COUNTABLE MODEL THEORY AND LARGE CARDINALS

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A familiar idea in core mathematics is to add a point at infinity, often in a canonical way.

We can look at this model theoretically as follows. By the linearly ordered predicate calculus, we simply mean ordinary predicate calculus with equality and a special binary relation symbol <. It is required that in all interpretations, < be a linear ordering on the domain. Thus we have the usual completeness theorem provided we add the axioms that assert that < is a linear ordering.

It will be convenient to consider subrational models M. These are models whose domain is a nonempty subset of the rationals, and whose ordering agrees with the usual ordering of rationals. We will be particularly interested in the subset N containedin Q of nonnegative integers; of course N may not be a subset of the domain of M, written dom(M).

It is obvious that by the downward Skolem-Lowenheim theorem, any consistent theory in linearly ordered predicate calculus has a subrational model.

Let T be a theory in linearly ordered predicate calculus. We say that “models have unique extensions (to models) at infinity” if and only if the following holds:

Let M = (D,<,…) be a model of T, which consists of a nonempty domain D, a linear ordering on D, and the components of M consisting of constants from M, relations of several variables on D, and functions of several variables from D into D, interpreting the constant, relation, and function symbols appearing in T. *Minor technical point* - we won’t allow M to interpret any constant, relation, and function symbols not appearing in T, although < must always be interpreted. Let ∞ be an object outside of D. We can adjoin ∞ to (D,<) to obtain $(D \cup \{\infty\},<)$, where ∞ is now the greatest element, and the order relation on D remains the same.

The requirement is that there is a unique model $M' = (D \cup \{\infty\},<,\ldots)$ extending M. I.e., all constants of M remain the same, and all relations and functions of M remain the same on D.

It is easy to see that this property of T can be expressed in predicate calculus in terms of validity. From this we see that the models of T have unique extensions at infinity if and only if the subrational models of T have unique extensions at infinity. In the same vein, this is true if and only if it is true of some finite subset of T. Also, by the completeness theorem for predicate calculus, we see that for finite T, this condition is recursively enumerable.

In light of these remarks, we can restrict attention to subrational models only. This makes the formulations more concrete and convenient.

Let M be a model. The surjective components of M are just the component functions of M whose range is the entire domain of M.

Let f be a k-ary function and X be a set. The range of f over X is simply $f[X^k]$.

PROPOSITION A. Every sentence in linearly ordered predicate calculus with a subrational model, whose subrational models have unique extensions at infinity, has a subrational model where the range of every surjective component over N contains infinitely many limit points of the domain. In fact, we can require that N be included in the domain.

Here is an obvious consequence.

PROPOSITION B. Every sentence in linearly ordered predicate calculus with a finite subrational model, whose subrational models have unique extensions at infinity, has a subrational model where the range over N of every surjective component contains infinitely many limit points of the domain. In fact, we can require that N be included in the domain.

Let ACA = arithmetic comprehension axiom scheme with full induction.

THEOREM 1. Propositions A and B are provable in ZFC + for all $n \in N$, there exists an n-Mahlo cardinal. However, neither A nor B is provable in the system MAH = ZFC + $\{\text{there exists an n-Mahlo cardinal}\}_n$, provided MAH is consistent.

THEOREM 2. The following is provable in ACA. A holds if and only if MAH is 1-consistent. And B holds if and only if MAH is consistent.

We can use an alternative conclusion for Propositions A,B. Let R be a k+1-ary relation on a linearly ordered set D and let $x \in D^k$.

We say that S is a cross sections of R if and only if there exists x such that $S = R_x = \{y:R(x,y)\}$. We say that S is a strongly regressive cross section of R if and only if there exists y < x such that $S = R_x \subseteq \{u: u \leq y\}$.

Finally, we say that S is a strongly regressive cross section of R over E if and only if there exists y < x, $y \in E^k$, such that $S = R_x \subseteq \{u: u \leq y\}$.

PROPOSITION C. Every sentence in linearly ordered predicate calculus with a subrational model, whose subrational models have unique extensions at infinity, has a subrational model M where each relational component has finitely many strongly regressive cross sections over $N \subseteq$ dom(M).

PROPOSITION D. Every sentence in linearly ordered predicate calculus with a finite subrational model, whose subrational models have unique extensions at infinity, has a subrational model M where each strongly regressive relational component has finitely many cross sections over $N \subseteq$ dom(M).

THEOREM 3. Theorems 1 and 2 hold for Propositions C and D if we replace Mahlo by subtle. Furthermore, if we restrict attention to $\forall\exists$ sentences in Propositions A - D, then the same results hold.

What happens if we drop “unique” in the hypothesis? First of all, the condition ceases to be first order - there are $\forall\exists$ sentences T such that subrational models have extensions at infinity, yet some (necessarily uncountable) models do not have extensions at infinity.

THEOREM 4. Propositions A - D are all refutable in ACA. They are all refutable in ZFC even if we replace “subrational models have unique extensions at infinity” with “models have unique extensions at infinity.”