COUNTABLE MODEL THEORY AND INCOMPLETENESS

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EXTENDED ABSTRACT

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Abstract. We present several natural statements in the realm of countable model theory, based on certain forms of elementary subsystem and quantifier elimination. We show that some of these statements are equivalent to the consistency of ZFC without the power set axiom, full ZFC, and ZFC with certain standard large cardinal hypotheses.

1. PRELIMINARIES.

DEFINITION 1.1. A system is an M = (D,...), where D is a nonempty set, and ... consists of finitely many constant and relation symbols, and no function symbols. An M definable subset of D is a set defined in first order predicate calculus with equality over M. An M atomic subset of D is a set defined by an atomic formula. In both cases, parameters are allowed unless stated otherwise.

DEFINITION 1.2. For systems M,R, M ≤ R if and only if M is an elementary subsystem of R.

DEFINITION 1.3. Let $M_1,...,M_t$ be systems. $M = (M_1,...,M_t)$ is the system with $dom(M) = dom(M_1) \cup ... \cup dom(M_t)$, the domains of $M_1,...,M_t$ appear as unary relations on dom(M), the constants of $M_1,...,M_t$ appear as constants of M, and the relations of $M_1,...,M_t$ appear as relations on M.

We use the following generalization of elementary subsystem.

DEFINITION 1.4. $M_1,...,M_t \leq R_1,...,R_t$ if and only if

i. $M_1,...,M_t$ and $R_1,...,R_t$ are systems of the same respective signatures.
ii. $dom(M_1) \subseteq dom(R_1)$.
iii. Any first order formula true over $(M_1,...,M_t)$, with parameters from $dom(M_1)$, is true over $(R_1,...,R_t)$.

DEFINITION 1.5. $M_1,M_2,... \leq R_1,R_2,...$ if and only if for all n, $M_1,...,M_n \leq R_1,...,R_n$.

2. M ≤ R

THEOREM 2.1. The following is false. There is a system M such that every M definable subset of dom(M) is M atomic.

THEROEM 2.2. There are countable systems M ≤ R such that every (M,R) definable subset of dom(M) is R atomic.

THEOREM 2.3. Theorem 2.2 is provably equivalent to the consistency of Z_2 (equivalently, of ZFC\P) over ACA.

We conclude that any proof of Theorem 2.2 is significantly set theoretic - which is entirely uncharacteristic of countable model theory.

3. M ≤ R ≤ S

PROPOSITION 3.1. There are countable systems M ≤ R ≤ S such that every (M,R,S) definable subset of dom(M) is R atomic.

PROPOSITIONS 3.2. There are countable systems $M_1 \leq M_2 \leq ...$ such that for all i < j, every $(M_1,...,M_j)$ definable subset of $dom(M_i)$ is M_{i+1} atomic.

THEROEM 3.3. Propositions 3.1, 3.2 are provably equivalent to the consistency of ZFC, over ACA. Propositions 3.1, 3.2 are not provable in ZFC (assuming ZFC is consistent). Propositions 3.1, 3.2 are independent of ZFC (assuming ZFC is 1-consistent). These results hold if we require that M,R,S are recursive in 0'.

4. (M,R) ≤ (R,S)

PROPOSITION 4.1. There are countable systems M,R ≤ R,S such that every (M,R,S) definable subset of dom(M) is R atomic.

PROPOSITION 4.2. There are countable systems $M_1,M_2,... \leq M_2,M_3,...$ such that for all $i < j$, every $(M_1,...,M_j)$ definable subset of M_i is M_{i+1} atomic.

THEROEM 4.3. Proposition 4.1 implies the consistency of ZFC + "there are totally indescribable cardinals" and follows from the consistency of ZFC + "there exists a subtle cardinal" over ACA. Proposition 4.2 is provably equivalent to the consistency of ZFC + $\{$there exists an n-subtle cardinal$\}_n$ over ACA. These results hold if we require that M,R,S and $(M_1,M_2,...)$, as an infinite tuple, are recursive in 0'.