CONSERVATIVE GROWTH: A UNIFIED APPROCH TO LOGICAL STRENGTH

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EXTENDED ABSTRACT

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ABSTRACT. In conservative growth, structures are enlarged in such a way that properties are maintained. We investigate a natural interpretation of this idea in terms of one structure M being elementarily extended to M', where M' "organizes" M definability. We show that this idea of conservative growth naturally develops into constructions and hypotheses corresponding to levels of set theory ranging from countable set theory through ZFC through various large cardinal hypotheses. We are working on some further variants that correspond to finite set theory (Peano arithmetic) and other interpretation levels.

1. INTRODUCTION.

Conservative Growth provides a unified foundational view which corresponds closely to the standard levels of logical strength that arise in the foundations of mathematics.

The basic idea is that we have an intellectual process whereby we enlarge a universe of objects once or a few or many times. We postulate that the relationships between objects remain stable undereach enlargement. The most familiar case is of course represented by elementary substructure.

It turns out that this idea leads to some very basic formalisms of wide ranging interpretation power, ranging from fragments of Peano arithmetic all the way through the upper regions of the large cardinal hierarchy.

The idea readily applies to set theory, where each structure is assumed to be some reasonable set theoretic universe - e.g., a model of ZFC. However, the idea in no way depends on the universes of objects being anything close to a universe of sets, or even having anything to do with set theory at all. In fact, in section 3 below, where we are thinking of set theory, we merely assume that each universe satisfies only bounded separation, and each universe is an element of the next universe. In other key applications, we will not be thinking of set theoretic universes at all.

I applied, what is in retrospect, conservative growth of countable sets of subsets of N, as a principal tool in my Borel incompleteness theorems from the 1980's. I was only thinking of Borel incompleteness at the time, and nowadays I focus on Π^0_1 incompleteness. In the old days, I was doing Π^1_3 and Π^1_2 incompleteness, so that I was at least concrete enough to be establishing unprovability from ZFC and fragments of ZFC + V = L.

We will now present Conservative Growth in four environments.

1. Digraph Theory. This is the most rudimentary setting that we consider. A digraph is a G = (V,E), where $E \subseteq V^2$.
2. Model Theory. This is the most general setting that we consider. The treatment is based on signatures SIG consisting of finitely many constant and relation symbols, and equality. No function symbols.
3. Set Theory. Here the treatment takes into account very basic elemental features of set theory based, as usual, on epsilon and equality. If we do not take into account any elemental features of set theory, then this environment is the same as the Digraph Theory environment.
4. As Theories. Here we present section 3 in terms of formal set theories.

DEFINITION 1.1. Let M be a relational structure. $S \subseteq dom(M)$

is M definable if and only if S is of the form {x: φ(x) holds in M}, where φ is a formula in the first order predicate calculus with equality, with parameters allowed from dom(M).

2. DIGRAPH THEORY.

DEFINITION 2.1. A digraph is a pair G = (V,E), where $E \subseteq V^2$. V = V(G) is the set of vertices, and E = E(G) is the edge relation. G ⊆ G′ if and only if V(G) ⊆ V(G′) ∧ E(G) = E(G′) $\cap V^2$. OUT(x,G) = {y: x E y}.

Thus a graph is a very special case of a relational structure in the sense of model theory - just one binary relation. So the G definable subsets of V(G) are available to us.

We now present two forms of one step conservative growth. There are some others that arise from the rudimentary set theoretic point of view, in sections 4 and 5. These other one step conservative growth statements are unprovable in ZFC.

THEOREM 2.1. There exist countable G,G′ such that
i. G is an elementary substructure of G′.
ii. Every G definable subset of V(G) is some OUT(x,G′), x ∈ V(G′).

THEOREM 2.2. There exist countable G,G′ such that
i. G is an elementary substructure of G′.
ii. Every (G,G′) definable subset of V(G) is some OUT(x,G′), x ∈ V(G′).

Here (G,G′) is the relational structure (V(G′),E(G′),V(G)), where V(G) is treated as a unary relation on V(G′).

Here i is the "conservation" and ii is the "growth".

Theorems 2.1 and 2.2 exhibit a general pattern for conservative growth statements. They are implicitly Π^0_1 in virtue of their form.

THEOREM 2.3. Using Gödel's Completeness Theorem, Theorems 2.1 and 2.2 are provably equivalent to a Π^0_1 sentences over WKL_0.

THEOREM 2.4. Theorems 2.1 and 2.2 are provable equivalent to Con(Z_2) over WKL_0. In particular, they cannot be proved in countable set theory as formalized by ZFC\P.

The most obvious form of conservative growth of finite length is as follows.

THEOREM 2.5. There exists countable $G_1,...,G_n$ such that
i. For all $1 \leq i < n$, G_i is an elementary substructure of G_{i+1}.
ii. For all $1 \leq i < n$, every G_i definable subset of $V(G_i)$ is some $OUT(x,G_{i+1})$, $x \in V(G_{i+1})$.

We give the following sharper form.

THEOREM 2.6. There exists countable $G_1,...,G_n$ such that
i. For all $1 \leq i < n$, G_i is an elementary substructure of G_{i+1}.
ii. For all $1 \leq i < n$, every $(G_1,...,G_n)$ definable subset of $V(G_i)$ is some $OUT(x,G_{i+1})$, $x \in V(G_{i+1})$.

THEOREM 2.7. Theorem 2.4 holds for Theorems 2.5 and 2.6.

But there is a much stronger kind of conservative growth of a fundamental nature. Consider G_1,G_2,G_3. When we grow from G_2 to G_3, we may "remember" that we started with G_1 and grew to G_2. Thus, looked at historically, we are actually growing from (G_1,G_2) to (G_2,G_3). We stipulate that this more comprehensive growth is conservative, in a natural sense.

To formalize this idea, we need a convenient way to combine digraphs. We have already used this for Theorems 2.2 and 2.6. We spell out this crucial notion in detail.

DEFINITION 2.2. $(G_1,...,G_n)$ is the relational structure $(V(G_n),E(G_n),V(G_1),...,V(G_{n-1}))$ with domain $V(G_n)$, binary relation $E(G_n)$ on $V(G_n)$, and unary relations $V(G_1),...,V(G_{n-1})$ on $V(G_n)$.

PROPOSITION 2.8. There exists countable $G_1 \subseteq G_2 \subseteq G_3$ such that
i. Every first order property that holds in (G_1,G_2) of elements of $V(G_1)$ also holds in (G_2,G_3).
ii. Every G_1 definable subset of $V(G_1)$ is some $OUT(x,G_2)$, $x \in V(G_2)$.

Again, i is the “conservation” and ii is the “growth”.

The conditions in Proposition 2.9 are quite robust.

THEROEM 2.9. Let $G_1 \subseteq G_2 \subseteq G_3$ have properties i,ii in Proposition 2.8. Then
i. G_1 is an elementary substructure of G_2 is an elementary substructure of G_3.
ii. Every (G_1,G_2) definable subset of $V(G_2)$ is some $OUT(x,G_3)$, $x \in V(G_3)$.

Just as in the case of Theorem 2.1, there are sharper versions of Proposition 2.8 where ii is strengthened. But we instead consider more digraphs as follows.

PROPOSITION 2.10. There exists countable $G_1 \subseteq \ldots \subseteq G_n$ such that
i. Every first order property that holds in $(G_1,\ldots,G_{n-1})$ of elements of $V(G_1)$ also holds in $(G_2,\ldots,G_n)$.
ii. Every G_1 definable subset of $V(G_1)$ is some $OUT(x,G_2)$, $x \in V(G_2)$.

We can sharpen Proposition 2.10 as follows.

PROPOSITION 2.11. There exists countable $G_1 \subseteq \ldots \subseteq G_n$ such that
i. $G_1,\ldots,G_n$ form an elementary chain.
ii. For all intervals $[i,j],[i',j'] \subseteq [1,n]$ of the same length, $i \leq i'$, every first order property that holds in $(G_i,\ldots,G_j)$ of elements of $V(G_i)$ also holds in $(G_{i'},\ldots,G_{j'})$.
iii. For all $1 \leq i \leq n-1$, every $(G_1,\ldots,G_n)$ definable subset of $V(G_i)$ is some $OUT(x,G_{i+1})$, $x \in V(G_{j+1})$.

THEOREM 2.12. Using Gödel's Completeness Theorem, Propositions 2.8, 2.10, 2.11 are provably equivalent to Π^0_1 sentences over WKL_0. This also holds for any fixed n.

THEOREM 2.13. Proposition 2.8 is provably equivalent to Con(ZFC + the scheme On is subtle) over WKL_0. Hence it is provable in ZFC + "there exists a subtle cardinal", but not in ZFC (assuming ZFC is consistent), and not in ZFC + "there exists a totally indescribable cardinal" (assuming ZFC + "there exists a totally indescribable cardinal" is consistent).

THEROEM 2.14. Propositions 2.10 and 2.11 are provably equivalent to Con(SRP) over WKL_0. Hence they are provable in SRP+ but not in any consistent SRP[k].

We now come to conservative growth of infinite length. We work with transfinite sequences of digraphs, $(G_\beta)_{\beta<\alpha}$.

DEFINITION 2.3. $(G_\beta)_{\beta<\alpha}$ is a countable chain if and only if for all $\gamma < \beta < \alpha$, $G_\gamma \subseteq G_\beta \wedge V(G_\beta)$ is countable. $[(G_\beta)_{\beta<\alpha}]$ is the relational structure (D,R,S] with domain $D = \cup V(G_\beta)_{\beta<\alpha}$ and binary relations R,S on D, where $R = \cup E(G_\beta)_{\beta<\alpha}$, $S(x,y) \leftrightarrow (\forall\beta<\alpha)(x \in V(G_\beta) \leftrightarrow y \in V(G_\beta))$. Here the S is needed if α is infinite.

PROPOSITION 2.14. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \alpha$, every $[(G_\gamma)_{\gamma<\beta}]$ definable subset of its domain is some $OUT(x,G_\beta)$, $x \in V(G_\beta)$.
ii. For all $\beta < \alpha$, every first order property true in $[(G_\gamma)_{\beta\leq\gamma<\alpha-1}]$ of elements of $V(G_\beta)$ remains true in $[(G_\gamma)_{\beta+1\leq\gamma<\alpha}]$.

There is an obvious (generally) sharper version of Proposition 2.14.

PROPOSITION 2.15. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta+1 < \alpha$, every $[(G_\gamma)_{\gamma<\alpha}]$ definable subset of $V(G_\beta)$ is some $OUT(x,G_{\beta+1})$, $x \in V(G_{\beta+1})$.
ii. For all $\beta < \alpha$, every first order property true in $[(G_\gamma)_{\beta\leq\gamma<\alpha-1}]$ of elements of $V(G_\beta)$ remains true in $[(G_\gamma)_{\beta+1\leq\gamma<\alpha}]$.

We will work with CGS(α).

CGS(α). Proposition 2.14 holds for α.

Here CGS is read "conservative growth system".

The finite case is the same as Proposition 2.9.

THEROEM 2.16. $(\forall n)$(CGS(n)) is provably equivalent to Con(SRP) over WKL_0.

We now want to get past the level of ZFM = ZFC + "there exists a measurable cardinal".

THEOREM 2.17. ZFM proves CSG(ω+2) but not CSG(ω+3), assuming ZFM is consistent. The following is provable in WKL_0. CSG(ω+3) → Con(ZFM) → Con(ZFC + CSG(ω+2)).

THEOREM 2.18. ZF2M proves CSG($\omega+\omega$+2) but not CSG($\omega+\omega$+3), assuming ZF2M is consistent. ZFC + "there exists uncountably many measurable cardinals" proves $(\forall\alpha<\omega_1)$(CSG(α)). ZFC + "there exists infinitely many measurable cardinals" does not prove $(\forall\alpha<\omega_1)$(CSG(α)), assuming that this theory is consistent.

We now present two strengthenings of conservative growth systems. These correspond to strong forms of measurability that have been extensively analyzed in inner model theory.

PROPOSITION 2.19. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \alpha$, every $(G_\gamma)_{\gamma<\beta}$ definable subset of its domain is some OUT(x,G_β), x $\in$ V(G_β).
ii. For all intervals $[\beta,\delta),[\beta',\delta') \subseteq [0,\alpha)$ of the same ordinal length, every first order property true in $[(G_\gamma)_{\beta\leq\gamma<\delta}]$ of elements of V(G_β) remains true in $[(G_\gamma)_{\beta'\leq\gamma<\delta'}]$.

PROPOSITION 2.20. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \alpha$, every $(\gamma)_{\gamma<\beta}$ definable subset of its domain is some OUT(x,G_β), x $\in$ V(G_β).
ii. The obvious generalization of Proposition 2.19 ii for finite unions of intervals.

Propositions 2.19 and 2.20 can be proved using suitably strong forms of measurable cardinals due to Mitchell.

There is another natural strengthening of Theorem 2.1 - which only uses two digraphs M,M'.

PROPOSITION 2.21. There exist countable G,G' such that
i. G is an elementary substructure of G'.
ii. V(G) is some OUT(x,G'), x $\in$ V(G').
iii. Every G' definable subset of any OUT(x,G), x $\in$ V(G), is some OUT(y,G), y $\in$ V(G).

THEOREM 2.21. Proposition 2.21 is provably equivalent to Con(ZFC) over WKL_0.

For a truly vast increase in strength, we use a radical approach to conservative growth that is motivated by rudimentary set theoretic considerations. We present this in section 4.

3. MODEL THEORY

In this section, we recast the entire development in section 2 in terms of elementary model theory. Of course, section 2 already used elementary model theory - first order predicate calculus with equality applied to relational structures. But the structures used were either digraphs or digraphs with some unary relations and sometimes an equivalence relation.

Here we use M for any relational structure in finitely many constant. Relation, and function symbols, ***including*** a one-one binary function symbol.

In model theory, the notion of quantifier free definability plays a special role. Here we use the stronger notion of atomic definability.

DEFINITION 3.1. A subset of dom(M) is M atomically definable if and only if it is M definable by an atomic formula with at most one free variable and zero or more parameters.

There is a major difference between quantifier free definability and atomic definability. A crucially important property of some structures in model theory is that

every M definable subset of dom(M) is quantifier free definable

as well as the higher dimensional form. This is so called quantifier elimination.

THEOREM 3.1. For every M with at least two elements, some M definable subset of dom(M) is not M atomically definable.

Note that the second statement in Theorem 3.1 follows easily from the first, since finite M can be treated by a simple counting argument.

The entire development of section 2 goes through with the following simple modifications.

a. Digraphs are replaced by structures M (in finitely many constants and relations).
b. "is of the form OUT(x,G)" is replaced by "is M atomically definable".
c. V(G) is replaced by dom(M).
d. E(G) is replaced by the constants and relations of M.

To illustrate this adaptation, we will only present Theorems 2.1, 2.2 and Propositions 2.8, 2.10, 2.11 in model theoretic terms.

THEOREM 3.2. There exists countable M,M' such that
i. M is an elementary substructure of M'.
ii. Every M definable subset of dom(M) is M' atomically definable.
Furthermore, M,M' can be taken to be in any finite language with a relation symbol of arity ≥ 2.

THEOREM 3.3. There exists countable M,M' such that
i. M is an elementary substructure of M'.
ii. Every (M,M') definable subset of dom(M) is M' atomically definable.
Furthermore, M,M' can be taken to be in any finite language with a relation symbol of arity ≥ 2.

PROPOSITION 3.4. There exists countable $M_1 \subseteq M_2 \subseteq M_3$ such that
i. Every first order property that holds in (M_1,M_2) of elements of $dom(M_1)$ also holds in (M_2,M_3).
ii. Every M_1 definable subset of $dom(M_1)$ is M_2 atomically definable.
Furthermore, M_1,M_2,M_3 can be taken to be in any finite language with a relation symbol of arity ≥ 2.

PROPOSITION 3.5. There exists countable $M_1 \subseteq \ldots \subseteq M_n$ such that
i. Every first order property that holds in $(M_1,\ldots,M_{n-1})$ of elements of $dom(M_1)$ also holds in $(M_2,\ldots,M_n)$.
ii. Every M_1 definable subset of $dom(M_1)$ is M_2 atomically definable.
Furthermore, $M_1,\ldots,M_s$ can be taken to be in any finite language with a relation symbol of arity ≥ 2.

PROPOSITION 3.6. There exists countable $M_1 \subseteq \ldots \subseteq M_n$ such that

i. $M_1,...,M_n$ form an elementary chain.
ii. For all intervals $[i,j],[i',j'] \subseteq [1,n]$ of the same length, $i \leq i'$, every first order property that holds in $(G_M,...,M_j)$ of elements of $V(G_i)$ also holds in $(G_{i}',...,G_{j}')$.
iii. For all $1 \leq i \leq n-1$, every $(M_1,...,M_i)$ definable subset of $dom(M_i)$ is M_{i+1} atomically definable.
Furthermore, $M_1,...,M_n$ can be taken to be in any finite language with a relation symbol of arity ≥ 2.

The robustness given by Theorems 2.7 and 2.9 also hold here.

As in section 2, Theorems 3.2, 3.3 can be proved in ZFC but not in ZFC without the power set axiom. Propositions 3.4, 3.5, 3.6 can be proved using large cardinals but not in ZFC (assuming ZFC is consistent). There are also obvious adaptations of CSG(α) with the same results. We can also readily adapt Proposition 2.14, 2.15, 2.19, 2.20.

4. SET THEORY

We now formulate conservative growth from the point of view of set theory. Here we use pairs $M = (D,\in_M)$, where D is a nonempty domain and $\in_M$ is a binary relation on D. These are the same as the digraphs $G = (V,E)$ of section 2, but with the arguments reversed. I.e., $OUT(x,G) = \{y: y \in_M x\}$.

DEFINITION 4.1. $ELT(x,M) = \{y: y \in_M x\}$.

We can obviously repeat the development in section 2 under this change of notation, with the same results. We repeat this here. We then discuss a modification which already climbs to ZFC in the case of two structures.

THEOREM 4.1. There exist countable M,M' such that
i. M is an elementary substructure of M'.
ii. Every M definable subset of dom(M) is some $ELT(x,M')$, $x \in dom(M')$.

THEOREM 4.2. There exist countable M,M' such that
i. M is an elementary substructure of M'.
ii. Every (M,M') definable subset of dom(M) is some $ELT(x,M')$, $x \in dom(M')$.

PROPOSITION 4.3. There exists countable $M_1 \subseteq M_2 \subseteq M_3$ such that

i. Every first order property that holds in (M_1,M_2) of elements of $\mathrm{dom}(M_1)$ also holds in (M_2,M_3).
ii. Every M_1 definable subset of dom(M1) is some $\mathrm{ELT}(x,M_2)$, $x \in \mathrm{dom}(M_2)$.

PROPOSITION 4.4. There exists countable $M_1 \subseteq \ldots \subseteq M_n$ such that
i. Every first order property that holds in $(M_1,\ldots,M_{n-1})$ of elements of $\mathrm{dom}(M_1)$ also holds in $(M_2,\ldots,M_n)$.
ii. Every M_1 definable subset of $\mathrm{dom}(M_1)$ is some $\mathrm{ELT}(x,M_2)$, $x \in \mathrm{dom}(M_2)$.

PROPOSITION 4.5. Let $G_1 \subseteq \ldots \subseteq G_n$ have properties i,ii in Proposition 2.8. Then
i. $G_1,\ldots,G_n$ form an elementary chain.
ii. For all intervals $[i,j],[i',j'] \subseteq [1,n]$ of the same length, $i \leq i'$, every first order property that holds in $(G_i,\ldots,G_j)$ of elements of $V(G_i)$ also holds in $(G_i',\ldots,G_j')$.
iii. For all $1 \leq i \leq n-1$, every $(M_1,\ldots,M_n)$ definable subset of $V(M_i)$ is some $\mathrm{ELT}(x,M_{i+1})$, $x \in V(M_{i+1})$.

PROPOSITION 4.6. There exists a countable chain $(M_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \alpha$, every $[(M_\gamma)_{\gamma<\beta}]$ definable subset of its domain is some $\mathrm{ELT}(x,M_\beta)$, $x \in \mathrm{dom}(M_\beta)$.
ii. For all $\beta < \alpha$, every first order property true in $[(M_\gamma)_{\beta\leq\gamma<\alpha-1}]$ of elements of $\mathrm{dom}(M_\beta)$ remains true in $[(M_\gamma)_{\beta+1\leq\gamma<\alpha}]$.

PROPOSITION 4.7. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \alpha$, every $(G_\gamma)_{\gamma<\beta}$ definable subset of its domain is some $\mathrm{OUT}(x,G_\beta)$, $x \in V(G_\beta)$.
ii. For all intervals $[\beta,\delta),[\beta',\delta') \subseteq [0,\alpha)$ of the same ordinal length, every first order property true in $[(G_\gamma)_{\beta\leq\gamma<\delta}]$ of elements of $V(G_\beta)$ remains true in $[(G_\gamma)_{\beta'\leq\gamma<\delta'}]$.

PROPOSITION 4.8. There exists a countable chain $(G_\beta)_{\beta<\alpha}$ such that
i. For all $\beta < \gamma < \alpha$, $G_\beta \subseteq G_\gamma$.
ii. For all $\beta < \alpha$, every $(\gamma)_{\gamma<\beta}$ definable subset of its domain is some $\mathrm{OUT}(x,G_\beta)$, $x \in V(G_\beta)$.
iii. The obvious generalization of Proposition 4.6 ii for finite unions of intervals.

PROPOSITION 4.9. There exist countable M,M′ such that
i. M is an elementary substructure of M′.
ii. dom(M) is some ELT(x,M′), x ∈ dom(M′).
iii. Every M′ definable subset of any ELT(x,M), x ∈ dom(M), is some ELT(y,M), y ∈ dom(M).

We obviously obtain the same results as in section 2, including the robustness theorems.

We now come to what we discussed at the end of section 2 relating to very strong statements involving M,M′. The conservative growth from M to M′ idea continues to include elementary substructure (M is an elementary substructure of M′). We will also take a set theoretic point of view, where extensionality is assumed.

PROPSOITION 4.10. There exists countable M,M′,h such that
i. M is an elementary substructure of M′ with respect to all formulas without h.
1i. h is an isomorphism from M onto M′.
iii. M,M′ satisfy extensionality and separation with respect to all formulas.
iv. {x: h(x) = x} is some ELT(x,M), x ∈ dom(M).

The idea behind iv is that there has been a substantial growth from M to M′. Clearly i-iii hold if M = M′ and h is the identity.

THEOREM 4.11. WKL0 proves
i. If Con(ZFC + there exists a nontrivial elementary embedding from a rank into itself) then Proposition 4.10 holds.
ii. If there is a conservative set theoretic growth system then Con(HUGE).

If we use an appropriate notion of bounded separation, then Proposition 4.10 is equivalent to Con(HUGE) over WKL_0.

5. AS THEORIES

We will use the language of set theory here. These axiom systems are simpler if we use constant symbols at various places instead of unary predicate symbols. Using constant symbols is natural from the point of view of axiomatic set theory.

We start with a formal system corresponding to Theorem 4.2. (The weaker Theorem 4.1 is more awkward to formalize).

The language of T_1 is $\in,=,c$, where c is a constant symbol.

1. $v_1,\ldots,v_n \in c \wedge \varphi \rightarrow (\exists v_{n+1} \in c)(\varphi)$, where φ is a formula without c, whose free variables are among $v_1,\ldots,v_{n+1}$.
2. $(\exists x)(\forall y)(y \in x \leftrightarrow y \in c \wedge \varphi)$, where φ is a formula whose free variables are among $v_1,\ldots,v_n$, in which x is not free.

We now come to Proposition 4.9. The language of T_2 is also $\in,=,c$.

1. $v_1,\ldots,v_n \in c \wedge \varphi \rightarrow (\exists v_{n+1} \in c)(\varphi)$, where φ is a formula without c, whose free variables are among $v_1,\ldots,v_{n+1}$.
2. $z \in c \rightarrow (\exists x \in c)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$, where φ is a formula in which x is not free.

We arrive at Proposition 4.4. (The weaker Proposition 4.3 is more awkward to formalize). The language of T_3 is $\in,=,c_1,c_2,\ldots$.

1. $v_1,\ldots,v_n \in c_1 \rightarrow (\varphi \leftrightarrow \varphi[c_1/c_2,c_2/c_3,\ldots])$, where φ is a formula whose free variables are among $v_1,\ldots,v_n$.
2. $(\exists x \in c_2)(\forall y)(y \in x \leftrightarrow y \in c_1 \wedge \varphi)$, where x is not free in φ.

T_4 is a more comprehensive version of T_3 in the same language.

1. $v_1,\ldots,v_n \in c_n \rightarrow (\varphi \leftrightarrow \varphi[c_n/c_{n+1},c_{n+1}/c_{n+2},\ldots])$, where φ is a formula whose free variables are among $v_1,\ldots,v_n$.
2. $z \in c_n \rightarrow (\exists x \in c_n)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$, where φ is a formula in which x is not free.
3. Extensionality.
4. $c_n \in c_m$, where $1 \leq n < m$.

We now come to a convenient formalism that goes past a measurable cardinal. This will correspond to a transfinite sequence of length $\omega+\omega$. The language of T_5 is $\in,=,P$ where P is unary. The idea is that the extension of P is a proper class of order type $\omega+\omega$ consisting of the worlds which have conservatively grown.

1. The extension of P is strictly linearly ordered by $\in$ with a limit point.
2. Let $x \in y$ both have P, and $v_1,\ldots,v_n \in x$. $(\exists z \in y)(\forall w)(w \in z \leftrightarrow w \in x \wedge \varphi^x)$. Here φ is a formula whose free variables are among $v_1,\ldots,v_n$, in which x,z are not free. Here φ^x is the result of relativizing the quantifiers in φ to x.
3. Let $x \in y$ both have P, and $v_1,\ldots,v_n \in x$. Then $\varphi \leftrightarrow \varphi^*$, where φ^* is the result of replacing each P(w) by $P(w) \wedge (y = w \vee y \in w)$.

We now arrive at Proposition 4.10. Proposition 4.8 clearly is within the realm of conservative growth, liberally interpreted. However, we can rearrange the content so that it no longer is within the realm of conservative growth, but instead is simpler, and becomes a version of elementary self embeddings. This is done by making M' the ground model, and making the inverse of the isomorphism h the elementary embedding. The language of T_6 is $\in,=,j$, where j is a unary function symbol.

1. Extensionality.
2. $(\exists x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$, where φ is a formula in which x is not free.
3. $(\exists x)(\forall y)(j(y) = y \rightarrow y \in x)$.
4. $\varphi \leftrightarrow \varphi[v_1/j(v_1),\ldots,v_n/j(v_n)]$, where φ is a formula without j whose free variables are among $v_1,\ldots,v_n$.

THEOREM 5.1. T_1 is mutually interpretable with ZFC\P. T_2 is mutually interpretable with ZFC. T_3, T_4 are mutually interpretable with SRP. T_5 interprets ZFC + (there exists a measurable cardinal of Mitchell order $n)_n$, and is interpretable in ZFC + there exists a measurable cardinal of Mitchell order ω_1 (order ω should suffice). T_6 interprets HUGE and is interpretable in I2.