOUNDED RANGES. Every bounded range of suitable pairs is the range of values of F,G over some bounded time interval. Here we use L[3,T] to present the bounded range of suitable pairs.

We now present a more subtle form of Arbitrary Bounded Ranges. This is a key idea needed to obtain high logical strength in this section.

Let us digress into an informal discussion. Recall the informal idea from the Introduction,

time is so vast, that any possibility will eventually occur

A more refined form for our purposes is

time is so vast, that any given possible behavior over time intervals will be realized over some time interval

Clearly Arbitrary Bounded Ranges is a special case of this general principle, where the behavior over a time interval focuses on the range of pairs of quantities.

However, there is a more subtle kind of behavior over a time interval, that we can consider.

Let J be a time interval and x lie in the second scale (the first quantity scale). We can consider the range of values in the third scale (the second quantity scale) that occur at some time in J where the second scale value is x. This gives what we call a range of cross sections (in the third scale)
over J. One cross section for each x.

We want to formulate a principle concerning the realization of a (suitably bounded) arbitrary range of cross sections over some time interval.

How are we going to specify "ranges of cross sections" in order to formulate this principle? How are we going to present "ranges of cross sections"?

The idea is very simple. Suppose we are presented an n-ary relation R, n ≥ 2, where the last argument lies within the third scale, and the earlier arguments are all from various scales. It is clear what we mean by the range of cross sections of R (obtained by fixing the first n-1 arguments in any way).

ARBITRARY BOUNDED CROSS SECTIONS. For any given bounded relation R with sorted arguments, the last sort being the third scale, there is a bounded time interval J such that the range of cross sections of R is the same as the range of cross sections over J. Furthermore, we can reverse the role of the second and third scales in this principle. We use L[3,T] to present R.

We say that a transition t is later than a point in any of the three scales if and only if

i. If the point is a time t_1 then $t_1 < t$; and
ii. if the point is a quantity then that quantity appears before time t.

TRANSITION SIMILARITY. Any true statement involving points z,y,z in the various three scales and a transition later than these points remains true if the transition is replaced by any later transition. Here we use L[3] to present the statement.

THEOREM 2.2.1. Linearity + Boundedness + Transitions + Arbitrary Bounded Ranges + Arbitrary Bounded Cross Sections + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

TAIL TRANSITION SIMILARITY. Let x,y,z,w be points in the various three scales, and let a true statement involving x,y,z, the relations $<_1,<_2,<_3$, and "being a transition later than x,y,z". Then the statement remains true about x,y,z, the relations $<_1,<_2,<_3$, and "being a transition later than x,y,z,w".

THEOREM 2.2.2. Linearity + Boundedness + Transitions + Arbitrary Bounded Ranges + Arbitrary Bounded Cross Sections + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in ZFC.

TRANSITION ACCUMULATION. There is a time which is the limit of earlier transitions.

THEOREM 2.2.3. Linearity + Boundedness + Transitions + Arbitrary Bounded Ranges + Arbitrary Bounded Cross Sections + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

THEOREM 2.2.4. Linearity + Boundedness + Transitions + Arbitrary Bounded Ranges + Arbitrary Bounded Cross Sections + Tail Transition Similarity + Strong Transition Accumulation interprets interpret ZFC + "there exists a measurable cardinal □ of order ≥ □" and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinal”. This is provable in EFA.

2.3. Varying Bit.

In this section, we work with a bit varying over time. This is obviously far more elemental than the varying quantities considered in sections 2.1, 2.2.

In order to get logical strength out of this very elemental situation, we need to use forward translations of time. This is most conveniently introduced through an addition function.

Forward time translation is based on the idea that for any times a,b, time going forward from a looks the same as time going forward from b.

Conceptually, a+b from the point of view of a looking forward, corresponds exactly to b from the point of view of -infinity looking forward. I.e., the "amount of time from a to a+b is measured by b".

We also say that a+b is the result of translating a by b. Let A be a set of points. The translation of A by b is the set of all a+b such that a lies in A.

We work in the usual predicate calculus with equality with the binary relation symbol <, the unary predicates T,P, and the binary operation +.

Here, as before, T(t) means that "t is a transition".

P corresponds to the varying bit. P(t) means "the varying bit at time t is 1".

We can give several informal interpretations of this setup.

a. A light is flashing on and off, forever. P(t) means that the light is on at time t.

b. An agent is active or inactive, at any given time. P(t) means that the agent is active at time t.

c. Instead of a time scale, we think of a physical ray, or one dimensional space with a direction. There are pointmasses at some positions. P(t) means that there is a pointmass at position t. Or P(t) means that the mass density at time t is 1; otherwise 0.

All three of these interpretations suggest some natural and fundamental restrictions on behavior, that are not considered in this section. There are considered in section 2.4 and chapter 5.

LINEARITY. < is a linear ordering.

TRANSITIONS. Every time is earlier than some transition.

BOUNDED TIME TRANSLATION. For every given range of times before a given time b, there exists a translation time c such that a time before b lies in the range of times if and only if the bit at time b+c is 1. Here we use L(<,T,P,+) to present the range of times.

The idea of Bounded Time Translation is our usual one: that the behavior of P over bounded intervals is arbitrary, up to translation. As mentioned above, various reasonable objections can be made to this idea in the full generality used here, some of which are met in the next section on Flashing Lights.

TRANSITION SIMILARITY. Any true statement involving x,y and a transition z > x,y, remains true if z is replaced by any transition w > z. Here we use L(<,P,+) to present the statement.

TAIL TRANSITION SIMILARITY. Let x,y,z be specific times, and let a true statement involving x,y, the notions <,P,+, and "being a transition > x,y" be given. The statement remains true about x,y, the notions <,P,+, and "being a transition > x,y,z".

TRANSITION ACCUMULATION. There is a point which is the limit of earlier transitions.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

Note that these axioms correspond to those given in section 2.1, except that we have incorporated addition in Bounded Time Translation (instead of the previous Arbitrary Bounded Ranges).

Note also that we do not need any fact whatsoever about addition, except that which is embodied in Bounded Time Translation.

THEOREM 2.3.1. Linearity + Transitions + Bounded Time Translation + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 2.3.2. Linearity + Transitions + Bounded Time Translation + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

THEOREM 2.3.3. Linearity + Transitions + Bounded Time Translation + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

THEOREM 2.3.4. Linearity + Transitions + Bounded Time Translation + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □" and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinals". This is provable in EFA.

We would like to know that these theories are compatible with substantial principles concerning time and addition. This turns out to be the case, as we see in the next two sections.

2.4. Persistently Varying Bit.

In this section, we weaken Bounded Time Translation considerably to meet objections concerning what is viewed as 'unrealistic' bit variation.

The idea is to reflect the view that a varying bit must be persistent - or alternatively, that one is only interested in persistently varying bits.

Specifically, the value of the bit at any time persists up to some later time. I.e., if a bit is on then it remains on for a while, and if a bit is off then it remains off for a while.

In the case of discrete time, where every time has an immediate successor, persistence is not a restriction. However, for continuous time, or even dense time (where between any two times there is a third), this is a major restriction.

More specifically, we say that a range of times is persistent if and only if for any time t there exists $t_1 > t$ such that i) if t lies in the range of times then every time in the interval $[t,t_1)$ lies in the range of times; if t does not lie in the range of times then no time in the interval $[t,t_1)$ lies in the range of times.

PERSISTENT TIME TRANSLATION. For any time b and persistent range of times before b, there exists a translation time c such that any time before b lies in the range of times if and only if the bit at time b+c is 1. Here we use L(<,T,P,+) to present the range of times.

The idea can be given in two closely related alternative forms:

a. Any possible behavior of a varying bit over a bounded time interval will occur + the behavior of a varying bit must be persistent.

b. Any possible behavior of a persistently varying bit over a bounded time interval will occur.

We cannot carry out the development in section 2.3 with this fundamentally weakened Time Translation axiom scheme. However, we can carry it out if we add

ADDITION. y < z implies x+y < x+z.

ORDER COMPLETENESS. Every nonempty range of times with an upper bound has a least upper bound. Here we use L(<,T,P,+) to present the nonempty range of times.

THEOREM 2.4.1. Linearity + Addition + Order Completeness + Transitions + Persistent Time Translation + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 2.4.2. Linearity + Addition + Order Completeness + Transitions + Persistent Time Translation + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

THEOREM 2.4.3. Linearity + Addition + Order Completeness + Transitions + Persistent Time Translation + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

THEOREM 2.4.4. Linearity + Addition + Order Completeness + Transitions + Persistent Time Translation + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □" and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinals”. This is provable in EFA.

2.5. Naive Time.

In sections 2.3 and 2.4, we use an addition function. In section 2.4, we used some properties of <,+. These properties are fragments of a natural theory of Naive Time, which is of interest in its own right.

In sections 2.3 and 2.4, we can add Naive Time to all of the theories considered, and obtain the same results.

Naive Time uses <,+ (with identity). When we apply it, we will generally have a larger language, and possibly sorts in addition to the time scale sort.

NAIVE LINEARITY. < is a linear ordering with left endpoint and no right endpoint.

NAIVE COMPLETENESS. Every range of times with an upper bound has a least upper bound. Here we use L(<,+), or the underlying language of the theory being axiomatized, to present the nonempty range of times.

NAIVE ADDITION. For every x, the function x+y of y is strictly increasing from all points onto the points ≥ x. We call this the translation function at x. 0+x = x. x+(y+z) = (x+y)+z. Here 0 is the left endpoint.

This completes the presentation of the axioms of Naive Time.

Here are some consequences of Naive Time.

THEOREM 2.5.1. Every nonempty range of times has a greatest lower bound. Here we use L(<,+) to present the nonempty range of times, or whatever the underlying language of the theory being considered.

THEOREM 2.5.2. If x □ y then there is a unique z such that x+z = y.

THEOREM 2.5.3. 0+x = x+0 = x. y < z □ x+y < x+z. y = z □ x+y = x+z. For y > 0, x+y is the least strict upper bound of the x+z, z < y. x □ y □ x+z □ y+z.

Naive Time splits into two branches.

DISCRETENESS. There is an immediate successor of the left endpoint.

DENSITY. There is no immediate successor of the left endpoint.

THEOREM 2.5.4. Naive Time + Discreteness proves every point has an immediate successor. Every nonempty range of values has a least element. Here we use L(<,+), or the underlying language of the theory being axiomatized, to present the nonempty range of times.

THEOREM 2.5.5. Naive Time + Density proves that between any two points there is a third.

In sections 2.3, 2.4, we can add Naive Time + Discreteness to the various axiom systems considered and obtain the same results. The same is true of Naïve Time + Density.

3. BINARY RELATIONS

3.1. Binary Relation, Single Scale.

Here we consider a time scale and a binary relation on points in the time scale.

We can think of this as a naive Cartesian plane, with a naively random pointset. Or we can think of it as naive two dimensional space with a naively random distribution of matter, where the point densities are 0 or 1.

Under such naive physical interpretations, it is important to reflect the symmetry between first and second coordinates.

Or we can think of the scale as the time scale, and a single mind is reflecting on multiple past events as time proceeds. Then $R(t,t_1)$ has the interpretation: the mind at time t is reflecting on the past at time t_1.

This immediately suggests a variant: that instead of a binary relation on a scale, we look at a function f on the scale, obeying $f(x) < x$. E.g., the mind, at any time, is reflecting on exactly one past event.

Yet another variant: $f(x) < x$ when defined. So we then use a partial function. This corresponds to the mind, at any time, reflecting on at most one past event.

Recall that a function f on a scale was already discussed in detail in sections 1 and 2, and simplified in section 3. We will discuss such variants in the future. This goes under the heading: restricted (partial) functions on a scale.

The axioms are formulated in the usual first order predicate calculus with equality, based on the following:

1. The binary relation symbol <.
2. The binary relation R.
3. The unary relation T.

Here, as usual, T(x) means "x is a transition". We write this language as L(<,R,T).

If we are thinking of the naive plane, or naive two dimensional space, then transitions are like the demarcations of physical or spatial horizons. For example, consider

a. Looking out over the ocean.
b. Looking out over the solar system.
c. Looking out over the galaxy.
d. Looking out over the local group.
e. Looking out over the visible universe.
f. "Looking" out over the universe.
g. "Looking" out over the multiverse.

Similarity phenomena have been noted by everyone. A particularly pure form of it is the kind of similarity embodied in our Transition Similarity axioms.

LINEARITY. < is a linear ordering with no right endpoint.

TRANSITIONS. Every point is earlier than some transition.

ARBITRARY BOUNDED RANGES (asymmetric). Every given range of points □ x is precisely the points □ x that some y is related to. Here we use L(<,R,T) to present the range of points.

ARBITRARY BOUNDED RANGES (symmetric). Every given range of points □ x is precisely the points □ x that some y is related to, and precisely the points □ x that is related to some z. Here we use L(<,R,T) to present the range of points.

Note how appropriate Arbitrary Bounded Ranges (symmetric) is under the "naive random" interpretation.

Note that there is a sharper form of Arbitrary Bounded Ranges (asymmetric) that is obtained by removing the second occurrence of "□ x" in Arbitrary Bounded Ranges. This strengthening is not appropriate if we think of R as naively random. However, it makes sense under some mind interpretations.

TRANSITION SIMILARITY. Any true statement involving x,y and a transition z > x,y, remains true if z is replaced by any transition w > z. Here we use L(<,R) to present the statement.

TAIL TRANSITION SIMILARITY. Let x,y,z be specific points, and let a true statement involving x,y, the relation <, and "being a transition > x,y" be given. The statement remains true about x,y, the relation <, and "being a transition > x,y,z".

TRANSITION ACCUMULATION. There is a point which is the limit of earlier transitions.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

THEOREM 3.1.1. Linearity + Transitions + Arbitrary Bounded Ranges (symmetric) + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 3.1.2. Linearity + Transitions + Arbitrary Bounded Ranges (symmetric) + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in ZFC.

THEOREM 3.1.3. Linearity + Transitions + Arbitrary Bounded Ranges (symmetric) + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

THEOREM 3.1.4. Linearity + Transitions + Arbitrary Bounded Ranges (symmetric) + Tail Transition Similarity + Strong

Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □" and is interpretable in ZFC + "there exists a measurable cardinal above infinitely Woodin cardinals". This is provable in EFA.

We now consider

SYMMETRY. R is symmetric. I.e., R(x,y) □ R(y,x).

This makes good sense if we are thinking of two communicating agents - i.e., two agents communicating over time.

THEOREM 3.1.5. Theorems 3.1.1 - 3.1.4 hold if we add Symmetry to all of the theories. We can also use Arbitrary Bounded Ranges (asymmetric).

3.2. Binary Relation, Two Separate Scales.

Here we do not assume that the scale for one axis is the same as the scale for the other.

We can also think of a binary relation on two separate scales as an ensemble of data. I.e., we can plot a diagram of pairs (height, weight) of persons. We can assert that the two parts - height and weight - are completely independent (which is of course not actually the case). More abstractly, we can speak of

naive independence.

This leads of course to the idea that we can rework the whole of probability and statistics as

naive probability theory.
naive statistics.

where the study will of course require that we go well beyond the usual axioms for mathematics (ZFC), and even use large cardinals.

We work with a two sorted predicate calculus, with equality on each sort. These correspond to the two separate scales.

We use x,x1,x2,... for variables over the first scale.
We use y,y1,y2,... for variables over the second scale.

We use the binary relation symbol $<_1$ on the first scale.
We use the binary relation symbol $<_2$ on the second scale.

We use the binary relation symbol R whose first arguments are from the first scale, and whose second arguments are from the second scale.

We use the unary relation symbol T_1 over the first scale.
We use the unary relation symbol T_2 over the second scale.

$T_1(x)$ means that x is a transition in the first scale.
$T_2(y)$ means that y is a transition in the second scale.

The idea is that transitions demarcate horizons. Transitions are unimaginably far apart. E.g., people say

they both play chess, but George is a true chess professional. There is a transition from amateur to professional chess player.

There are rating systems that demarcate different leagues of chess players. There are also different leagues in baseball.

But as we shall see, we are going to rely on the idea that at the upper reaches of the two independent scales, there are "more" fine gradations.

In scales used for measuring finite populations, this is not going to be the case. As one moves up the scales, there are fewer and fewer examples.

However, the situation is arguably quite different for "abstract populations", which consider all possibilities.

LINEARITY. $<_1,<_2$ are linear orderings on the first and second scales, respectively, with no right endpoints.

TRANSITIONS. Every point in the first scale is less than some first scale transition. Every point in the second scale is less than some second scale transition.

ARBITRARY BOUNDED RANGES. Every bounded range of points from the second scale is precisely the points that some point from the first scale is related to. Every bounded range of points from the first scale is precisely the points that are related to some point from the first scale.

Here we use $L(<_1,<_2,T_1,T_2,R)$ to present bounded ranges of points.

TRANSITION SIMILARITY. Any true statement involving points x_1,x_2 in the first scale and a transition $x > x_1,x_2$ remains true if x is replaced by any transition > x. Any true statement involving points y_1,y_2 in the second scale and a transition $y > y_1,y_2$ remains true if y is replaced by any transition > y. Here we use $L(<_1,<_2,R)$ to present the true statements.

TAIL TRANSITION SIMILARITY. Let x,y,z be points in the first scale, and let a true statement involving x,y, the relations $<_1,<_2$,R, and "being a transition > x,y in the first scale" be given. The statement remains true about x,y, the relations $<_1,<_2$,R, and "being a transition > x,y,z in the first scale". Let x,y,z be points in the second scale, and let a true statement involving x,y, the relations $<_1,<_2$,R, and "being a transition > x,y in the second scale" be given. The statement remains true about x,y, the relations $<_1,<_2$,R, and "being a transition > x,y,z in the second scale".

TRANSITION ACCUMULATION. There is a point in the first scale which is the limit of earlier transitions. There is a point in the second scale which is the limit of earlier transitions.

STRONG TRANSITION ACCUMULATION. There is a transition in the first scale which is the limit of earlier transitions. There is a transition in the second scale which is the limit of earlier transitions.

THEOREM 3.2.1. Linearity + Transitions + Arbitrary Bounded Ranges + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 3.2.2. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

THEOREM 3.2.3. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal with many measurable cardinals below " and is interpretable in ZFC + "there exists a measurable cardinal with normal measure 1 measurable cardinals below ". This is provable in EFA.

THEOREM 3.2.4. Linearity + Transitions + Arbitrary Bounded Ranges + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal of order ≥ ”, and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinals”. This is provable in EFA.

4. MULTIPLE AGENTS, TWO STATES.

We use a two sorted predicate calculus, with equality on each sort. The two sorts are the time sort, and the agent (mind) sort.

We use

i. Variables $t,t_1,t_2,...$ over times.
ii. Variables $x,x_1,x_2,...$ over agents (minds).
iii. Binary relation < on the time sort.
iv. Binary relation A relating agents and times.
v. Unary relation T on the time sort.

Here A(x,t) means "mind x is active at time t". T(t) means "t is a transition".

The idea is that an agent will be active at some times and not at other times. The first time an agent is active is regarded as its date of its creation. Although it is natural to do so, we will not assume that every agent has a birthdate (date of creation). However, the axiom of Continual Creation asserts that every time is the birthdate of some agent.

LINEARITY. < is a linear ordering.

TRANSITIONS. Every time is < some transition.

CONTINUAL CREATION. At any time there is some agent which is active at that time but not previously.

UNRESTRICTED ACTIVITY. Let a time t be given. As time varies, the then active agents that have been active previous to t, are arbitrary. Here we use L(<,A,T) to present the arbitrary condition.

TRANSITION SIMILARITY. Any true statement involving times t_1,t_2 and a later transition t remains true t is replaced by any later transition. Here we use L(<,A) to present the true statement.

TAIL TRANSITION SIMILARITY. Lte times t_1,t_2,t_3 be given, and let a true statement involving times t_1,t_2, the relations <,A, and "being a transition > t_1,t_2" be given. The statement remains true about t_1,t_2, the relations <,A, and "being a transition > t_1,t_2,t_3".

TRANSITION ACCUMULATION. There is a point which is the limit of earlier transitions.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

THEOREM 4.1. Linearity + Transitions + Continual Creation + Unrestricted Activity + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 4.2. Linearity + Transitions + Continual Creation + Unrestricted Activity + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in ZFC.

THEOREM 4.3. Linearity + Transitions + Continual Creation + Unrestricted Activity + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

THEOREM 4.4. Linearity + Transitions + Continual Creation + Unrestricted Activity + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □", and is interpretable in ZFC + "there is a measurable cardinal above infinitely many Woodin cardinals". This is provable in EFA.

5. POINT MASSES

5.1. Discrete Point Masses In One Dimensional Space.

We will treat one dimensional space as we did time - a linearly ordered ray, with no right endpoint. Here one dimensional space is made up of points. At some points there lies a point mass. So the point masses form a subclass of the points.

Throughout this chapter, our treatment of point masses is very primitive – we do not assign masses to point masses. Nor do these point masses move.

Consider the case of point masses whose distribution is unrestricted. This is exactly the case of a Varying Bit treated in section 2.3.

We want to consider the case of point masses which are discrete (discretely arranged). I.e., they are isolated from each other.

This situation has much in common with section 2.4, but is a little bit different. So we restate the results.

The language is again predicate calculus with equality in the language <,T,P,+. T(x) means "x is a transition (horizon) point". P(x) means "there is a point mass at position x".

LINEARITY. < is a linear ordering.

ADDITION. y < z implies x+y < x+z.

ORDER COMPLETENESS. Every nonempty range of points with an upper bound has a least upper bound. Here we use L(<,T,P,+) to present the nonempty range of points.

TRANSITIONS. Every point is earlier than some transition.

We say that a range of points is discrete if and only if for any x in the range, there exists y,z such that y < x <

z and x is the only point in the range in the open interval (y,z).

DISCRETE POINT MASS TRANSLATION. For any point b and discrete range of points before b, there exists a translation distance c such that any point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,T,P,+) to present the discrete range of points.

TRANSITION SIMILARITY. Any true statement involving x,y and a transition z > x,y, remains true if z is replaced by any transition w > z. Here we use L(<,P,+) to present the statement.

TAIL TRANSITION SIMILARITY. Let x,y,z be points, and let a true statement involving x,y, the notions <,P,+, and "being a transition > x,y" be given. The statement remains true about x,y, the notions <,P,+, and "being a transition > x,y,z".

TRANSITION ACCUMULATION. There is a point which is the limit of earlier transitions.

STRONG TRANSITION ACCUMULATION. There is a transition which is the limit of earlier transitions.

THEOREM 5.1.1. Linearity + Addition + Order Completeness + Transitions + Discrete Point Mass Translation + Transition Similarity interprets ZFC + "there exists a subtle cardinal" and is interpretable in ZFC + "there exists an almost ineffable cardinal". This is provable in EFA.

THEOREM 5.1.2. Linearity + Addition + Order Completeness + Transitions + Discrete Point Mass Translation + Tail Transition Similarity interprets ZFC + "for all x □ □, x# exists" and is interpretable in ZFC + "there exists a measurable cardinal". This is provable in EFA.

THEOREM 5.1.3. Linearity + Addition + Order Completeness + Transitions + Discrete Point Mass Translation + Tail Transition Similarity + Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ with □ many measurable cardinals below □" and is interpretable in ZFC + "there exists a measurable cardinal □ with normal measure 1 measurable cardinals below □". This is provable in EFA.

THEOREM 5.1.4. Linearity + Addition + Order Completeness + Transitions + Discrete Point Mass Translation + Tail Transition Similarity + Strong Transition Accumulation interprets ZFC + "there exists a measurable cardinal □ of order ≥ □" and is interpretable in ZFC + "there exists a measurable cardinal above infinitely many Woodin cardinals". This is provable in EFA.

As in section 2.4, we can add the naive time principles reformulated as naive one dimensional space principles, in the sense of section 2.5, without changing any of the above results. We can choose between Discreteness and Density. We can also add the axiom "the point masses are discrete" without changing the results.

We now come to bodies. We take this to be non overlapping closed intervals, where the endpoints form a discrete set. This is very similar to what we have just done, although not quite the same. The same results hold.

5.2. Discrete Point Masses With End Expansion.

Here we will consider exactly two snapshots of one dimensional space. The first snapshot we will call "the present". The second snapshot we will call "the future". Every present point is a future point, but not vice versa. The variables range over the future points.

We will also assume that we have a set of point masses. Every point mass that exists at present also exists at the future, in the same position. However, new point masses may be created, at new points.

We do not take into account motion of point masses.

We use predicate calculus with equality, with the following additional symbols.

1. The binary relation symbol < on all points; i.e., points of one dimensional space at the future.
2. The unary relation symbol P where P(x) means "there is a point mass at position x in one dimensional space".
3. The unary relation symbol R where R(x) means "x is a point in one dimensional space at the present".
4. Addition, +.

Note that we do not use transition points.

LINEARITY. < is a linear ordering.

ADDITION. y < z implies x+y < x+z.

ORDER COMPLETENESS (present). Every nonempty range of points in the present, with an upper bound in the present, has a least upper bound in the sense of the present. Here we use L(<,P,R,+) to present the nonempty range of points.

ORDER COMPLETENESS (future). Every nonempty range of points, with an upper bound, has a least upper bound. Here we use L(<,P,R,+) to present the nonempty range of points.

END EXPANSION. If a point is before some point that exists in the present, then that point also exists in the present. There is a point that is not in the present.

PROPERTY PRESERVATION. Any true statement stated in terms of the points existing at the present, and involving a given point existing at the present, remains true when stated in terms of all points, and the given point. Here we use L(<,P,+) to present the true statement.

The idea of Property Preservation is that the expansion of space does not effect any property of points.

DISCRETE POINT MASS TRANSLATION (present). For any point b in the present, and discrete range of points before b in the present, there exists a translation distance c in the present such that any present point x lies in the range of points if and only if there is a point mass at position x+c. Here
we use L(<,P,R,+) to present the discrete range of points.

DISCRETE POINT MASS TRANSLATION (future). For any point b and discrete range of points before b, there exists a translation distance c such that any point x before b lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,+) to present the discrete range of points.

It can be seen that Discrete Point Mass Translation (future) and Order Completeness (present) follow from the preceding axioms.

THEOREM 5.2.1. Linearity + Addition + Order Completeness (present,future) + End Expansion + Property Preservation + Discrete Point Mass Translation (present,future) is mutually interpretable with ZFC. This is provable in EFA.

We can add the naive time principles reformulated as naive one dimensional space principles, in the sense of section 2.5, without changing any of the above results. We can choose between Discreteness and Density. We can also add the axiom "the point masses are discrete" without changing this result.

5.3. Discrete Point Masses With Inner Expansion.

We carry out the development in section 5.2, but with the idea that there exists a point not in the present which is earlier than some point in the present.

We use the same language as in section 5.2.

NAIVE LINEARITY. < is a linear ordering with left endpoint and no right endpoint.

NAIVE ADDITION. For every x, the function x+y of y is strictly increasing from all points onto the points >= x. We call this the translation function at x. 0+x = x. x+(y+z) = (x+y)+z. Here 0 is the left endpoint.

ORDER COMPLETENESS (present). Every nonempty range of points in the present, with an upper bound in the present, has a least upper bound in the sense of the present. Here we use L(<,P,R,+) to present the nonempty range of points.

ORDER COMPLETENESS (future). Every nonempty range of points, with an upper bound, has a least upper bound. Here we use L(<,P,R,+) to present the nonempty range of points.

INNER EXPANSION. There are points x < y such that [x,y] contains no points in the present.

PROPERTY PRESERVATION. Any true statement stated in terms of the points existing at the present, and involving a given point existing at the present, remains true when stated in terms of all points, and the given point. Here we use L(<,P,+) to present the true statement.

POINT MASS TRANSLATION (present). For any point b in the present, and range of points before b in the present, there exists a translation distance c in the present such that any present point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,+) to present the range of points.

POINT MASS TRANSLATION (future). For any point b and discrete range of points before b, there exists a translation distance c such that any point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,+) to present the range of points.

THEOREM 5.3.1. Naive Linearity + Naive Addition + Order Completeness (present,future) + Inner Expansion + Property Preservation + Point Mass Translation (present,future) interprets ZFC + "there exists a Ramsey cardinal" and is interpretable in ZFC + "there exists a measurable cardinal".

We have a number of choices of additional axioms, without changing the results.

i. Discreteness of points.
ii. Density of points.
iii. Discreteness of point masses.
iv. Every point is earlier than some present point.
v. There is a point later than all present points.

Of course, if we add ii then we cannot add i. If we add iv then we cannot add v.

5.4. Point Masses With Inner Expansion.

Here we carry out the development of section 5.3 without restricting to discreteness. We aim for extremely high logical strength.

We will use "transition points of the present". It makes sense to also have transition points associated with the future, but we will not need these.

We use predicate calculus with equality, with the following additional symbols.

1. The binary relation symbol < on all points; i.e., points of one dimensional space at the future.
2. The unary relation symbol P where P(x) means "there is a point mass at position x in one dimensional space".
3. The unary relation symbol R where R(x) means "x is a point in one dimensional space at the present".
4. Addition, +.
5. The unary relation symbol T, where T(x) means "x is a transition point of the present".

NAIVE LINEARITY. < is a linear ordering with left endpoint and no right
endpoint.

NAIVE ADDITION. For every x, the function x+y of y is strictly increasing from all points onto the points >= x. We call this the translation function at x. 0+x = x. x+(y+z) = (x+y)+z. Here 0 is the left endpoint.

ORDER COMPLETENESS (present). Every nonempty range of points in the present, with an upper bound in the present, has a least upper bound in the sense of the present. Here we use L(<,P,R,T,+) to present the nonempty range of points.

ORDER COMPLETENESS (future). Every nonempty range of points, with an upper bound, has a least upper bound. Here we use L(<,P,R,T,+) to present the nonempty range of points.

INNER EXPANSION. There are points x < y such that [x,y] contains no points in the present.

PROPERTY PRESERVATION. Any true statement stated in terms of the points existing at the present, and involving a given point existing at the present, remains true when stated in terms of all points, and the given point. Here we use L(<,P,T,+) to present the true statement.

The idea of Property Preservation is that the expansion of space does not effect the relationship between present points and present point masses.

POINT MASS TRANSLATION (present). For any point b in the present, and range of points in the present before b, there exists a translation distance c in the present such that any present point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,T,+) to present the range of points.

POINT MASS TRANSLATION (future). For any point b and discrete range of points before b, there exists a translation distance c such that any point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,T,+) to present the range of points.

TRANSITIONS. Every transition is at the present. Every present point is earlier than some transition point.

We now come to Transition Similarity. We need a natural strengthening of this principle.

EXTENDED TRANSITION SIMILARITY. Any true statement involving a range of points before a point before a transition, and that transition, remains true if we use any later transition. Here we use L(<,P,R,T,+) and any points, to present the range of points, and L(<,P,R,+) to present the true statement.

THEOREM 5.4.1. Naive Linearity + Naïve Addition + Order Completeness (present,future) + Inner Expansion + Property Preservation + Point Mass Translation (present,future) + Transitions + Extended Transition Similarity interprets ZF + "there exists a nontrivial elementary embedding from some V(□) into V(□), where V(□) is an elementary substructure of V (scheme)". Hugh Woodin has shown some time ago that ZFC + "there exists a nontrivial elementary embedding from some successor rank into itself" is interpretable in this theory, and hence into Naive Linearity + Naive Addition + Order Completeness (present,future) + Inner Expansion + Property Preservation + Point Mass Translation (present,future) + Transitions + Extended Transition Similarity.

We have a number of choices of additional axioms, without changing the results.

i. Discreteness of points.
ii. Density of points.
iii. Every point is earlier than some present point.
iv. There is a point later than all present points.

Of course, if we add iv then we cannot add v.

5.5. Discrete Point Masses With Inner Expansion Revisited.

We carry out the development in section 5.4, but with the idea that the point masses are discrete. In order to carry this off, we need to consider three snapshots of one dimensional space. We call these, respectively, the present points, the intermediate points, and the future points. Every present point is an intermediate point, and every intermediate point is a future point. The variables range over the future points.

We use the language

1. The binary relation symbol < on all points; i.e., points of one dimensional space at the future.
2. The unary relation symbol P where P(x) means "there is a point mass at position x in one dimensional space".
3. The unary relation symbol R where R(x) means "x is a point in one dimensional space at the present".
4. The unary relation symbol S where S(x) means "x is a point in one dimensional space at the intermediate".
5. Addition, +.
6. The unary relation symbol T, where T(x) means "x is a transition point of the present".

NAIVE LINEARITY. < is a linear ordering with left endpoint and no right endpoint.

NAIVE ADDITION. For every x, the function x+y of y is strictly increasing from all points onto the points ≥ x. We call this the translation function at x. 0+x = x. x+(y+z) = (x+y)+z. Here 0 is the left endpoint.

ORDER COMPLETENESS (present). Every nonempty range of points in the present, with an upper bound in the present, has a least upper bound in the sense of the present. Here we use L(<,P,R,S,T,+) to present the nonempty range of points.

ORDER COMPLETENESS (intermediate). Every nonempty range of points in the intermediate, with an upper bound in the intermediate, has a least upper bound in the sense of the intermediate. Here we use L(<,P,R,S,T,+) to present the nonempty range of points.

ORDER COMPLETENESS (future). Every nonempty range of points, with an upper bound, has a least upper bound. Here we use L(<,P,R,S,T,+) to present the nonempty range of points.

INNER EXPANSION. There are points x < y such that [x,y] contains no points in the present.

DOUBLE PROPERTY PRESERVATION. Any true statement stated in terms of the points existing at the present, the points existing at the intermediate, and involving a given point existing at the present, remains true when stated in terms of the points existing at the intermediate, the points existing at the future (i.e., all points), and the given point. Here we use L(<,P,T,+) to present the true statement.

DISCRETE POINT MASS TRANSLATION (present). For any point b in the present, and discrete range of points in the present before b, there exists a translation distance c in the present such that any present point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,S,T,+) to present the discrete range of points.

DISCRETE POINT MASS TRANSLATION (intermediate). For any point b in the intermediate, and discrete range of points in the intermediate before b, there exists a translation distance c in the intermediate such that any intermediate point lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,S,T,+) to present the discrete range of points.

DISCRETE POINT MASS TRANSLATION (future). For any point b and discrete range of points before b, there exists a translation distance c such that any point x lies in the range of points if and only if there is a point mass at position x+c. Here we use L(<,P,R,S,T,+) to present the discrete range of points.

TRANSITIONS. Every transition is at the present. Every present point is earlier than some transition point.

EXTENDED TRANSITION SIMILARITY. Any true statement involving a range of points before a point before a transition, and that transition, remains true if we use any later transition. Here we use L(<,P,R,S,T,+) and any points, to present the range of points, and L(<,P,R,S,+) to present the true statement.

THEOREM 5.5.1. Naive Linearity + Naive Addition + Order Completeness (present,intermediate,future) + Inner Expansion + Double Property Preservation + Point Mass Translation (present,intermediate,future) + Transitions + Extended Transition Similarity interprets NBG + "there is a nontrivial elementary embedding from V into V" interprets ZF + "there exists a nontrivial elementary embedding from some V(□) into V(□), where V(□) is an elementary substructure of V (scheme)". Hugh Woodin has shown some time ago that ZFC + "there exists a nontrivial elementary embedding from some successor rank into itself" is interpretable in this theory, and hence into Naive Linearity + Naive Addition + Order Completeness (present,intermediate,future) + Inner Expansion + Double Property Preservation + Point Mass Translation (present,intermediate,future) + Transitions + Extended Transition Similarity.

We have a number of choices of additional axioms, without changing the results.

i. Discreteness of points.
ii. Density of points.
iii. Discreteness of point masses.
iv. Every point is earlier than some present point.
v. There is a point later than all present points.

Of course, if we add ii then we cannot add i. If we add iv then we cannot add v.

6. TOWARDS THE MEROLOGICAL

In all of the theories thus far, with the exception of the theories in section 1.2, we have used a unary predicate P on the time scale, where P(t) means "t is a transition". The idea is that transition points mark out epochs.

However, we can take the view that changing epochs are not marked by single points. Thus epochs are regions of time.

We view this as a small step towards a fully mereological reworking of our results.

Under this approach, instead of the unary predicate T, we have a binary relation E, where E(x,y) means that x,y are in the same epoch. Thus we have the axiom

EPOCHS. E forms an equivalence relation, where every equivalence class under E forms a bounded interval (however, the endpoints of these equivalence classes may or may not exist).

Transition Similarity is replaced by

EPOCH SIMILARITY. Let x,y,z,w be three given times, where z,w reside in an epoch later than x,y. Let a true statement involving x,y, the relevant notions, and "being in the epoch in which z resides" be given. Then the statement remains true about x,y, the relevant notions, and "being in the epoch in which w resides".

Tail Transition Similarity is replaced by two different forms, one weaker than the other:

TAIL EPOCH SIMILARITY (1). Let x,y,z be three given times, and let a true statement involving x,y, the < relation and other relevant notions, and "being in an epoch later than x,y". The statement remains true involving x,y, the relevant notions, and "being in an epoch later than x,y,z".

TAIL EPOCH SIMILARITY (2). Let x,y,z be three given times, and let a true statement involving x,y, the < relation and other relevant notions, and the relation E restricted to points in an epoch later than x,y". The statement remains true involving x,y, the relevant notions, and E restricted to points in an epoch later than x,y,z.

Transition Accumulation is replaced by

EPOCH ACCUMULATION. There is an epoch which is a limit of earlier epochs.

The results conform to the following analogies.

Transitions: Epochs.
Transition Similarity: Epoch Similarity (2).
Tail Transition Similarity: Tail Epoch Similarity.
Strong Transition Accumulation: Epoch Accumulation.

Transition Accumulation does not have an obvious analog in the epoch formulation.

Fully mereological formulations are expected to be of clear importance and require the banishment of points in favor of intervals (where endpoints have no significance).