

EMBEDDED MAXIMAL CLIQUES AND INCOMPLETENESS EXTENDED ABSTRACT

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Abstract. We investigate "every order invariant graph on every Cartesian power of J has an f embedded maximal clique", where J be a rational interval and $f:J \rightarrow J$ is partial. For f with finite domain, we give an elegant necessary and sufficient condition. For general f , it is necessary that f be strictly increasing, and the investigation of sufficient conditions leads to statements with unexpected logical properties. The statements have the ZFC refutation property - it is provable from the usual ZFC axioms for mathematics, that if the statement is false then it is refutable in ZFC. The statements are demonstrably equivalent to the generally believed $\text{Con}(\text{SRP})$, which asserts the consistency of ZFC augmented by a well studied large cardinal hypothesis. Thus the statements cannot be proved in ZFC (assuming ZFC is consistent). This phenomena emerges even for numerical shift functions $f:Q_{\geq 0} \rightarrow Q_{\geq 0}$. We investigate associated infinite sequential constructions and show that "such infinite constructions have output" is also demonstrably equivalent to $\text{Con}(\text{SRP})$. We present five finite approximations to these constructions, within the realm of finite mathematics, with the ZFC refutation property. These statements are in explicitly Π^0_2 , Π^0_1 , Π^0_4 , Σ^0_2 , Σ^0_2 form. The last four are provably equivalent to $\text{Con}(\text{SRP})$. Iterated exponential bounds on the existential quantifiers put the last three in explicitly Π^0_1 form. We conclude with a discussion of associated computer investigations, leading to the prospect of the confirmation of $\text{Con}(\text{SRP})$ by actual practical computer implementations.

1. Graphs, cliques, embeddings, and tame sets.
2. Embedded Maximal Clique Theorems.

3. Infinite sequential constructions.
4. Finite sequential constructions.
5. Computer investigations.

1. Graphs, cliques, embeddings, and tame sets.

A graph is a pair $G = (V, E)$, where V is the vertex set, $E \subseteq V \times V = V^2$ is the edge set (edge relation, adjacency relation), and E is irreflexive and symmetric.

We say that x, y are adjacent in G if and only if $E(v, w)$. A clique in G is an $S \subseteq V$ such that any two distinct elements are adjacent in G . We also consider sequences from V to be cliques in G if and only if its set of terms is a clique in G .

A maximal clique in G is a clique in G which is not a proper subset of any clique in G . Note that S is a maximal clique in G if and only if S is a clique in G , where every vertex is nonadjacent, in G , to some element of S .

THEOREM 1.1. Every graph has a maximal clique. Every clique in every graph can be extended to a maximal clique. Both of these statements are provably equivalent to the axiom of choice over ZF. For countable graphs, the first statement is provable in RCA_0 , and the second statement is provably equivalent to ACA_0 over RCA_0 .

We will be placing embedding conditions on maximal cliques. Let $S \subseteq V$. We say that f embeds S if and only if $f: V \rightarrow V$ is a one-one partial function, and for all $x \in \text{dom}(f)$, $f(x) \in S \Leftrightarrow x \in S$.

We will be using graphs with multidimensional vertex sets. In this context, we use the coordinate action of f . I.e., let $S \subseteq A^k$. We say that f embeds S if and only if $f: A \rightarrow A$ is a one-one partial function, and for all $x_1, \dots, x_k \in \text{dom}(f)$, $(f(x_1), \dots, f(x_k)) \in S \Leftrightarrow (x_1, \dots, x_k) \in S$. The previous paragraph corresponds to the case $k = 1$.

We use Q for the set of all rational numbers. We say that rational vectors x, y are order equivalent if and only if they have the same length, and $(\forall i, j) (1 \leq i, j \leq \text{length}(x) \rightarrow (x_i < x_j \Leftrightarrow y_i < y_j))$.

We say that $S \subseteq Q^k$ is order invariant if and only if for all order equivalent $x, y \in Q^k$, $x \in S \Leftrightarrow y \in S$. More generally, S is an order invariant subset of $A \subseteq Q^k$ if and only if for

all order equivalent $x, y \in A$, $x \in S \leftrightarrow y \in S$. For each $A \subseteq Q^k$, there are finitely many order invariant subsets of A .

An order invariant graph on $V \subseteq Q^k$ is a graph (V, E) such that E is an order invariant subset of $V^2 \subseteq Q^{2k}$. Thus there are finitely many order invariant graphs on $V \subseteq Q^k$.

A rational interval is a $J \subseteq Q$ such that for all $x < y < z$, if $x, z \in J$ then $y \in J$.

We have given all of the definitions needed for the Embedded Maximal Clique statement, which appears near the beginning of section 2. There, we also give the tame form of the Embedded Maximal Clique Statement.

For the tame form, it is best to introduce the tame theory of $(Q, <)$. We say that $S \subseteq Q^k$ is order theoretic if and only if $S \subseteq Q^k$ that can be presented as a Boolean combination of inequalities

$$x_i < x_j, x_i < p, p < x_i$$

where $1 \leq i \leq k$, and p is a constant from Q . In actual presentations, it is convenient to use the wider collection of comparisons, $<, >, \leq, \geq, =, \neq$, and logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ for Boolean combinations.

Partial functions play a crucial role in the theory. A partial $f: Q^k \rightarrow Q^r$ is considered order theoretic if and only if its graph is order theoretic as a subset of Q^{k+r} .

THEOREM 1.2. $S \subseteq Q^k$ is order invariant if and only if $S \subseteq Q^k$ is order theoretic without constants. I.e., S can be presented as above, without the use of constants p .

THEOREM 1.3. The order theoretic $S \subseteq Q$ are the finite disjoint unions of rational intervals. Every order theoretic $S \subseteq Q$ is the union of the maximal intervals that it contains. These finitely many maximal intervals are pairwise disjoint rational intervals.

We say that $f: Q \rightarrow Q$ is basic if and only if $\text{dom}(f)$ is a nonempty open rational interval, or a singleton, and f is either constant or the identity on $\text{dom}(f)$.

THEOREM 1.4. Partial $f: Q \rightarrow Q$ is order theoretic if and only if it is the finite union of basic functions. Every order theoretic partial $f: Q \rightarrow Q$ is the union of the maximal basic

functions that it contains. These finitely many basic functions have pairwise disjoint domains. Every order theoretic $f:Q \rightarrow Q$ moves finitely many points.

2. Embedded maximal clique theorems.

All of the statements investigated here have an important metamathematical property. We say that a statement φ has the T refutation property if and only if

T proves that if φ is false then T proves that φ is false.

Since ZFC represents the usual axioms for mathematics, the ZFC refutation property assumes special importance. It is clear that the T refutation property becomes stronger as we weaken T. However, generally speaking, whenever a natural mathematical assertion is seen to have the ZFC refutation property, it can also be seen to have the T refutation property, for much weaker T.

Here we find that all of our statements have the WKLO refutation property, which is much stronger than the ZFC refutation property. WKLO is the second system in the list of the most commonly used systems of Reverse Mathematics that we set up in our ASL abstract. In increasing order of proving power, these are RCA_0 , WKL_0 , ACA_0 , ATR_0 , and $\Pi^1_1\text{-}CA_0$.

The T refutation property conveniently avoids consideration of logical forms - in particular, the logical form Π^0_1 .

THEOREM 2.1. Let T be a consistent extension of RCA_0 .

- i. Suppose that φ is a sentence such that there is a Π^0_1 sentence ψ with T proves $\varphi \leftrightarrow \psi$. Then φ has the T refutation property.
- ii. There is a sentence φ with the T refutation property, yet there is no Π^0_1 sentence ψ such that T proves $\varphi \leftrightarrow \psi$.

Moreover, we shall see that all of our embedded maximal clique statements are provably equivalent to Π^0_1 sentences over WKL_0 , and hence automatically have not only the ZFC refutation property, but the WKL_0 refutation property.

Let J be a rational interval and $f:J \rightarrow J$ be partial. We investigate the statement

Every order invariant graph on every J^k has an f embedded maximal clique.

THEOREM 2.2. Let J, f be as given. A necessary but not sufficient condition for the statement is that f be strictly increasing.

We have been able to give a necessary and sufficient condition for finite f . We say that f is finite (infinite) if and only if $\text{dom}(f)$ is finite (infinite).

FINITE EMBEDDED MAXIMAL CLIQUE THEOREM. FEMCT. Let J be a rational interval and $f: J \rightarrow J$ be a finite partial function. The following are equivalent.

- i. Every order invariant graph on every J^k has an f embedded maximal clique.
- ii. f is strictly increasing, and the iterates of f at any endpoint do not include a second endpoint.

There are only three nondegenerate rational intervals, $Q, Q_{\geq 0}, Q[0,1]$ up to isomorphism (the degenerate empty set and singletons are of no interest). Here $Q_{\geq 0}$ is the set of nonnegative rationals, and $Q[0,1] = Q \cap [0,1]$. We give the following versions of FEMCT, for these three intervals. Note the simplifications in the first two below.

FINITE EMBEDDED MAXIMAL CLIQUE THEOREM (Q). FEMCT(Q). Let $f: Q \rightarrow Q$ be a finite partial function. The following are equivalent.

- i. Every order invariant graph on every Q^k has an f embedded maximal clique.
- ii. f is strictly increasing.

FINITE EMBEDDED MAXIMAL CLIQUE THEOREM ($Q_{\geq 0}$). FEMCT($Q_{\geq 0}$). Let $f: Q_{\geq 0} \rightarrow Q_{\geq 0}$ be a finite partial function. The following are equivalent.

- i. Every order invariant graph on every $Q_{\geq 0}^k$ has an f embedded maximal clique.
- ii. f is strictly increasing.

FINITE EMBEDDED MAXIMAL CLIQUE THEOREM ($Q[0,1]$). FEMCT($Q[0,1]$). Let $f: Q[0,1] \rightarrow Q[0,1]$ be a finite partial function. The following are equivalent.

- i. Every order invariant graph on every $Q[0,1]^k$ has an f embedded maximal clique.
- ii. f is strictly increasing, and the iterates of f at any endpoint do not include a second endpoint.

THEOREM 2.3. FEMCT(Q) is provable in RCA_0 . FEMCT, FEMCT($Q_{\geq 0}$), and FEMCT($Q[0,1]$), and $\text{Con}(\text{PA})$ are provably equivalent over

RCA0. For each of these three versions, this equivalence holds for $ii \rightarrow i$, but $i \rightarrow ii$ is provable in RCA_0 .

We now consider the case of infinite f .

EMBEDDED MAXIMAL CLIQUE (gen). EMC(gen). Let J be a rational interval and $f:J \rightarrow J$ be an infinite partial function, where f is strictly increasing, continuous, and moves finitely many points, at most one of which is an endpoint. Every order invariant graph on every J^k has an f embedded maximal clique.

We write "gen" for "general", since EMC(gen) involves pathological f , as there is no restriction on its domain. However, EMC is really about order theoretic f . Thus we emphasize the following tame version.

EMBEDDED MAXIMAL CLIQUE. EMC. Let J be a rational interval and $f:J \rightarrow J$ be an infinite order theoretic partial function, where f is strictly increasing, continuous, and moves at most one endpoint. Every order invariant graph on every J^k has an f embedded maximal clique.

The following has an easy direct proof.

THEOREM 2.4. RCA0 proves $\text{EMC}(\text{gen}) \leftrightarrow \text{EMC}$. Furthermore, RCA0 proves that this equivalence holds for any J, k, G given in advance.

Henceforth, we will consider only EMC and some tame variants.

EMBEDDED MAXIMAL CLIQUE (\mathbb{Q}). EMC(\mathbb{Q}). Let $f:\mathbb{Q} \rightarrow \mathbb{Q}$ be an order theoretic partial function, where f is strictly increasing and continuous. Every order invariant graph on every \mathbb{Q}^k has an f embedded maximal clique.

EMBEDDED MAXIMAL CLIQUE ($\mathbb{Q}_{\geq 0}$). EMC($\mathbb{Q}_{\geq 0}$). Let $f:\mathbb{Q}_{\geq 0} \rightarrow \mathbb{Q}_{\geq 0}$ be an order theoretic partial function, where f is strictly increasing and continuous. Every order invariant graph on every $\mathbb{Q}_{\geq 0}^k$ has an f embedded maximal clique.

EMBEDDED MAXIMAL CLIQUE ($\mathbb{Q}[0,1]$). EMC($\mathbb{Q}[0,1]$). Let $f:\mathbb{Q}[0,1] \rightarrow \mathbb{Q}[0,1]$ be an infinite order theoretic partial function, where f is strictly increasing and continuous, and moves at most one endpoint. Every order invariant graph on every $\mathbb{Q}[0,1]^k$ has an f embedded maximal clique.

SRP is the system extending ZFC by the large cardinal hypothesis scheme

(there exists a limit ordinal with k -SRP) $_k$.

We say that λ has the k -SRP (stationary Ramsey property of order k) if and only if for every partition of the unordered k -tuples from λ into two cells, there is a stationary $E \subseteq \lambda$ such that all unordered k -tuples from E lie in the same cell. In the large cardinal hierarchy, 2-SRP is stronger than weakly compact which is stronger than any reasonable kind of Mahloness. Also, the entire SRP hierarchy is weaker than the weakest Ramsey cardinal $\kappa \rightarrow \omega$, which is in turn much weaker than cardinals that are incompatible with $V = L$ such as $\kappa \rightarrow \omega_1$, $\kappa \rightarrow \kappa$, and measurable cardinals.

Con(SRP) asserts that the system SRP is consistent (i.e., free of contradiction).

THEOREM 2.5. EMC, EMC(Q), EMC($Q_{\geq 0}$), EMC(Q[0,1]) have the WKL₀ refutation property, and are implied by Con(SRP) over WKL₀. EMC, EMC($Q_{\geq 0}$), and EMC(Q[0,1]) are provably equivalent to Con(SRP) over WKL₀.

CONJECTURE. EMC(Q) is provably equivalent to Con(SRP) over WKL₀.

Now consider the following templates. In the headers, J is a rational interval, $f:J \rightarrow J$ is infinite, partial, and order theoretic, and G is an order invariant graph on the Cartesian power of some rational interval. Note that J is considered to be a component of f .

EMC/ k , EMC/ J , EMC/ f , EMC/ G , EMC/ k,J , EMC/ k,f , EMC/ f,G , are EMC with k , J , f , G , k,J , k,f , f,G respectively fixed.

THEOREM 2.6. Each EMC/ k , EMC/ J , EMC/ f , EMC/ G , EMC/ k,J , EMC/ k,f , EMC/ f,G has the WKL₀ refutation property, and is provable in WKL₀ + Con(SRP).

CONJECTURE. Every instance of EMC/ k , EMC/ J , EMC/ f , EMC/ G , EMC/ k,J , EMC/ k,f , EMC/ f,G is either provable in WKL₀ + Con(SRP) or refutable in RCA₀. Furthermore, this statement is provable in RCA₀. Moreover, there is a mathematically elegant algorithm for each of these 7, which, provably in ZFC + Con(SRP), correctly determines the truth values.

THEOREM 2.7. There are k, J, f, G such that $EMC/k, EMC/J, EMC/f, EMC/G, EMC/k, J, EMC/k, f, EMC/f, G$ are each not provable in ZFC (assuming ZFC is consistent).

Here are some basic examples of order theoretic partial $f: Q_{\geq 0} \rightarrow Q_{\geq 0}$ that are continuous and strictly increasing.

$SH[n](x) = x+1$ if $x \in \{0, \dots, n\}$; undefined otherwise.
I.e., shift discretely.

$LSH[n](x) = x+1$ if $x \in \{0, \dots, n\}$; x if $x > n+1$; undefined otherwise.
I.e., shift discretely, and then flatten out higher up.

$LSHG[n](x) = x+1$ if $x \in \{0, \dots, n\}$; x if $x > n+2$; undefined otherwise.
I.e., shift discretely, and then flatten out higher up with a gap.

SH, LSH, LSHG are read "shift", "lower shift", "lower shift with gap". For the rest of this manuscript, we focus on the following three forms of EMC.

EMBEDDED MAXIMAL CLIQUE $(Q_{\geq 0}, SH)$. $EMC(Q_{\geq 0}, SH)$. For all $k, n \geq 1$, every order invariant graph on $Q_{\geq 0}^k$ has an $SH[n]$ embedded maximal clique.

EMBEDDED MAXIMAL CLIQUE $(Q_{\geq 0}, LSH)$. $EMC(Q_{\geq 0}, LSH)$. For all $k, n \geq 1$, every order invariant graph on $Q_{\geq 0}^k$ has an $LSH[n]$ embedded maximal clique.

EMBEDDED MAXIMAL CLIQUE $(Q_{\geq 0}, LSHG)$. $EMC(Q_{\geq 0}, LSHG)$. For all $k, n \geq 1$, every order invariant graph on $Q_{\geq 0}^k$ has an $LSHG[n]$ embedded maximal clique.

THEOREM 2.8. $EMC(Q_{\geq 0}, SH), EMC(Q_{\geq 0}, LSH), EMC(Q_{\geq 0}, LSHG)$ each have the WKL_0 refutation property. $EMC(Q_{\geq 0}, SH)$ is provably equivalent to $Con(PA)$ over WKL_0 . $EMC(Q_{\geq 0}, LSHG)$ is provable in Zermelo set theory. $EMC(Q_{\geq 0}, LSH)$ is provably equivalent to $Con(SRP)$ over WKL_0 .

3. Infinite sequential constructions.

Let G be a countable graph. Let v_0, \dots be a finite or infinite sequence from V . Consider the following deterministic construction of a subsequence of v_0, \dots .

DET(G, v_0, \dots)

Start at stage 0 with v_0, \dots in place.

At stage i , keep v_i if the remaining v_0, \dots, v_{i-1} ,
and v_i form a clique in G .

Otherwise, remove v_i .

Go to stage $i+1$. End after all v 's are processed.

THEOREM 3.1. Assumes $V = \{v_0, \dots\}$. The output of
DET(G, v_0, \dots) is a maximal clique in G .

We will want to construct maximal cliques with embedding properties, and we shall see that the above kind of deterministic construction will not suffice. For greater flexibility, we introduce the following nondeterministic construction. In this construction, it is simplest to replace vertices, instead of removing them. We consider v to be a replacement of v .

NONDET(G, v_0, \dots)

Start at stage 0, with v_0, \dots in place.

At stage i , replace v_i with a vertex v_i' nonadjacent to v_i ,
such that v_0', \dots, v_i' is a clique in G .

Go to stage $i+1$. End after all v 's are processed.

THEOREM 3.2. Assume $V = \{v_0, \dots\}$. NONDET(G, v_0, \dots) never gets blocked. Suppose $V = \{v_0, \dots\}$. The outputs of NONDET(G, v_0, \dots), as sets of vectors, are exactly the maximal cliques in G . For some G and $V = \{v_0, \dots\}$, some output of NONDET(G, v_0, \dots) is not the same as the output of DET(G, v_0, \dots).

Until after Theorem 3.6, we fix a graph $G = (V, E)$, a one-one partial $f: V \rightarrow V$, and $V = \{v_0, \dots\}$. We consider sequences from V to be f embedded if and only if their set of terms is f embedded. We want to construct an f embedded maximal clique in G .

WARNING: The f embedded maximal cliques in G are not the same as the maximal f embedded cliques in G . Maximal f embedded cliques in G always exist, but may not be maximal cliques in G .

There may be no f embedded maximal cliques in G . In fact, it is possible that all f embedded cliques in G are empty. However, we present a nondeterministic algorithm for

constructing an f embedded maximal clique in G , if there is one, on the basis of any $V = \{v_0, \dots\}$.

We have been using finite cliques in G as approximates to maximal cliques in G . We cannot use f embedded cliques in G as approximates to f embedded maximal cliques in G . It might be the case that all finite f embedded maximal cliques in G are empty. So we make the following definition.

S is an f friendly clique in G if and only if
 $S \cup f[S] \cup f^{-1}[S]$ is a clique in G .

Note that since f is one-one, if S is finite then $S \cup f[S] \cup f^{-1}[S]$ is finite.

We consider a sequence from V to be an f friendly clique in G if and only if its set of terms is an f friendly clique in G .

Let us try the following deterministic greedy construction.

DET(G, f, v_0, \dots)

Start at stage 0, with v_0, \dots in place.

At stage i , keep v_i if the remaining v_0, \dots, v_{i-1} and
 v_i is an f friendly clique in G .

Otherwise, remove v_i .

Go to stage $i+1$. End after all v 's are processed.

THEOREM 3.3. Assume $V = \{v_0, \dots\}$. Any output of DET(G, f, v_0, \dots) is a maximal f embedded clique in G . There exists G, f , and $V = \{v_0, \dots\}$ such that there is an f embedded maximal clique in G , and the output of DET(G, f, v_0, \dots) is not a maximal clique in G .

Therefore we need more flexibility to construct f embedded maximal cliques. Consider the following nondeterministic construction.

NONDET(G, f, v_0, \dots)

Start at stage 0, with v_0, \dots in place.

At stage i , replace v_i with a vertex v_i' nonadjacent to v_i ,
such that v_0', \dots, v_i' is an f friendly clique in G .

Go to stage $i+1$. End after all v 's are processed.

THEOREM 3.4. Assume $V = \{v_0, \dots\}$. The outputs of $\text{NONDET}(G, f, v_0, \dots)$, as sets of vectors, are exactly the f embedded maximal cliques in G .

In the course of implementing $\text{NONDET}(G, f, v_0, \dots)$, a block occurs at stage i if and only if we have reached stage i (i.e., completed the first $i-1$ stages), and there is no such v_i as required.

THEOREM 3.5. Assume $V = \{v_0, \dots\}$. If there is a maximal f embedded clique in G , then $\text{NONDET}(G, f, v_0, \dots)$ cannot have a block at stage 0. On the other hand, there exists G, f , and $V = \{v_0, \dots\}$ such that there is a maximal f embedded clique in G , and $\text{NONDET}(G, f, v_0, \dots)$ can be implemented with a block at stage 1.

Theorem 3.5 indicates that we can make fatal errors in the implementation of $\text{NONDET}(G, f, v_0, \dots)$ from which we cannot recover, even if there is a maximal f embedded clique in G .

We now apply this construction to $\text{EMC}(Q_{\geq 0}, \text{LSH})$ from section 2. Fix $k, n \geq 1$, and an order invariant graph G on $Q_{\geq 0}^k$. Note that $\text{LSH}[n]$ lifts to a one-one partial function from each $Q_{\geq 0}^k$ into $Q_{\geq 0}^k$ via the coordinate action. Let $Q_{\geq 0}^k = \{v_0, \dots\}$. We use $\text{NONDET}(G, f, v_s, \dots)$.

We now introduce the following statement.

INFINITE CONSTRUCTION (LSH). $\text{INFCON}(Q_{\geq 0}, \text{LSH})$. Let $k, n \geq 1$, G be an order invariant graph on $Q_{\geq 0}^k$, and $Q_{\geq 0}^k = \{v_0, \dots\}$. $\text{NONDET}(G, \text{LSH}[n], v_0, \dots)$ has an output.

THEOREM 3.6. $\text{INFCON}(Q_{\geq 0}, \text{LSH})$ is provably equivalent to $\text{Con}(\text{SRP})$ over WKL_0 .

4. Finite sequential constructions.

Obviously $\text{INFCON}(Q_{\geq 0}, \text{LSH})$ lies outside the realm of finite mathematics since the construction $\text{NONDET}(G, \text{LSH}[n], v_0, \dots)$ is of infinite length.

We can obviously stay within the realm of finite mathematics by simply restricting to finite sets of vectors.

FINITE CONSTRUCTION ($Q_{\geq 0}, \text{LSH}$). $\text{FINCON}(Q_{\geq 0}, \text{LSH})$. Let $k, n \geq 1$, G be an order invariant graph on $Q_{\geq 0}^k$, and $v_0, \dots, v_m \in Q_{\geq 0}^k$. $\text{NONDET}(G, \text{LSH}[n], v_0, \dots, v_m)$ has an output.

Note that $\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ is explicitly Π_2^0 .

THEOREM 4.1. $\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ is provable in $\text{EFA} + \text{Con}(\text{PA})$.

Theorem 4.1 indicates that $\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ is only a very weak approximation to $\text{INFCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$. We now present four much stronger approximations to $\text{INFCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$. In fact, they are all provably equivalent to $\text{INFCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$, and hence to $\text{Con}(\text{SRP})$, over WKL_0 .

All of these four strong approximations have their advantages. They provide a range of perhaps equally attractive finite forms.

For the first strong approximation, we add a numerical requirement in the choice of v_i' in $\text{NONDET}(G, \text{LSH}[n], v_0, \dots, v_m)$.

The norm of a finite sequence from \mathbb{Q} is the least positive integer t such that all terms can be written with numerators and denominators of magnitude $\leq t$.

We say that $v_0, \dots, v_r \in \mathbb{Q}^k$ is sharp if and only if for all $0 \leq i \leq r$, if the concatenations of (v_0, \dots, v_i) and $(v_0, \dots, v_{i-1}, v_i')$ are order equivalent, then the norm of v_i' is at most the norm of v_i . Controlling the norms in this way is natural when considering computer implementations.

SHARP/NONDET (G, f, v_0, \dots)

Start at stage 0, with v_0, \dots in place.

At stage i , replace v_i with a vertex v_i' nonadjacent to v_i , such that v_0', \dots, v_i' is a sharp f friendly clique in G .

Go to stage $i+1$. End after all v 's are processed.

SHARP FINITE CONSTRUCTION $(\mathbb{Q}_{\geq 0}, \text{LSH})$. SHARP/FINCON $(\mathbb{Q}_{\geq 0}, \text{LSH})$. Let $k, n \geq 1$, G be an order invariant graph on $\mathbb{Q}_{\geq 0}^k$, and $v_0, \dots, v_m \in \mathbb{Q}_{\geq 0}^k$. SHARP/NONDET $(G, \text{LSH}[n], v_0, \dots, v_m)$ has an output.

Note that SHARP/FINCON $(\mathbb{Q}_{\geq 0}, \text{LSH})$ is explicitly Π_1^0 , as there is an obvious upper bound to the norms of all vectors in the output relative to the norms of the given v_0, \dots, v_m , in terms of an iterated exponential expression.

THEOREM 4.2. SHARP/FINCON($Q_{\geq 0}$, LSH) is provably equivalent to Con(SRP) over EFA.

The remaining three strong approximations to INFCON(Q_s , LSH) do not involve numerical considerations (arguably an advantage). In the second strong approximation, we use a basic form of interactivity that is nicely described by typical student teacher interaction.

Let G be an order invariant graph on $Q_{\geq 0}^k$. The teacher hands the student $v_0, \dots, v_m \in Q_{\geq 0}^k$. The student follows $\text{NONDET}(G, \text{LSH}[n], v_0, \dots, v_m)$. After the student outputs $v_0', \dots, v_m' \in Q_{\geq 0}^k$, the teacher looks at this output, and chooses additional $v_{m+1}, \dots, v_r \in Q^k$. The student is required to follow $\text{NONDET}(G, \text{LSH}[n], v_0, \dots, v_r)$, continuing where they left off, at stage $m+1$, with earlier output v_0', \dots, v_m' .

In a more demanding second version of the interactive algorithm, the graph and n, m are set in advance. The teacher decides r only after looking at the student's output. In the first version, r is set in advance.

INTERACTIVE FINITE CONSTRUCTION ($Q_{\geq 0}$, LSH).

INTER/FINCON($Q_{\geq 0}$, LSH). Let $k, n \geq 1$, and G be an order invariant graph on $Q_{\geq 0}^k$. Student success is achievable, as indicated above, where the length of the second list of teacher generated vectors depends on the first output.

WEAK INTERACTIVE FINITE CONSTRUCTION ($Q_{\geq 0}$, LSH).

W/INTER/FINCON($Q_{\geq 0}$, LSH). Let $k, n, r \geq 1$, and G be an order invariant graph on $Q_{\geq 0}^k$. Student success is achievable, as indicated above, where the length of the second list of teacher generated vectors is r .

This interactivity can be extended in the following natural way. First the teacher gives a finite sequence of vectors. The student gives the output. The teacher looks at the output and gives a second finite sequence of vectors. The student continues work and gives more output. The teacher looks at the additional output, and gives a third sequence of vectors. And so forth, for any number of interactions given in advance.

EXTENDED INTERACTIVE FINITE CONSTRUCTION ($Q_{\geq 0}$, LSH).

EXT/INTER/FINCON($Q_{\geq 0}$, LSH). Let $k, n, t \geq 1$, and G be an order invariant graph on $Q_{\geq 0}^k$. Student success is achievable for t rounds of interactivity as indicated above, where the length of lists is dependent on outputs.

THEOREM 4.3. $\text{INTER}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$, $\text{W}/\text{INTER}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$, $\text{EXT}/\text{INTER}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$, and $\text{Con}(\text{SRP})$ are provably equivalent over EFA.

Note that $\text{INTER}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ and $\text{W}/\text{INTER}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ are in explicitly Π^0_4 form. However, there is an easy way to put these statements in Π^0_1 form. This is by controlling the magnitudes of the numerators and denominators generated by the student in the coordinates of the vectors they respond with. Note that we are bringing in numerical considerations after the fact as part of a logical analysis of these statements, as opposed to bringing numerical considerations into the statement itself as in $\text{SHARP}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$. Thus we get a statement, in terms of nonnegative integer codes, in the form

$$(\forall t_1) (\exists t_2 < \alpha(t_1)) (\forall t_3) (\exists t_4 < \beta(t_1, t_2, t_3)) (\varphi(t_1, t_2, t_3, t_4))$$

where α, β are iterated exponential expressions. This is in Π^0_1 form by standard quantifier manipulations. These ideas can also be adapted to put $\text{EXST}/\text{INTER}/\text{FINCON}(\mathbb{Q}_s, \text{LSH})$ in Π^0_1 form.

In the third strong approximation, we start with order invariant graphs G_1, \dots, G_r on $\mathbb{Q}_{\geq 0}^k$, partial $f: \mathbb{Q}_{\geq 0} \rightarrow \mathbb{Q}_{\geq 0}$, and v_0, \dots, v_m .

$$\text{NONDET}(G_1, \dots, G_r, f, v_0, \dots, v_m)$$

Run $\text{NONDET}(G_1, f, v_0, \dots, v_m)$ with output $v[1, 0], \dots, v[1, m]$

Run $\text{NONDET}(G_2, f, v[1, 0], \dots, v[1, m])$ with output
 $v[2, 0], \dots, v[2, m]$.

...

Run $\text{NONDET}(G_r, f, v[r-1, 0], \dots, v[r-1, m])$ with output
 $v[r, 0], \dots, v[r, m]$.

MULTIPLE FINITE CONSTRUCTION $(\mathbb{Q}_{\geq 0}, \text{LSH})$. $\text{MULT}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$. Let $k, n, r \geq 1$, and G_1, \dots, G_r be order invariant graphs on $\mathbb{Q}_{\geq 0}^k$. $\text{NONDET}(G_1, \dots, G_r, \text{LSH}[n])$ has an output.

THEOREM 4.4. $\text{MULT}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ and $\text{Con}(\text{SRP})$ are provably equivalent over EFA.

Note that $\text{MULT}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{LSH})$ is explicitly Π^0_2 . However, by a straightforward bound on the norm of the output by an

iterated exponential expression, $\text{MULT}/\text{FINCON}(Q_{\geq 0}, \text{LSH})$ is easily put in Π_1^0 form.

The fourth and final strong approximation allows the greatest flexibility and will drive the computer investigations discussed in section 5.

We use a scheduling function fixed in advance. We use scheduling functions $\alpha: (Q_{\geq 0}^k)^{\leq t} \rightarrow Q_{\geq 0}^k$, where $\alpha: (Q_{\geq 0}^k)^{\leq t}$ is the set of all sequences from $Q_{\geq 0}^k$ of length at most t , including length 0.

NONDET (G, f, α)

Start at stage 0, with nothing in place.

At stage i , output some v_i nonadjacent to $\alpha(v_0, \dots, v_{i-1})$
such that v_0, \dots, v_i is an f friendly clique in G .

Go to stage $i+1$. End after stage t .

Of course, this use of arbitrary α puts the above construction outside the realm of finite mathematics. But we will insist that α be order theoretic in the sense that each component function $\alpha: (Q_{\geq 0}^k)^i \rightarrow Q_{\geq 0}^k$ is order theoretic, viewed as a function from $Q_{\geq 0}^{ki}$ into $Q_{\geq 0}^k$.

$\text{SCHEDULED FINCON}(Q_{\geq 0}, \text{LSH})$. $\text{SCHED}/\text{FINCON}(Q_{\geq 0}, \text{LSH})$. Let $k, n, t \geq 1$, G be an order invariant graph on $Q_{\geq 0}^k$, and $\alpha: (Q_{\geq 0}^k)^{\leq t} \rightarrow Q_{\geq 0}^k$ be order theoretic. $\text{NONDET}(G, \text{LSH}, \alpha)$ has an output.

THEOREM 4.5. $\text{SCHED}/\text{FINCON}(Q_{\geq 0}, \text{LSH})$ and $\text{Con}(\text{SRP})$ are provably equivalent over EFA.

Note that $\text{SCHED}/\text{FINCON}(Q_{\geq 0}, \text{LSH})$ is explicitly Π_2^0 . However, by a straightforward bound on the norm of the output by an iterated exponential expression, $\text{SCHED}/\text{FINCON}(Q_{\geq 0}, \text{LSH})$ is easily put in Π_1^0 form.

5. Computer Investigations.

It would appear that even the actual construction of short outputs for the various nondeterministic constructions in section 4 is highly computer intensive. Of course, we know that short outputs exist using $\text{Con}(\text{SRP})$. But this knowledge (relative to $\text{Con}(\text{SRP})$) does not appear to provide a method of constructing short outputs by hand, or even by computer without doing an exhaustive search. Of course, there appear

to be plenty of opportunities for clever reduction in computer resources used for the search.

The most convenient and flexible setup for computer investigations seems to be SCHEDULING FINCON($Q_{\geq 0}$, LSH). We also strengthen the LSH friendly clique requirement using the following binary relation REL on finite sequences from Q . REL is routinely used in the theory of indiscernibles in set theory.

Let x, y be finite sequences from Q . $REL(x, y)$ if and only if

- i. x and y are order equivalent.
- ii. $(\forall i) (x_i \neq y_i \rightarrow \text{every } x_j \geq x_i \text{ and } y_j \geq y_i \text{ lies in } N)$.

Here N is the set of all nonnegative integers. It is easy to see that REL is an equivalence relation on the set of all finite sequences of Q .

Let J be a rational interval. We say that $S \subseteq J^k$ is REL invariant if and only if for all REL equivalent $x, y \in J^k$, we have $x \in S \Leftrightarrow y \in S$.

EMBEDDED MAXIMAL CLIQUE ($Q_{\geq 0}$, REL). EMC($Q_{\geq 0}$, REL). Every order invariant graph on every $Q_{\geq 0}^k$ has a REL invariant maximal clique.

THEOREM 5.1. RCA_0 proves $EMC(Q_{\geq 0}, REL) \Leftrightarrow EMC(Q_{\geq 0}, LSH)$.
 $EMC(Q_{\geq 0}, REL)$, $EMC(Q_{\geq 0}, LSH)$ and $Con(SRP)$ are provably equivalent over WKL_0 .

Let G be a graph on $Q_{\geq 0}^k$. We make the following definition.

$S \subseteq Q_{\geq 0}^k$ is a REL friendly clique in G if and only if
 $REL[S] \cap Q_{\geq 0}^k$ is a clique in G

where $REL[S]$ is the set of all vectors related to an element of S by REL. We consider a sequence from $Q_{\geq 0}$ to be a REL friendly clique if and only if the set of its terms is a REL friendly clique.

For our computer investigations, we will be adapting

NONDET(G, REL, α)

Start at stage 0, with nothing in place.
 At stage i , output some v_i nonadjacent to $\alpha(v_0, \dots, v_{i-1})$
 such that v_0, \dots, v_i is a REL friendly clique in G .

Go to stage $i+1$. End after stage $t+1$.

where G is an order invariant graph on $Q_{\geq 0}^k$ and $\alpha: (Q_{\geq 0}^k)^{\leq t} \rightarrow Q_{\geq 0}^k$ is order theoretic. Since $\text{REL}[S] \cap Q_{\geq 0}^k$ is generally infinite even if S is finite, there is an issue as to whether we have left the realm of finite mathematics. However, it is easy to see that REL friendliness only involves finitely many vectors from $\text{REL}[S]$.

SCHEDULED FINCON($Q_{\geq 0}$, REL). SCHED/FINCON($Q_{\geq 0}$, REL). Let $k, t \geq 1$, G be an order invariant graph on $Q_{\geq 0}^k$, and $\alpha: (Q_{\geq 0}^k)^{\leq t} \rightarrow Q_{\geq 0}^k$ be order theoretic. NONDET(G, REL, α) has an output.

THEOREM 5.2. SCED/FINCON($Q_{\geq 0}$, REL) and Con(SRP) are provably equivalent over EFA.

We are now ready to design the computer investigation. First note that any order invariant $B \subseteq Q_{\geq 0}^k$ has a canonical presentation $W \subseteq \{1, \dots, k\}^k$, where the coordinates of every $x \in W$ form an initial segment of $1, \dots, k$. B is the set of $x \in Q_{\geq 0}^k$ that is order equivalent to some element of W .

1. Create an order invariant graph G on $Q_{\geq 0}^k$, where k is chosen to be an experimentally adjustable small integer. Here G is given by the canonical presentation W of its edge set. W is chosen experimentally, with a considerable amount of randomness present. A human is not going to make any good sense of G .

2. Choose an experimentally adjustable small integer t . Then experimentally choose an order theoretic $\alpha: (Q_{\geq 0}^k)^{\leq t} \rightarrow Q_{\geq 0}^k$. Generally, α is going to be much too big to be held in a table. Instead, experimentally develop an algorithm for α as a piece of software, with the help of random number generators and experimental heuristics. Thus α is presented as an involved algorithm which can be experimentally adjusted so as to be far from humanly digestible. Because of the use of REL, we should use the integers $0, \dots, 2k$ as parameters (as well as a few other fractions), and considerably overweight their probability of being coordinates in values of α .

2'. An alternative to 2 is to construct an attractive algorithm for α based on a clear mathematical idea. We still want to overweight $0, \dots, 2k$ in the outputs. Both 2 and 2' seem to have their advantages, depending on whether we want to showcase the power of Con(SRP), or confirm the truth of Con(SRP) - see 4 and 6 below.

3. By Theorem 5.2, we know that $\text{SCHED}/\text{FINCON}(\mathbb{Q}_{\geq 0}, \text{REL})$ has an output, assuming the widely believed $\text{Con}(\text{SRP})$. The goal is to create an actual output of $\text{NONDET}(G, \text{REL}, \alpha)$. The only way we know how to do this is by computer search. In the execution of $\text{NONDET}(G, \text{REL}, \alpha)$, the testing of v_0, \dots, v_i for being a REL friendly clique can be computationally intensive, and there seems to be great opportunities for clever resource reducing algorithmic ideas. If the parameters are chosen so that this testing is impractical, then we can require only that our output passes a reduced form of this testing. That is tantamount to an adjusted form of the nondeterministic algorithm, which is still subject to exhaustive search. In this way, we can experiment with a huge range parameters without worrying about resource blowup. Just seek lower quality output.

4. A successful search for an output might be viewed as experimental confirmation of $\text{Con}(\text{SRP})$. One important feature of confirmations that is often cited is that if the experiment fails then the theory is refuted. This is the case here. I.e., failure to find an output during the exhaustive search does immediately imply that SRP is inconsistent. In particular, it provides a refutation of certain well studied large cardinal hypotheses.

5. We can also adjust the parameters so that the computer search is beyond reach of present day computers, but under predicted computer developments, will be within reach at more or less definite times in the future. Thus $\text{Con}(\text{SRP})$ predicts that these future searches will be successful. This suggests a range of futuristic predications coming from $\text{Con}(\text{SRP})$. Again, if the future experiments fail, then they will refute $\text{Con}(\text{SRP})$ and also the large cardinal hypotheses will be directly refuted within ZFC (and much less).

6. Although $\text{Con}(\text{SRP})$ is widely believed, it just might be the case that humans only know how to find proofs that are reasonably short and have digestible structure. The kind of refutations that could conceivably arise from failed searches for outputs, as discussed above, are neither reasonably short nor have digestible structure. So such challenges to $\text{Con}(\text{SRP})$ by computer just might yield astonishing results.