ON THE PRINCIPLE OF CONSISTENT TRUTH

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ABSTRACT. Let L be a language for expressing mathematical statements. CT(L) is the principle of consistent truth for L, asserting that "if a sentence in L is consistent with ZC then it is true". We show, for certain L, that CT(L) is provably equivalent to the negation of Cantor's continuum hypothesis, over ZFC.

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1. INTRODUCTION

Let L be a language for expressing mathematical statements. CT(L) is the principle of consistent truth for L. CT(L) is the assertion "if a sentence in L is consistent with ZC then it is true". We show that for certain L, CT(L) is provably equivalent to ¬CH over ZFC. (ZC is Zermelo set theory with the axiom of choice, which is ZFC with Frankel's axiom scheme of replacement.)

Now obviously ZFC + CT(L) is inconsistent if L contains a sentence and its negation. Therefore, CT(L) is to be explored for languages L that are not closed under negation. But how should appropriate languages L be chosen for the investigation of CT(L)?

Here we consider languages L that are motivated by the grammar of classical mathematics. We have been focusing on a particularly simple statement that refutes the continuum hypothesis = CH. $ below has the same general grammar of a considerable part of classical mathematics.

$. For all $f:\Re \to \Re$, there exists $x,y \in \Re$, neither of which is the value of f at an integral shift of the other.
$. $(\forall f:\Re \to \Re)(\exists x,y \in \Re)(\forall n \in Z)(x \neq f(y+n) \wedge y \neq f(x+n))$.

We also note that the closely related

$'. For all $f:\Re \to \Re$, there exists $x,y \in \Re$, neither of which is an integral shift of the value of f at the other.
$'. $(\forall f:\Re \to \Re)(\exists x,y \in \Re)(\forall n \in Z)(x+n \neq f(y) \wedge y+n \neq f(x))$.

is provable in ZC.

$ follows immediately from a well known equivalence of ¬CH:

PNCU. The plane is not a countable union of functions $y = f(x)$ or $x = f(y)$.

Here PNCU = "plane not countable union".

PNII. The plane is not the union of the integral isometries of any function $y = f(x)$.

In fact, $ also follows immediately from PNII = "plane not integral isometries". Thus we see that $,PNCU,PNII are all equivalent to the negation of the continuum hypothesis.

The natural logical form this variant $ takes is critical, and provides the starting point for the Consistent Truth program.

A substantial portion of celebrated mathematical theorems concern certain kinds of functions from one space to another, and have the following general grammar:

for every function from one space to another, of a certain kind, something holds involving elements of the spaces and their values under the function

Many celebrated theorems of classical analysis have this general shape, including

1. $(\forall \text{ continuous } f:[0,1] \to [0,1])(\exists x \in [0,1])(f(x) = x)$.
2. $(\forall \text{ continuous } f:[0,1] \to \Re)(\exists x \in [0,1])(f(x) = (f(0)+f(1))/2)$.

3. $(\forall \text{ continuous } f:[0,1] \to \Re)(\exists x \in [0,1])(f(0) \leq 0 \leq f(1) \to f(x) = 0)$.
5. $(\forall \text{ differentiable } f:\Re \to \Re)(\exists x \in \Re)(f(0) = f(1) \to f'(x) = 0)$.
4. $(\forall \text{ continuous } f:[0,1] \to \Re)(\exists x \in [0,1])(\forall y \in [0,1])(f(y) \leq f(x))$.
5. $(\forall \text{ continuous } f:\Re \to \Re)(\forall x,y,z \in \Re)(f(x) \leq y \leq f(z) \to (\exists w)(f(w) = y))$.

There are many more examples, all of which suggest very interesting little languages to investigate, in the realm of single and several real and complex variable classical analysis. This takes us well beyond the scope of this paper.

What we are emphasizing here is the outside universal quantifier over functions, inside quantifiers over arguments, and simplicity.

This motivation from elementary classical mathematics occurred to us after reflecting on $ above.

The plan is to investigate languages L specifically surrounding $, of ever increasing scope. In particular, to determine the status of CT(L).

We also investigate the principle of Continuous truth for L. COT(L) is the assertion "if a sentence of L is true for all continuous functions then it is true". Whereas CT(L) makes sense for essentially any language, COT(L) makes sense only for languages of roughly the sort under consideration here, involving functions between Polish spaces and point evaluation. We also seek to determine the status of COT(L).

In this paper, we focus on a very primitive language L_1 which readily incorporates $,$'. In particular, we show that CT(L_1) is equivalent to ¬CH over ZFC. We also show that COT(L_1) is equivalent to ¬CH over ZFC. We conjecture that all sentences of L1 are in the following three categories.

W1. Sentences which are provable in Z, and whose Borel form is provable in ATR_0.
W2. Sentences which are refutable in Z, and whose Borel form is refutable in ATR_0.

W3. Sentences which are provably equivalent to ¬CH over ZC, and whose Borel form is provable in ATR_0.

Should we construe the equivalence of $CT(L_1)$ with ¬CH for an anticipated series of ever more expressive languages L of the type under consideration here as a kind of refutation of CH?

Probably not from a fully Platonist view of set theory, where statements are viewed as having a matter of fact definite truth value, independently of human contemplation.

However, more nuanced points of view have been put forth, as weaker forms of realism, wherein a definite objective criteria for determining set theoretic truth is believed to exist, and a principal goal of set theorists is to explicate such criteria and apply it to set theoretic problems such as CH. E.g., see
https://www.math.ucla.edu/~asl/bsl/1102/1102-008.ps
http://logic.harvard.edu/EFI_Martin_CompletenessOrIncompleteness.pdf

We have also done some investigation from above, where we construct ever simpler languages L of this rough kind where ZC + CT(L) is inconsistent. In particular, constructing "simple" sentences of the kind being considered here which, provably in ZC, implies CH. For some of the relevant languages, there are open questions as to whether CT(L) is consistent. But even for the languages where there are statements implying CH, the particular instances are of rather high complexity, involving coding of a non straightforward nature. In this way, this asymmetry between CH and ¬CH in favor of ¬CH, will likely survive even as the inconsistencies keep arising. In particular, we can restrict our languages to only sentences of rather low complexity, thereby resurrecting the equivalence of CT(L) with ¬CH.

We also see a new area of continuous real variables and descriptive set theory as a spinoff here. We can look at the sort of languages under consideration where we quantify over only continuous functions, or Borel functions, and don't even consider quantifying over all functions. We can hope to determine the values of all of the statements that arise. In the case here of L_1 and quantifying over continuous functions, RCA_0 is easily sufficient to determine all such truth values. We expect that for more expressive languages, there will be sharp differences when quantifying over continuous versus Borel functions.

2. $ ↔ ¬CH

$. For all $f:\mathfrak{R} \to \mathfrak{R}$, there exists x,y ∈ Â, neither of which is the value of f at an integral shift of the other.

$. $(\forall f:\mathfrak{R} \to \mathfrak{R})(\exists x,y \in \mathfrak{R})(\forall n \in Z)(x \neq f(y+n) \wedge y \neq f(x+n))$.

$'. For all $f:\mathfrak{R} \to \mathfrak{R}$, there exists $x,y \in \mathfrak{R}$, neither of which is an integral shift of the value of f at the other.

$'. $(\forall f:\mathfrak{R} \to \mathfrak{R})(\exists x,y \in \mathfrak{R})(\forall n \in Z)(x+n \neq f(y) \wedge y+n \neq f(x))$.

PNCU. The plane is not a countable union of functions $y = f(x)$ or $x = f(y)$.

Thus we are using what we call two "types" of functions, namely $y = f(x)$ and $x = f(y)$.

PNII. The plane is not the union of the integral isometries of any function $y = f(x)$.

Here an integral isometry is an isometry of $\mathfrak{R}^2$ mapping Z^2 to Z^2. An integral isometry of a subset of $\mathfrak{R}^2$ is the image of the set under an integral isometry.

THEOREM 2.1. ZC proves $'.

Proof: Let $f:\mathfrak{R} \to \mathfrak{R}$. We first try setting $x = y$, and hope that $(\forall n \in Z)(x+n \neq f(x) \wedge x+n \neq f(x))$. We can assume that, unfortunately, $(\exists n \in Z)(x+n = f(x))$. I.e., we can assume that $(\forall x)(\exists n \in Z)(x+n = f(x))$. Now let $x-y$ be irrational. Then for all $n \in Z$, $x+n \neq f(y) \wedge y+n \neq f(x)$, because $f(y)-y$ and $f(x)-x$ are integers, and so $f(y)-x$ and $f(x)-y$ are not. QED

PNCU (plane not countable union) is well known to be equivalent to ¬CH over ZC.

LEMMA 2.2. $,PNCU,PNII are provable in ZC + ¬CH.

Proof: Obviously PNII is an immediate consequence of PNCU, so we can remove it from consideration here.

We first prove PNCU in ZC + ¬CH. Let $\mathfrak{R}^2$ be the union of countably many functions. These functions are of one of two possible types. Let $A \subseteq \mathfrak{R}$ have cardinality ω_1. Let $x \in \mathfrak{R}$ not be the value of any of the functions at an argument in A, no matter which type. Let $y \in A$ not be the value of any of the functions at x, no matter which type. Now (x,y) must be an element of one of the functions no matter which type. I.e., one of the functions maps x to y, or y to x, and can be of any type. By the choice of y, none of the functions map x to y. By the choice of x, and that $y \in A$, none of the functions map y to x.

\$ is an immediately consequence of PNCU, and even PNII. Let $f:\mathfrak{R} \rightarrow \mathfrak{R}$. Use the functions f(x+n), f(y+n), $n \in Z$ fixed. The f(x+n) are functions of the first type, and the f(y+n) are functions of the second type. Furthermore, both are obvious integral isometries of f. The conclusion of \$ just says that these functions do not cover the plane. QED

THEOREM 2.3. ZC proves \$,PNCU,PNII ↔ ¬CH.

Proof: By Lemma 2.2, it suffices to show \$ → ¬CH since PNII immediately implies \$ (see the proof of Lemma 2.2). We assume CH and refute \$.

We find a counterexample $f:\mathfrak{R} \rightarrow \mathfrak{R}$ to \$. We use the equivalence relation: x,y are equivalent iff x-y is an integer.

We begin by defining f(x+2n) for $0 \leq x < 1$. Set f(x+2n) = x+n. Note that we already have this: If x,y are equivalent then x is the value of f at an integral shift of y.

Now enumerate the equivalence class without repetition of length ω_1. We have not yet defined f on infinitely many elements of each equivalence class. So we are free to define f on these remaining elements so that f maps each equivalence class onto at least the union of all of the previous equivalence classes.

Now let $x,y \in \Re$. If x,y are equivalent then x,y is as required. Otherwise, x appears earlier or later than y in the transfinite enumeration. If x appears earlier, then x is the value of f at some integral shift of y. If x appears later, then y is the value of f at some integral shift of x. QED

THEOREM 2.4. RCA0 proves \$,PNCU,PNII for continuous functions. ATR_0 proves \$,PNCU,PNII for Borel functions.

Proof: The derivation of PNCU for continuous functions, within RCA_0, is by a diagonal argument where a pair of reals is constructed in stages, easiest as base 2 reals in [0,1]. To prove PNCU for Borel functions within ATR_0, it is quickest to simply rely on the development of measure theory for Borel functions within ATR_0. note that every Borel function of either type is of Lebesgue measure 0 in $\Re^2$. Therefore the union of these Borel functions has Lebesgue measure 0 in $\Re^2$, and therefore cannot cover $\Re^2$. QED

3. THE LANGUAGES

The language L_1 consists of the sentences of the form

$$(\forall f:\Re \rightarrow \Re)(\exists x,y \in \Re)(\forall n \in Z)(s_1 \neq t_1 \wedge \ldots \wedge s_k \neq t_k)$$

where the s_i, t_i are from the following list of terms.

L1 TERM LIST

ax+by+cn+d
f(ax+bn+c)
f(ay+bn+c)

where a,b,c are integer constants (coefficients), x,y are variables ranging over R, n is a variable ranging over Z, and f is a variable ranging over functions from $\Re$ to $\Re$. We keep the monomials in a term even if the coefficient is 0.

In section 7, we discuss some candidates for more expressive languages to be used in this Consistent Truth Program.

DEFINITION 3.1. The language $L_1(\omega)$ is the same as L_1 except that conjunctions of length ω are allowed. $L_1(C),L_1(C,\omega)$ are the same as $L_1,L_1(\omega)$, respectively, except that the function quantifier is only over continuous $f:\Re \to \Re$.

DEFINITION 3.2. $CT(L_1)$ is the assertion "every L_1 sentence consistent with ZC is true". $CT(L_1(\omega))$ is the assertion "every $L_1(\omega)$ sentence satisfiable with ZC is true". $BT(L_1)$ is the assertion "every true $L_1(C)$ sentence remains true if the function quantifier is changed to be over all $f:\Re \to \Re$". $COT(L_1(\omega))$ is the assertion "every true $L_1(C,\omega)$ sentence remains true if the function quantifier is changed to be over all $f:\Re \to \Re$".

Here we determine the status of $CT(L_1), CT(L_1(\omega)), COT(L_1), COT(L_1(\omega))$. We show that they are all provably equivalent to $\neg$CH over ZFC. We conjecture that this is also the case for similar languages with much greater expressive power.

DEFINITION 3.3. The L_1 conjuncts are the $s \neq t$ where s,t are L1 terms. For L_1 conjuncts $\alpha_1,\ldots,\alpha_k$, $k \geq 1$, $(\alpha_1,\ldots,\alpha_k)^*$ is the L_1 sentence $(\forall f:\Re \to \Re)(\exists x,y \in \Re)(\forall n \in Z)(\alpha_1 \wedge \ldots \wedge \alpha_k)$. For L_1 conjuncts $\alpha_1,\alpha_2,\ldots$, $(\alpha_1,\alpha_2,\ldots)^*$ is the $L_1(\omega)$ sentence $(\forall f:\Re \to \Re)(\exists x,y \in \Re)(\forall n \in Z)(\alpha_1 \wedge \alpha_2 \wedge \ldots)$. $C(\alpha_1,\ldots,\alpha_k)^*$, $C(\alpha_1,\alpha_2,\ldots)^*$ are the same except that we quantify over continuous $f:\Re \to \Re$. The rank of an L_1 sentence is the number of conjuncts that appear, counting repetitions.

4. RANK ONE SENTENCES

W1. Sentences which are provable in Z, and whose continuous form is provable in RCA_0.
W2. Sentences which are refutable in Z, and whose continuous form is refutable in RCA_0.

We show that every rank one L1 sentence is in $W_1 \cup W_2$.

We use the notation $(x_1,\ldots,x_k)$ for the k-tuple, and not for gcd's.

LEMMA 4.1. Let $a \neq a'$. $(ax+by+cn+d \neq a'x+b'y+c'n+d')^* \in W_1$. Let $b \neq b'$. $(ax+by+cn+d \neq a'x+b'y+c'n+d')^* \in W_1$.

Proof: Let $f:\Re \to \Re$. Let x be irrational, $y = 0$. $(\forall n \in Z)(ax+cn+d \neq a'x+c'n+d')$. For the second claim, switch x,y.

QED

LEMMA 4.2. Let a = a' and b ≠ b'. (ax+by+cn+d ≠ a'x+b'y+c'n+d')* ∈ W_1.

Proof: Let f be identically 0. Let x = 0 and y be irrational. (∀n ∈ Z)(ax+by+cn+d ≠ ax+b'y+c'n+d'). QED

LEMMA 4.3. Let a = a' and b = b' and c'-c be a multiple of b'-b. (ax+by+cn+d ≠ a'x+b'y+c'n+d')* ∈ W_2.

Proof: Let f be identically 0. Let x,y ∈ ℜ. (∃n ∈ Z)(ax+by+cn+d = ax+by+c'n+d'). QED

LEMMA 4.4. Let a = a' and b = b' and c'-c not a multiple of b'-b. (ax+by+cn+d ≠ a'x+b'y+c'n+d')* ∈ W_1.

Proof: Let f:ℜ → ℜ. Let x = y = 0. (∃n ∈ Z)(ax+by+cn+d = ax+by+c'n+d'). QED

LEMMA 4.5. Let b ≠ 0. (ax+by+cn+d ≠ f(a'x+b'n+c')* ∈ W1. Let a ≠ 0. (ax+by+cn+d) ≠ f(a'y+b'n+c')* ∈ W_1.

Proof: Let f:ℜ → ℜ. Let x = 0 and y outside the least f closed subfield of ℜ. (∀n ∈ Z)(by+cn+d ≠ f(b'n+c')). For the second claim, switch x,y.

LEMMA 4.6. Let a',b',c' be integers, a',b' nonzero. There exists Borel g:ℜ → ℜ such that the following holds. For all x ∈ ℜ, there exists n ∈ Z, such that g(a'x+b'n+c') = x.

Proof: We first construct a continuous h:ℜ → ℜ such that (∀x ∈ ℜ)(∃n ∈ Z)(h(x+n) = x). First define h on each interval $[2^n3^m+(1/n)),[2^n3^m+1-(1/n)]$, n ≥ 3 ∧ m ≥ 0, by h(x) = the integer shift of x that lies in [m,m+1). Now define h on each interval $[5^n3^m+(1/n)),[5^n3^m+1-(1/n)]$, n ≥ 3 ∧ m ≥ 0, by h(x) = the integer shift of x that lies in [-m,-m+1). Now define h to be the identity on each integer. Finally extend h continuously on all of ℜ. Let x ∈ [m,m+1), m ∈ Z. If x = m then h(x+0) = x, and so we assume x ∈ (m,m+1). Assuming m ≥ 0, let x+i be an integer shift of x lying in some $[2^n3^m+(1/n)),[2^n3^m+1-(1/n)]$, n ≥ 3. Then h(x+i) = x. If m < 0, let x+j be an integer shift lying in some $[5^n3^{-m}+(1/n)),[5^n3^{-m}+1-(1/n)]$, n ≥ 3. Then h(x+j) = x.

Now let $h'(x) = b'h(x/b')$. Then $h'(x+b'n) = b'h((x/b')+n)$. Hence $(\forall x \in \Re)(\exists n \in Z)(h'(x+b'n) = x)$. Define $g(x) = h'(x-c')/a'$. Then $g(a'x+b'n+c') = h'(a'x+b'n)/a'$, which is, for some $n \in Z$, $a'x/a' = x$. Obviously h' and g are continuous. QED

LEMMA 4.7. Let $b = 0$ and $a',b' \neq 0$. $(ax+by+cn+d \neq f(a'x+b'n+c')^* \in W_2$. Let $a = 0$ and $a',b' \neq 0$. $(ax+by+cn+d \neq f(a'y+b'n+c') \in W_2$.

Proof: Define $f(x) = ag(x)+cn+d$, where g is given by Lemma 4.6'. Let $x,y \in \Re$, and choose n such that $g(a'x+b'n+c') = x$. Then
$f(a'x+b'n+c') = ax+cn+d$ and f is Borel. For the second claim, switch x,y. QED

LEMMA 4.8. Let $b,a' = 0$ and $a \neq 0$. $(ax+by+cn+d \neq f(a'x+b'n+c'))^* \in W_1$. Let $a,a' = 0$ and $b \neq 0$. $(ax+by+cn+d \neq f(a'y+b'n+c'))^* \in W_1$.

Proof: Let $f:\Re \to \Re$. Let x lie outside the least subfield of $\Re$ closed under f, and $y = 0$. $(\forall n \in Z)(ax+cn+d \neq f(b'n+c'))$. For the second claim, switch x,y. QED

LEMMA 4.9. Let $b,a' = 0$ and $a = 0$. $(ax+by+cn+d \neq f(a'x+b'n+c'))^* \in W_2$. Let $a,a' = 0$ and $b = 0$. $(ax+by+cn+d \neq f(a'y+b'n+c'))^* \in W_2$.

Proof: Let f be constantly 1/2. Let $x,y \in \Re$. $(\exists n \in Z)(cn+d = f(b'n+c))$. For the second claim, switch x,y. QED

LEMMA 4.10. Let $b,b' = 0$ and $a' \neq 0$. $(ax+by+cn+d \neq f(a'x+b'n+c')^* \in W_2$. Let $a,b' = 0$ and $a' \neq 0$. $(ax+by+cn+d \neq f(a'y+b'n+c')^* \in W_2$.

Proof: Let $f(x) = a(x-c')/a' + d$. Let $x,y \in \Re$. $(\exists n \in Z)(ax+cn+d = f(a'x+c'))$. For the second claim, switch x,y. QED

LEMMA 4.11. Let $f(s),f(t)$ be L_1 terms. $(f(s) \neq f(t))^* \in W_2$.

Proof: Let f be identically 0. Let $x,y \in \Re$. $(\exists n \in Z)(f(s) = f(t))$. QED

THEOREM 4.12. Every rank one sentence of L1 lies in $W_1 \cup W_2$.

Proof: It remains to check that all rank 1 L_1 sentences (α)* have been treated in the above lemmas. This is not strictly true. But we show that every such or its twin has been treated.

case 1. Conjuncts without f. These are handled in Lemmas 4.1 - 4.4. The case a ≠ a' is handled. It remains to handle the case a = a'. Within this, there are two cases, b ≠ b' and b = b'. The former case is handled. The latter case is handled with the case split "c'-c is a multiple of b'-b".

case 2. Conjuncts with exactly one f, with no "y" under f. These are handled in the first claims of Lemmas 4.5,4.7-4.10. The case b ≠ 0 is handled. It remains to handle the case b = 0. The cases a',b' ≠ 0, a' = 0 ∧ a ≠ 0, a' = 0 ∧ a = 0, b' = 0 ∧ a' ≠ 0, are handled. Therefore the case a' = 0 is handled. It remains to handled the case a' ≠ 0. This is handled for b' ≠ 0 and for b' = 0.

case 3. Conjuncts with exactly one f, with no "x" under f. These are handled in the second claims of Lemmas 4.5,4.7-4.10. The case a ≠ 0 is handled. It remains to handle the case a = 0. The cases a',b' ≠ 0, a' = 0 ∧ b ≠ 0, a' = 0 ∧ b = 0, b' = 0 ∧ a' ≠ 0, are handled. Therefore the case a' = 0 is handled. It remains to handled the case a' ≠ 0. This is handled for b' ≠ 0 and for b' = 0.

case 4. Conjuncts with exactly two f's. These are handled by Lemma 4.11.

QED

5. CONSISTENT TRUTH

Upon examination of the development in section 4, we have the following compilation. Here we have removed monomials with coefficient 0, and removed pairs of like terms from both sides.

P1. ax+by+cn+d ≠ a'x+b'y+c'n+d', a ≠ a'. Lemma 4.1, first claim.
P1a. ax+by+cn+d ≠ a'x+b'y+c'n+d', b ≠ b'. Lemma 4.1, second claim.
P2. by+cn+d ≠ b'y+c'n+d', b ≠ b'. Lemma 4.2, first part.

P2a. $bx+cn+d \neq b'x+c'n+d'$, $b \neq b'$. Lemma 4.2, second part.
P3. $cn+d \neq c'n+d'$, $c'-c$ not a multiple of $b-b'$. Lemma 4.4.
P4. $ax+by+cn+d \neq f(a'x+b'n+c')$, $b \neq 0$. Lemma 4.5, first part.
P4a. $ax+by+cn+d \neq f(a'y+b'n+c')$, $a \neq 0$. Lemma 4.5, second part.
P5. $ax+cn+d \neq f(b'n+c')$, $a \neq 0$. Lemma 4.8, first part.
P5a. $ay+cn+d \neq f(b'n+c')$, $a \neq 0$. Lemma 4.9, second part.

DEFINITION 5.1. A P formula is an L_1 conjunct listed among the P's above. A sentence in $L_1(\omega)$ or $L_1(C,\omega)$ is normal if and only if all of the L_1 conjuncts appearing are P formulas. It is abnormal otherwise.

THEOREM 5.1. Normal rank one L_1 sentences lie in W_1, and abnormal rank one sentences lie in W_2.

Proof: The conjuncts in the above listing were shown to lie in W_1 by the indicated lemmas. Now note that the lemmas cited in the above list comprise all of the lemmas which place L_1 conjuncts in W_1. The remaining lemmas, with the exception of Lemma 4.6, place the remaining L_1 conjuncts in W_2. QED

Obviously there is a strongest normal sentence in $L_1(\omega)$ and $L_1(C,\omega)$ where every normal L1 conjunct appears. But we will use this definition of normality in the discussion in section 7.

THEOREM 5.2. ATR_0 proves "a sentence in $L_1(C,\omega)$ is true if and only if it is normal".

Proof: Suppose we have a normal sentence in $L_1(C,\omega)$. We can assume that its conjuncts are exactly the P formulas. To show that this normal sentence is true, let $f:\mathfrak{R} \to \mathfrak{R}$ be continuous. We need to find $x,y \in \mathfrak{R}$ such that for all $n \in Z$ and P formulas φ, $\varphi(n,x,y)$ holds. If this is false, then we obtain a covering of $\mathfrak{R}^2$ by the sets $\{(x,y) \in \mathfrak{R}^2: \neg\alpha(n,x,y)\}$, indexed by $n \in Z$ and P conjuncts α. However, by inspecting each P formula, we see that these subsets of $\mathfrak{R}^2$ are continuous functions of both types. By PNCU for continuous functions, Theorem 2.4, this is impossible.

Suppose be have an abnormal sentence in $L_1(C,\omega)$. By Theorem

5.1, we see that the abnormal sentence is false, using RCA_0. QED

THEOREM 5.3. ZC + ¬CH proves "a sentence in $L_1(\omega)$ is true if and only if it is normal". ZFC proves $CT(L_1) \leftrightarrow CT(L_1(\omega)) \leftrightarrow \neg CH$.

Proof: From the first paragraph of the proof of Theorem 5.2, we that ZC + PNUF proves "every normal sentence in $L_1(\omega)$ is true". Hence ZC + ¬CH proves "every normal sentence in $L_1(\omega)$ is true", by Theorem 2.3. Also ZC + ¬CH proves "every abnormal sentence in $L_1(\omega)$ is false" by Theorem 5.1.

For the second claim, we first prove $CT(L_1(\omega))$ in ZFC + ¬CH. Let φ be a sentence in $L_1(\omega)$ that is satisfiable with ZC. By Theorem 5.1, φ is normal. By the first claim, φ is true.

Finally, we have ZFC proves $CT(L_1) \rightarrow \neg CH$ since ZFC proves the consistency of ZC + $, and $ → ¬CH, both of which follow from Theorem 2.3. QED

6. INCONSISTENT TRUTH

We seek to understand how little we can expand the language L_1 so that $CT(L_1)$ becomes inconsistent. We will focus on what expansions L prove CH from ZFC + CT(L).
Since $CT(L_1)$ already proves CH through $, such expansions L have inconsistent CT(L).

To be continued...

7. FUTURE RESEARCH

Of course, the choice of languages will need to be revised according to the unforeseen technical obstacles that arise.

To be continued.