MONADIC PARTIAL FUNCTIONS WITH OBJECTS AND SETS

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PRELIMINARY REPORT

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1. MPFS - MONADIC PARTIAL FUNCTIONS WITH SETS

In this section, we present the base theory MPFS. In the second section, we discuss an extension of MPFS.

We are interested in algorithmic decidability. I.e., decision procedures for determining whether a statement in the language is valid. We give a crude upper bound on the computational complexity of some algorithms. A lot remains to be done concerning the development of efficient algorithms.

All formulas considered here are quantifier free. However, in the second section, where we build on MPFS, we will be introducing abstraction (set formation) terms that hide underlying quantifiers.

MPFS has three sorts: objects, sets, and partial functions.

The object variables are $v_0,v_1,\ldots$. The set variables are $A_0,A_1,\ldots$. The (monadic partial) function variables are $f_0,f_1,\ldots$.

The object terms are defined inductively by
i. Every object variable is an object term.
ii. If f is a function variable and t is an object term, then ft is an object term.

The atomic formulas are as follows. Let s,t be object terms, A,B be set variables, and f,g be function variables.

$s = t$
$s \equiv t$
$s\downarrow$
$s\uparrow$
$A \subseteq B$
$t \in A$
$f \subseteq g$

Here = indicates both sides are defined and identical. ≡ indicates that both sides are undefined, or both sides are defined and equal. s↓ indicates that s is defined. s↑ indicates that s is undefined. ⊆ is the usual inclusion between sets. ∈ is the usual membership relation between objects and sets. We also use ⊆ for inclusion between the graph of f and the graph of g. Note that A = B and f = g are readily defined as $A \subseteq B \wedge B \subseteq A$, and $f \subseteq g \wedge g \subseteq f$, respectively.

Thus all set and function terms are variables, which according to the formal semantics below, must always be defined, whereas there are compound object terms, and although object variables are always defined, compound object terms may or may not be defined. It should be noted that if t is not defined then ft is not defined.

The MPFS formulas φ are, as usual, the propositional combinations of the atomic formulas.

An MPFS structure is a triple M = (X,Y,Z), where X is a set of objects containing 0, Y is a nonempty set of subsets of X, and Z is a nonempty set of partial functions from X into X. M is said to be finite if and only if X is finite. We write dom(M) = X.

An M assignment is a function α that maps object variables into X, set variables into ℘(X), and function variables to partial functions from X into X. I.e., functions whose domain and range are subsets of X.

We inductively define the valuation in M of object term t at M assignment α as follows. val(M,t,α) may or may not be defined.

val(M,v,α) = α(v), v any variable.
val(M,fs,α) ≡ (α(f))(val(M,s,α)).

Note that we have used ≡ is the metalanguage. This is of course a slight abuse of notation, as we have only officially introduced ≡ in the object language (language of MPFS). Note that this use of ≡ has the following effect: if α(f) is undefined at val(M,s,α) then val(M,fs,α) is undefined; otherwise val(M,fs,α) is the value of α(f) at val(M,s,α).

We define satisfaction for atomic formulas of MPFS as follows.

sat(M,s = t,α) if and only if val(M,s,α) and val(M,t,α) are defined and equal.
sat(M,s ≡ t,α) if and only if either val(M,s,α) and val(M,t,α) are both defined and equal, or val(M,s,α) and val(M,t,α) are both undefined.
sat(M,s↓,α) if and only if val(M,s,α) is defined.
sat(M,s↑,α) if and only if val(M,s,α) is undefined.
sat(M,A ⊆ B,α) if and only if α(A) ⊆ α(B).
sat(M,t ∈ A,α) if and only if val(M,t,α) ∈ α(A). This is considered false if val(M,t,α) is undefined.
sat(M,f ⊆ g,α) if and only if α(f) ⊆ α(g).

sat(M,φ,α) for formulas φ of MPFS is defined in the usual way by induction, according to the ¬,∨,∧,→,↔ cases.

A formula of MPFS is (finitely) valid if and only if it holds in all (finite) MPFS structures M under all M assignments.

A formula of MPFS is (finitely) realizable if and only if it holds in some (finite) MPFS structure M under some M assignment.

DEFINITION 1.1. Let f be a function and E be a set. f||E = f ∩ E^2, using the graph of f as a set of ordered pairs. Let M = (X,Y,Z) be an MPFS structure and E ⊆ X be nonempty. M|E is the MPFS structure (E,Y',Z') with domain E, where Y' = {S ∩ E: S ∈ Y}, and Z' = {f||E: f in Z}. Let α be an M assignment. α|E is the M|E assignment defined as follows.

For object variables v, $(\alpha|E)(v) = \alpha(v)$ if $\alpha(v) \in E$; 0 otherwise. For set variables v, $(\alpha|E)(v) = \alpha(v) \cap E$. For function variables v, $(\alpha|E)(f) = f||E$.

Let sat(M,φ,α). We want to cut M down to a finite M' and find α' such that sat(M',φ,α'). The basic idea is to simply cut down with | to the interpretations by α of subterms of φ. However, this may change the equality relation as two different partial functions may become the same when cut down. We have the same problem with sets. We can fix every instance of this problem by throwing in finitely many more elements into E. This will not create any new problems.

The subterms of φ are defined in the obvious way. E.g., fw is a subterm of the formula hgfw $\subseteq$ gfv $\rightarrow$ ggv$\uparrow$.

The cardinality of an MPFS structure is the cardinality of its domain X.

LEMMA 1.1. Assume sat(M,φ,α), φ a formula of MPFS. There exists finite E $\subseteq$ X such that sat(M|E,φ,α|E). For the E constructed in the proof, if E $\subseteq$ E' $\subseteq$ X, then sat(M|E',φ,α|E'). The cardinality of E is at most #(φ), a positive integer effectively extracted from φ.

Proof: Let sat(M,φ,α), M = (X,Y,Z). We effectively construct E $\subseteq$ X from M,φ as follows.
i. Take 0, which is used for default values.
ii. For each object subterm t of φ, take val(M,t,α) if it exists.
iii. For each subformula v $\subseteq$ w of φ, v,w set variables, where α(v) $\neg\subseteq$ α(w), take an element of α(v) not in α(w).
iv. For each subformula v $\subseteq$ w of φ, v,w function variables, where α(v) $\neg\subseteq$ α(w), pick an element of α(v) not in α(w), and take both coordinates.

Note that there is an obvious bound #(φ) on the cardinality of E that depends only on φ.

We claim that for all object subterms t of φ, val(M|E,t,α|E) $\equiv$ val(M,t,α). We prove this by induction on t.

case 1. t is an object variable v in φ. Since α(v) ∈ E, we have (α|E)(v) = α(v). Hence val(M|E,v,α|E) = val(M,v,α).

case 2. t = g(s), where g is a function symbol and s is a term in φ. Since s is in φ, we have val(M|E,s,α|E) ≡ val(M,s,α). Note that val(M,g,α) = α(g) ∧ val(M|E,g,α|E) = (α(g))||E.

case 2.1. val(M,s,α) is undefined. Then val(M,t,α), val(M|E,s,α|E), val(M|E,t,α|E) are undefined.

case 2.2. val(M,s,α) = u, (α(g))(u) ∈ E. Then val(M,t,α) = (α(g))(u). Now val(M|E,t,α|E) ≡ ((α|E)(g))(val(M|E,s,α|E)) ≡ ((α|E)(g))(u) ≡ (α(g)||E)(u) = (α(g))(u) ∈ E.

case 2.3. val(M,s,α) = u, ¬((α(g))(u) ∈ E). If val(M,t,α) is defined then by construction, val(M,t,α) ∈ E. Hence val(M,t,α) is undefined. Also (α(g)||E)(u) is undefined. Now val(M|E,t,α|E) ≡ (α|E)(g)(val(M|E,s,α|E)) ≡ ((α|E)(g))(u) ≡ (α(g)||E)(u) is undefined.

Next we claim that for all set variables v of φ, val(M|E,v,α|E) = val(M,v,α) ∩ E. This follows from (α|E)(v) = α(v) ∩ E.

Next we claim that for all function variables f of φ, val(M|E,f,α|E) = val(M,f,α)||E. This follows from (α|E)(f) = α(f)||E.

Next we claim that for all atomic subformulas ψ of φ, sat(M|E,ψ,α|E) ↔ sat(M,ψ,α).

case 3. ψ is s = t, s ≡ t, s↑, s↓, where s,t are object terms. This follows from the first claim.

case 4. ψ is t ∈ A, where t is an object term and A is a set term. sat(M|E,ψ,α|E) ↔ val(M|E,t,α|E) ∈ (α|E)(A) ↔ val(M,t,α) ∈ α(A) ∩ E. Now if val(M,t,α) is defined then val(M,t,α) ∈ E. Hence val(M,t,α) ∈ α(A) ∩ E ↔ val(M,t,α) ∈ α(A).

case 5. ψ is $v \subseteq w$, where v,w are set variables. By the second claim, $sat(M|E,\psi,\alpha|E) \leftrightarrow val(M|E,v,\alpha|E) \subseteq val(M|E,w,\alpha|E) \leftrightarrow val(M,v,\alpha) \cap E \subseteq val(M,w,\alpha) \cap E$. Now if $val(M,v,\alpha) \neg\subseteq val(M,w,\alpha)$ then we have thrown a counterexample into E. Hence $val(M,v,\alpha) \cap E \subseteq val(M,w,\alpha) \cap E \leftrightarrow val(M,v,\alpha) \subseteq val(M,w,\alpha)$.

case 6. ψ is $f \subseteq g$, where f,g are function variables. By the third claim, $sat(M|E,\psi,\alpha|E) \leftrightarrow val(M|E,f,\alpha|E) \subseteq val(M|E,g,\alpha|E) \leftrightarrow val(M,f,\alpha)||E \subseteq val(M,g,\alpha)||E$. Now if $val(M,f,\alpha) \neg\subseteq val(M,g,\alpha)$ then we have thrown both coordinates of a counterexample into E. Hence $val(M,f,\alpha)||E \subseteq val(M,g,\alpha)||E \leftrightarrow val(M,f,\alpha) \subseteq val(M,g,\alpha)$.

It is now clear that $sat(M\}E),\varphi,\alpha|E) \leftrightarrow sat(M,\varphi,\alpha)$ by the usual induction argument according to the $\neg,\vee,\wedge,\rightarrow,\leftrightarrow$ cases.

The last claim is clear since we have not used any limitation on E. QED

LEMMA 1.2. Let φ be an MPFS formula. The following are equivalent.
i. φ is valid.
ii. φ is finitely valid.
iii. For every MPFS structure M of cardinality at most $\#(\varphi)$ and every M assignment α, $sat(M,\varphi,\alpha)$.

Proof: i $\rightarrow$ ii $\rightarrow$ iii is immediate. Suppose iii. Suppose φ is not valid. Let $sat(M,\neg\varphi,\alpha)$. According to Lemma 1.1, $sat(M|E,\neg\varphi,\alpha|E)$, $|E| \leq \#(\neg\varphi) = \#(\varphi)$, contradicting iii. QED

THEOREM 1.3. Validity and finite validity are equivalent. There is a decision procedure for determining validity of formulas in MPFS.

Proof: Use Lemma 1.2. We have only to verify that condition iii can be algorithmically tested. But we can obviously enumerate all MPFS structures of cardinality $\leq$ n up to isomorphism, uniformly in n, along with all M assignments, and also satisfaction is effectively testable. QED

Of course, the algorithm in the proof of Theorem 1.3 is crude and grossly impractical for almost any φ. It would seem that the procedure can be greatly refined, both theoretically and practically, and that with even very nonoptimal refinements, it should be efficient in practical situations.

2. MPFS* - MONADIC PARTIAL FUNCTIONS WITH SET ABSTRACTION

The sorts and variables of MPFS* are the same as for MPFS.

The object terms and function terms of MPFS* are the same as for MPFS.

MPFS* has more set terms. The set terms of MPFS* are

i. The set terms of MPFS.
ii. {v: φ}, where φ is a formula of MPFS and v is an object variable.

Note that in ii we steer clear of iteration. Just how much (and what kind of) iteration can be tolerated until we run into recursive unsolvability is a matter for future research.

The atomic formulas of MPFS* are as follows. Let s,t be object terms of MPFS* (MPFS), s*,t* be set terms of MPFS*, and f,g be function variables.

s = t
s ≡ t
s↓
s↑
s* ⊆ t*
t ∈ t*
f ⊆ g

This list is the same as for MPFS except that we use s* ⊆ t* instead of the special case A ⊆ B, and also t ∈ t* instead of the special case t ∈ A.

MPFS* is quite a bit more powerful than MPFS. For instance, we can readily express that two partial functions f,g agree on a set A by

A ⊆ {v: fx ≡ gx}.

Or that f is everywhere defined on A, by

$A \subseteq \{v: fv\downarrow\}$.

Or that f is constantly c on A, by

$A \subseteq \{v: fv = c\}$.

The MPFS* structures M are the same as the MPFS structures. We use the same M assignments. We define

$val(M,v,\alpha) = \alpha(v)$, v any variable.
$val(M,fs,\alpha) \equiv (\alpha(f))(val(M,s,\alpha))$.
$val(M,\{v: \varphi\},\alpha)$ is the set of all $u \in dom(M)$ such that $sat(M,\varphi,\alpha[v/x])$, where $\alpha[v/x]$ is the result of reassigning the object variable v to be u.

We define satisfaction for atomic formulas of MPFS* as follows (following the list above of atomic formulas).

$sat(M,s = t,\alpha)$ if and only if $val(M,s,\alpha)$ and $val(M,t,\alpha)$ are defined and equal.
$sat(M,s \equiv t,\alpha)$ if and only if either $val(M,s,\alpha)$ and $val(M,t,\alpha)$ are both defined and equal, or $val(M,s,\alpha)$ and $val(M,t,\alpha)$ are both undefined.
$sat(M,s\downarrow,\alpha)$ if and only if $val(M,s,\alpha)$ is defined.
$sat(M,s\uparrow,\alpha)$ if and only if $val(M,s,\alpha)$ is undefined.
$sat(M,s^* \subseteq t^*,\alpha)$ if and only if $val(M,s^*,\alpha) \subseteq val(M,t^*,\alpha)$.
$sat(M,t \in t^*,\alpha)$ if and only if $val(M,t,\alpha) \in val(M,t^*,\alpha)$.
$sat(M,f \subseteq g,\alpha)$ if and only if $\alpha(f) \subseteq \alpha(g)$.

$sat(M,\varphi,\alpha)$ for formulas φ of MPFS is defined in the usual way by induction.

A formula of MPFS* is (finitely) valid if and only if it holds in all (finite) MPFS structures M under all M assignments.

A formula of MPFS* is (finitely) realizable if and only if it holds in some (finite) MPFS structure M under some M assignment.

Also Definition 1.1 is repeated without change for MPFS*.

LEMMA 2.1. Let M be an MPFS structure, $\alpha_1,\ldots,\alpha_k$ be M assignments, and $\varphi_1,\ldots,\varphi_k$ be formulas of MPFS. Suppose for all i, sat(M,φ_i,α_i). There exists finite E ⊆ X such that each sat(M|E,φ_i,α_i|E). For the E constructed in the proof, if E ⊆ E' ⊆ X, then each sat(M|E',φ_i,α_i|E'). The cardinality of E is at most ##($\varphi_1,\ldots,\varphi_k$), a positive integer effectively extracted from $\varphi_1,\ldots,\varphi_k$.

Proof: Let each sat(M,φi,αi), M = (X,Y,Z). We construct E ⊆ X as follows.
i. Take 0, which is used for default values.
ii. For each object subterm t of each φ_i, take val(M,t,α_i) if it exists.
iii. For each subformula v ⊆ w of each φ_i, v,w set variables, where α(v) ¬⊆ α(w), take an element of α(v) not in α(w).
iv. For each subformula v ⊆ w of each φ_i, v,w function variables, where α_i(v) ¬⊆ α_i(w), pick an element of α_i(v) not in α_i(w), and take both coordinates.

We claim that for all object subterms t of φ_i, val(M|E,t,α_i|E) ≡ val(M,t,α_i). We prove this by induction on t as we did in section 1.

Next we claim that for all set variables v of φ_i, val(M|E,v,α_i|E) = val(M,v,α_i) ∩ E. This follows from (α_i|E)(v) = α_i(v) ∩ E.

Next we claim that for all function variables f of each φ_i, val(M|E,f,α_i|E) = val(M,f,α)||E. This follows from (α_i|E)(f) = α_i(f)||E.

Next we claim that for all atomic subformulas ψ of each φ_i, sat(M|E,ψ,α_i|E) ↔ sat(M,ψ,α_i). Follow the proof in section 1.

It is now clear that sat(M}E,φ_i,α_i|E) iff sat(M,φ_i,α_i) as in section 1.

The last claim is clear since we have not used any limitation on E. QED

LEMMA 2.2. Let M be an MPFS structure, α an M assignment, and φ be a formula of MPFS*. Suppose sat(M,φ,α). There exists finite $E \subseteq X$ such that each $sat(M|E,\varphi,\alpha|E)$. The cardinality of E is at most ###(φ), a positive integer effectively extracted from $\varphi_1,...,\varphi_k$.

Proof: We first construct $\varphi_1,...,\varphi_k$, and $\alpha_1,...,\alpha_k$, where each $sat(M,\varphi_i,\alpha_i)$. Look at all of the subformulas of φ of the forms

i. $\{v: \psi\} \subseteq w$, w a set variable.
ii. $w \subseteq \{v: \psi\}$, w a set variable.
iii. $\{v: \psi\} \subseteq \{w: \rho\}$.

which are satisfied to be false in M under assignment α. For each instance of i, take ψ and $\alpha[v/u]$ such that $sat(M,\psi,\alpha[v/u]) \wedge u \notin \alpha(w)$. For each instance of ii, take ψ and $\alpha[v/u]$ such that $sat(M,\neg\psi,\alpha[v/u]) \wedge u \in \alpha(w)$. For each instance of iii, take ψ and $\alpha[v/u]$ such that $sat(M,\psi,\alpha[v/u])$, and take $\neg\rho$ and $\alpha[w/u]$ such that $sat(M,\neg\rho,\alpha[w/u])$.

Now apply Lemma 2.1 to $\varphi_1,...,\varphi_k$ and $\alpha_1,...,\alpha_k$ to obtain finite $E \subseteq X$ such that each $sat(M||E,\varphi_i,\alpha_i||E)$. Then the negation of each of i,ii,iii is satisfied in M|E under $\alpha|E$. According to Lemma 2.1, this is also the case if we enlarge $E \subseteq X$. Now we can simplify φ to φ' by eliminating i,ii,iii from φ in M under α by replacing them with absurdity and absorbing with the connectives. Then φ' is a formula of MPFS and $\varphi \leftrightarrow \varphi'$ holds under all $\alpha|E'$, $E \subseteq E' \subseteq X$. Now choose finite $E \subseteq E' \subseteq X$ such that $sat(M|E',\varphi',\alpha|E')$ as in the proof of Lemma 1.1. QED

THEOREM 2.3. Validity and finite validity in MPFS* are equivalent. There is a decision procedure for determining validity of formulas in MPFS*.

Proof: By Lemma 2.2 as we obtained Theorem 1.3 from Lemma 1.2. QED