TANGIBLE MATHEMATICAL INCOMPLETENESS OF ZFC

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Abstract. We present the lead statements in the new Emulation Theory that provide the first tangible examples of the mathematical incompleteness of the usual ZFC axioms for mathematics. We focus on three forms: Maximal Emulation Stability (MES), Maximal Duplication Stability (MDS), and Maximal Clique Stability (MCS). MES (MDS) asserts that every finite subset of $Q[0,k]^k$ (k-tuples of rationals in [0,k]) has a stable maximal emulator (duplicator). MCS asserts that every order invariant graph on $Q[0,k]^k$ has a stable maximal clique. Stability of $S \subseteq Q[0,k]^k$ here means that for all $p_1,...,p_i < i$, $(p_1,...,p_i,i,...,k-1) \in S \leftrightarrow (p_1,...,p_i,i+1,...,k) \in S$. The strongest such stability notion is full stability, which can also be used here. Although graph theorists are expected to prefer MCS, MES and MDS bypass the middleman of graphs and reflect the intuitive ideas of emulation and duplication. MES and MDS are sufficiently natural, transparent, concrete, elementary, interesting, memorable, flexible, teachable, and rich in varied intricate examples and weaker and stronger forms, that they merit being classified in the category of Everybody's Mathematics. This is particularly evident for $Q[0,2]^2$ where (full) stability is particularly vivid. MES, MDS, MCS are readily seen, via Gödel's Completeness Theorem, to be implicitly Π^0_1. MES, MDS, MCS are provable from SRP^+ but not from ZFC (or SRP). SRP^+ is an extension of ZFC by certain large cardinal hypotheses that are well investigated and accepted by set theorists. In fact, MES, MDS, MCS are equivalent to the consistency of SRP.

1. INTRODUCTION

Maximal Emulation Stability (MES) was first presented publicly in [Fr18a] to a software engineering audience. It used the current notion of emulator, but a weaker form of stable than we use here (and on $Q[0,k]^k$). It is a particularly slow and careful presentation suitable for very wide mathematically oriented audiences. Here we also discuss the closely related MDS and MCS. $Q[p,q] = Q \cap [p,q]$.

MAXIMAL EMULATION STABILITY. MES. Every finite subset of $Q[0,k]^k$ has a stable maximal emulator.

MAXIMAL DUPLICATION STABILITY. MDS. Every finite subset of $Q[0,k]^k$ has a stable maximal duplicator.

MAXIMAL CLIQUE STABILITY. MCS. Every order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

Here $S \subseteq Q[0,k]^k$ is stable if and only if

$$\text{2) for all } p_1,\ldots,p_i < i, \quad (p_1,\ldots,p_i,i,\ldots,k-1) \in S \leftrightarrow (p_1,\ldots,p_i,i+1,\ldots,k) \in S)$$

EXAMPLE. $(1.38,2.99,2,3,4,5) \in S \leftrightarrow (1.38,2.99,2,4,5,6) \in S$.

For the other definitions used in MES, MDS, MCS, see section 2.1. Stability is an immediately transparent special case of the much stronger full stability discussed in section 4.

We claim that MES and MDS are sufficiently natural, transparent, concrete, elementary, interesting, memorable, flexible, teachable, and rich in varied intricate examples and weaker and stronger forms, that they merit being classified in the category of Everybody's Mathematics. A small sample of calculations of maximal emulators can be found in [Fr18a], and are repeated here in section 2.1. Plans are under way to confirm this assessment through documented engagement with mathematically gifted high school students and undergraduate mathematics majors starting in 2019.

The definitions supporting MES, MDS are trivial by normal mathematical standards as can be seen from section 2. They involve only the comparison of rational numbers by size, and sets of rational tuples, and no other aspects of rational numbers such as addition, subtraction, multiplication, or divisibility.

But even for a small handful of ordered pairs from Q[0,2], determining the emulators (duplicators), and maximal emulators (maximal duplicators) can range from trivial to highly intricate.

The proof of MES from large cardinals (SRP^+), and in WKL_0 + Con(SRP), already appears in [Fr17] with different (and comparatively clumsy) terminology. The same proof also establishes MDS, MCS. This is explained in section 2. The converse, that MES is not provable in ZFC, is in preparation with a target date of the end of calendar year 2018, expected to appear in [Fr19]. The proof follows the same general strategy that is used in Boolean Relation Theory, [Fr18b], Chapter 5. Namely, we start with MES, and follow a long process by which we progressively build a model of ZFC (even SRP). This proves the consistency of ZFC (even SRP) from MES. Therefore, by Gödel's Second Incompleteness Theorem, ZFC (even SRP) does not prove MES (assuming ZFC and SRP, are respectively consistent). It is clear that this applies to MDS, MCS as they imply MES (see section 2).

In section 3, we consider two generalizations. One is a strengthened form of maximality called step maximality. The other is a strengthened form of emulation and duplication. With emulation and duplication, we only consider pairs of k-tuples. In r-emulation and r-duplication, we consider r-tuples of k-tuples. The corresponding generalization for graphs is to use r-hypergraphs.

In section 4, we introduce the fundamental equivalence relation "critically equivalent" on Q^k. After a lot of thought, we have settled on a particularly transparent definition of critical equivalence. Full stability of $S \subseteq Q[0,k]^k$ is then defined as being completely invariant with respect to critical equivalence. The use of full stability in MES, MDS, MCS also results in statements equivalent to Con(SRP) over WKL_0.

Determination of which stability notions lead to independence from ZFC, and classifying the levels of associated logical strength, appears to be a fundamentally exciting but highly challenging task.

In section 5, we present the more general forms of MES, MDS, MCS based on sections 3,4.

In section 6, we present a finite towered form of MES which prove explicitly Π^0_2 and explicitly Π^0_1 forms with the same metamathematical properties.

In section 7, we give very brief accounts of further developments that are or will be documented elsewhere.

2. TANGIBLES

We present three main tangible statements independent of the usual ZFC axioms for mathematics: MES, MDS, MCS.

MES and MDS directly apply the notion of order equivalence in order to present conceptually very clear notions of emulation and duplication. The general idea is that S is an emulator of E if and only if any pattern of a certain kind present in S is already present in E. S is a duplicator of E if and only if E,S have the same patterns of a certain kind.

MCS may be preferable to MES, MDS for graph theorists. Instead of directly applying order equivalence to E,S, we start with a substantial mathematical object: an order invariant graph. The order equivalence is encapsulated in that object. Then we employ the standard notion of maximal clique, which plays the role of a maximal emulator (but not obviously of a maximal duplicator). The given order invariant graph can be viewed as a middleman - where MES, MDS dispenses with the middleman.

We give an elementary proof of MDS → MES and MCS → MES in sections 2.2, 2.3. There are obstacles to giving elementary proofs of the remaining implications.

In section 2.3, we present the weakened form MCS(res) of MCS, and show MES → MCS(res). MCS(res), read "MCS restricted" is the real point of departure for the crucial reversal in [Fr19]. We have the length five chain of implications in Theorem 2.3.4, establishing the equivalence of MES, MDS, MCS, MCS(res) over WKL_0.

In section 2.4, we consolidate matters with MES(ext), MDS(ext), MCS(ext), which sharpen MES, MDS, MCS analogously to the way "every clique extends to a maximal clique" sharpens "there is a maximal clique". We give an elementary proof that MES(ext) ↔ MCS(ext) → MDS(ext). They are also equivalent to Con(SRP) over WKL_0.

All of the implications proved in section 2 work for any stability notion. In fact for any attribute on the $S \subseteq Q[1,k]^k$ provided order invariant $S \subseteq Q[1,k]^k$ fall under the attribute. Furthermore, all of the implications preserve dimension k.

2.1. MES

DEFINITION 2.1.1. N,Z,Z^+,Q is the set of all nonnegative integers, integers, positive integers, rationals, respectively. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers unless otherwise indicated. We use p,q with or without subscripts for rationals unless otherwise indicated. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

Evidently, order equivalence implies that for all $1 \leq i,j \leq k$, $(x_i = x_j \leftrightarrow y_i = y_j) \wedge (x_i \leq x_j \leftrightarrow y_i \leq y_j)$.

EXAMPLE. (1,4,3),(2,8,3) are order equivalent, but (1,4,3),(2,8,2) are not.

DEFINITION 2.1.2. $Q[p,q] = Q \cap [p,q]$. S is an emulator of $E \subseteq Q[1,k]^k$ if and only if $S \subseteq Q[1,k]^k$ and every element of S^2 is order equivalent to an element of E^2. S is a maximal emulator of $E \subseteq Q[0,k]^k$ if and only if S is an emulator of $E \subseteq Q[0,k]^k$ which is not a proper subset of any emulator of $E \subseteq Q[0,k]^k$.

THEOREM 2.1.1. (RCA_0) Every subset of $Q[0,k]^k$ has a maximal emulator. Every subset of $Q[0,k]^k$ has a (elementary) recursive maximal emulator. Every subset of $Q[0,k]^k$ has the same emulators as some finite subset of $Q[0,k]^k$.

Proof: Start with a standard enumeration of $Q[0,k]^k$ and perform the obvious greedy algorithm, retaining a term in the enumeration if doing so maintains having an emulator, and skipping over that term otherwise. For the last claim, note that only the set of pairs of elements of the subset of $Q[0,k]^k$, up to order equivalence, is relevant for emulation. QED

Before moving on, we present six particularly simple examples of maximal emulator determinations on $Q[0,2]^2$. We use Q[0,2] here because the stability notions are particularly simple for $Q[0,2]^2$. Of course, the emulators and maximal emulators are purely order theoretic notions and the actual choice of closed interval here is irrelevant. It becomes relevant only when stability is considered - not just yet.

EXAMPLE 1. Emulators of $\emptyset \subseteq Q[0,2]^2$. These are $\emptyset$. Maximal emulator is $\emptyset$.

EXAMPLE 2. Emulators of $\{x\} \subseteq Q[0,2]^2$. These are $\emptyset$, $\{y\}$, y order equivalent to x. Maximal emulators are these $\{y\}$.

EXAMPLE 3. Emulators of $\{(0,0),(1,1)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$ where all $(p,q) \in S$ has $p = q$. Maximal emulator is $\{(p,p): 0 \leq p \leq 2\}$.

EXAMPLE 4. Emulators of $\{(0,1),(0,2)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$ where all $(p,q),(r,s) \in S$ have $p < q \wedge p = r < s$. Maximal emulators are the $\{(p,q): p < q\}$, where $0 \leq p < 2$ is fixed.

EXAMPLE 5. Emulators of $\{(0,1),(1,2)\} \subseteq Q[0,2]^2$ are $\emptyset$, $\{(p,q)\}$, $p < q$, and the $\{(r,s),(s,t)\} \subseteq Q[0,2]^2$, $r < s < t$. Maximal emulators are $\{(0,2)\}$ and these $\{(r,s),(s,t)\}$.

EXAMPLE 6. Emulators of $\{(0,1),(3/2,2)\} \subseteq Q[0,2]^2$ are the S $\subseteq Q[0,2]^2$ where all distinct $(p,q),(r,s) \in S$ have $p < q < r < s \vee r < s < p < q$. Maximal emulators are these S with no gaps.

DEFINITION 2.1.3. $S \subseteq Q[0,k]^k$ is stable if and only if for all $p_1,\ldots,p_i < i$, $(p_1,\ldots,p_i,i,\ldots,k-1) \in S \leftrightarrow (p_1,\ldots,p_i,i+1,\ldots,k) \in S$.

EXAMPLE. $(1.38,2.99,2,3,4,5) \in S \leftrightarrow (1.38,2.99,2,4,5,6) \in S$.

MAXIMAL EMULATION STABILITY. MES. Every finite subset of $Q[0,k]^k$ has a stable maximal emulator.

From Theorem 2.1.1, we know that in MES we can remove "finite" without change. However, we choose to retain "finite" in order to emphasize the concrete nature of MES.

This notion of stable is explicitly given. There is a very natural but more abstract form that immediately implies stable.

DEFINITION 2.1.4. $S \subseteq Q[0,k]^k$ is stable↑ if and only if no matter how you extend any given length $0 \leq i \leq k$ sequence of rationals by k-i strictly greater strictly increasing integers from $\{1,\ldots,k\}$, membership in S is equivalent.

The natural stopping point for stability in the sense intended here seems to be fully stable. This notion is presented and discussed in section 4.

These three stability notions are particularly simple in $Q[0,2]^2$.

$S \subseteq Q[0,2]^2$ is stable if and only if for all $p < 1$, $(p,1) \in S \leftrightarrow (p,2) \in S$.

$S \subseteq Q[0,2]^2$ is stable↑ if and only if for all $p < 1$, $(p,1) \in S \leftrightarrow (p,2) \in S$.

In section 4, we see that $S \subseteq Q[0,2]^2$ is fully stable if and only if

i. For $p < 1$, $(p,1) \in S \leftrightarrow (p,2) \in S$.
ii. For $p < 1$, $(1,p) \in S \leftrightarrow (2,p) \in S$.
iii. $(1,1) \in S \leftrightarrow (2,2) \in S$.

We can readily verify MES, with fully stable, in the six simple examples above:

EXAMPLE 1. $\emptyset$ is fully stable.

EXAMPLE 2. $\{y\}$ is fully stable if $\max(y) < 1$.

EXAMPLE 3. $\{(p,p): 0 \leq p \leq 2\}$ is fully stable.

EXAMPLE 4. $\{(1,q): 0 < q\}$ is fully stable.

EXAMPLE 5. $\{(r,s),(s,t)\} \subseteq Q[0,2]^2$, $r < s < t$, is fully stable if $r,s,t \notin \{1,2\}$.

EXAMPLE 6. Obviously there exist such S with no gaps where 1,2 do not appear as coordinates. These are fully stable.

We now derive MES (even with stable↑) from a version of MES referred to as MESU/2 in [Fr17]. MESU/2 is proved in [Fr17] in WKL_0 + Con(SRP). This shows that MES (even with stable↑) is provable in WKL0 + Con(SRP).

In section 4, we also derive our MES with full stability from MESU/2, thus establishing that our MES with full stability is provable in WKL_0 + Con(SRP).

We now quote from [Fr17], which we put in italics to avoid confusion with the present text. In [Fr17], we work exclusively in $Q[0,1]^k$, and the definition of emulator and maximal emulator is the same as it is here.

MAXIMAL EMULATION USE DEFINITION. MEU/DEF. (adapted from [Fr17]). $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable if and only if for all subsets of $Q[0,1]^k$, some maximal emulator contains its R image.

DEFINITION 3.4.1. [Fr17]. Let A ⊆ Q[0,1]. The relation $R_k(A)$ ⊆ $Q[0,1]^k \times Q[0,1]^k$ is given by $R_k(A)(x,y)$ if and only if
i. x,y are order equivalent.
ii. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A.

MAXIMAL EMULATION SMALL USE/2. MESU/2. [Fr17]. For finite A ⊆ Q(0,1], $R_k(A) \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable.

THEOREM 3.5.9. [Fr17]. MESU/2 for dimension k = 2 is provable in ZFC. In fact, Z and even Z_3 suffices.

THEOREM 3.5.13. (excerpt from [Fr17]). The following are provable in EFA.
ii. ZFC proves that for all k ≥ 1, if there is a max(k-1,0)-subtle cardinal then MESU/2 holds for dimension k+1.

We trivially adapt $R_k(A)$ to our setting $Q[0,k]^k$.

DEFINITION 2.1.6. Let A ⊆ Q[0,k]. The relation $R_k(A)^* \subseteq Q]0,k]^k \times Q[0,k]^k$ is given by $R_k(A)^*(x,y)$ if and only if
i. x,y are order equivalent.
ii. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A.

LEMMA 2.1.3. (RCA_0) MESU/2 of [Fr17] is equivalent to "MES holds in every $Q[0,k]^k$ using invariance for every $R_k(A)^*$, A ⊆ Q[0,k] finite". MESU/2 for fixed k is equivalent to "MES holds in $Q[0,k]^k$ using invariance for every $R_k(A)^*$, A ⊆ Q[0,k] finite".

Proof: This is simply a matter of quoting multiplication by k as an order preserving bijection from Q[0,1] onto Q[0,k]. QED

LEMMA 2.1.4. (EFA) Suppose $x,y \in Q[0,k]^k$ are the extension of a length 0 ≤ i ≤ k sequence of rationals by k-i strictly greater strictly increasing integers from {1,...,k}, as in the definition of stable↑. Then $R_k(A)^*(x,y)$, where A = {1,...,k}.

Proof: We have merely to verify i,ii in Definition 2.1.6. This is obvious by inspection. QED

THEOREM 2.1.5. (EFA) MES/2 of [Fr17] implies MES with stable↑. This holds for any fixed dimension k ≥ 2. It follows that

i. MES is equivalent to Con(SRP) over WKL_0.
ii. MES for dimension k = 2 is provable in Z_3.
iii. MES for dimension k = 3, is provable in ZFC + "there exists a subtle cardinal".
These results hold for MES with stable↑.

Proof: Immediate from Theorems 3.5.9, 3.5.13 of [Fr17], and Lemmas 2.1.3, 2.1.4. For MES → Con(SRP) in i, we are relying on this reversal, to appear in [Fr19]. QED

We write MES[k] for MES on $Q[0,k]^k$. By Theorem 2.1.3, MES[2] with stable↑ is provable in Z_3. We think it more likely than not that MES[2] with stable↑ is provable in Z_2, and plausibly in RCA_0, as we guess that a complete effective analysis of the maximal emulations of finite subsets of $Q[0,2]^2$ can be given.

The proof of MES[3] with stable↑ via Theorem 2.1.3 already uses ZFC + "there exists a subtle cardinal". We think it more likely than not that MES[3] with stable↑ cannot be proved in ZFC. See section 5 for discussion of some variants of MES[2] and MES[3] that we conjecture cannot be proved in Z_2 and ZFC, respectively.

2.2. MDS

DEFINITION 2.2.1. S is a duplicator of $E \subseteq Q[0,k]^k$ if and only if $S \subseteq Q[0,k]^k$, every element of S^2 is order equivalent to an element of E^2, and every element of E^2 is order equivalent to an element of S^2. S is a maximal duplicator of $E \subseteq Q[0,k]^k$ if and only if S is a duplicator of $E \subseteq Q[0,k]^k$ which is not a proper subset of any duplicator of $E \subseteq Q[0,k]^k$.

THEOREM 2.2.1. S is a duplicator of $E \subseteq Q[0,k]^k$ if and only if S is an emulator of $E \subseteq Q[0,k]^k$ and E is an emulator of $S \subseteq Q[0,k]^k$. If S is a maximal duplicator of $E \subseteq Q[0,k]^k$ then S is a maximal emulator of $E \subseteq Q[0,k]^k$.

Proof: The first claim is obvious. Let S be a maximal duplicator of $E \subseteq Q[0,k]^k$. Then S is an emulator of $E \subseteq Q[0,k]^k$. Let $S' \supseteq S$ be an emulator of $E \subseteq Q[0,k]^k$. Then S' is a duplicator of $E \subseteq Q[0,k]^k$, and so S' = S. QED

MAXIMAL DUPLICATOR STABILITY. MDS. Every finite subset of $Q[0,k]^k$ has a stable maximal duplicator.

THEOREM 2.2.2. RCA_0 proves MDS → MES.

Proof: Immediate from Theorem 2.2.1. QED

It is not clear how to give an elementary proof of MES → MDS. In the above Examples 1,2,3,4,6, all maximal emulators are maximal duplicators. In Example 5, this is not the case.

THEOREM 2.2.3. MDS is implicitly Π^0_1 via Gödel's Completeness Theorem. MDS is equivalent to Con(SRP) over WKL_0.

Proof: See Theorem 2.2.2 and remarks. The proof of the first claim and the derivation of MDS from Con(SRP) are straightfoward adaptations of the proofs of Theorem 2.2.2. By Theorems 2.2.2, the derivation of Con(SRP) from MDS is immediate from the derivation of Con(SRP) from MES (that derivation is to appear in [Fr19]). QED

2.3. MCS, MCS(res)

DEFINITION 2.3.1. A graph is an ordered pair G = (V,E), where V is the set of vertices and $E \subseteq V \times V = V^2$ is the set of edges. It is required that E is irreflexive and symmetric. We say that G is on V. x,y are adjacent if and only if x E y. y is a neighbor of x if and only if x,y are adjacent. A clique is an $S \subseteq V$ such that any two distinct elements of S are adjacent.

DEFINITION 2.3.2. $A \subseteq Q[0,k]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0,k]^k$, $x \in A \leftrightarrow y \in A$. An order invariant graph on $Q[0,k]^k$ is a graph on $Q[0,k]^k$ whose edge set is an order invariant subset of $Q[0,k]^{2k}$.

MAXIMAL CLIQUE STABILITY. MCS. Every order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

LEMMA 2.3.1. RCA_0 proves MCS → MES.

Proof: Assume MCS, and let $A \subseteq Q[0,k]^k$ be finite. We construct a stable maximal emulator of $A \subseteq Q[0,k]^k$. If A =

∅ then is a stable maximal emulator of A. If A = {x} then {x'} is a stable maximal emulator of A, where x,x' are order equivalent and $x' \in Q[0,1)^k$. Henceforth, we assume |A| ≥ 2.

Let G be the order invariant graph on $Q[0,k]^k$ whose edges are the $(x,y) \in Q[0,k]^k \times Q[0,k]^k$ with x ≠ y that are order equivalent to some (x',y') ∈ A × A. Hence G has at least one edge. Let X be the set of all $x \in Q[0,k]^k$ not order equivalent to any element of A. Then G and X are order invariant. Let G' be G together with all edges (x,y),(y,x), where x ≠ y and x ∈ X. Then G' is an order invariant graph on $Q[0,k]^k$.

We claim that if S is a clique in G' then S\X is a clique in G. Let S be a clique in G', and let x ≠ y be from S\X. Now (x,y) is an edge in G', and by the construction of G', (x,y) is an edge in G.

We claim that if S is a clique in G then S ∪ X is a clique in G'. Let S be a clique in G, and let x ≠ y be from S ∪ X. If x,y ∈ S then (x,y) is an edge in G and therefore in G'. Otherwise, (x,y) is an edge in G'.

We claim that if S is a maximal clique in G' then X ⊆≠ S and S\X is a maximal clique in G. To see this, let S be a maximal clique in G'. S ⊆ X is impossible since G has at least one edge. The insertion of elements of X cannot destroy cliquedom in G', so X ⊆≠ S. Now S\X is a clique in G since S is a clique in G'. Suppose S\X ∪ {x} is a clique in G. Then S\X ∪ {x} ∪ X is a clique in G'. Hence S ∪ {x} is a clique in G'. Therefore x ∈ S. Also from S\X ∪ {x} being a clique in G, and S\X ≠ ∅, we have x ∉ X. Hence x ∈ S\X.

Now let $S \subseteq Q[0,k]^k$ be a stable maximal clique in G'. We now show that S\X is a stable maximal emulator of $A \subseteq Q[0,k]^k$. Note that X ⊆≠ S and S\X is a maximal clique in G. We show the following.

1. S\X is an emulator of $A \subseteq Q[0,k]^k$. Let (x,y) ∈ S\X × S\X. If x ≠ y then (x,y) is an edge in G, and so (x,y) is order equivalent to an element of A^2. Suppose x = y. Since x ∉ X, x is order equivalent to an element of A. Hence (x,x) is

order equivalent to an element of A^2.

2. S\X is a maximal emulator of $A \subseteq Q[0,k]^k$. Let S\X ∪. {x} be an emulator of $A \subseteq Q[0,k]^k$. We claim that S\X ∪. {x} is a clique in G. Let y,z ∈ S\X ∪. {x}, y ≠ z. Then (y,z) is order equivalent to an element of A^2. Since y ≠ z, (y,z) is an edge in G. We have a contradiction since S\X is a maximal clique in G.

3. S\X is stable. This follows from S being stable and X being order invariant. This is because equivalent tuples in the definition of stable are also order equivalent.

QED

There are difficulties giving an elementary proof of MES → MCS. We now weaken MCS as follows.

MAXIMAL CLIQUE STABILITY(restricted). MCS(res). Let G be an order invariant graph on $Q[0,k]^k$, where max(v) < min(w) → v,w are adjacent. G has a stable maximal clique.

THEOREM 2.3.2. RCA_0 proves MES → MCS(res).

Proof: Assume MES. Let G be an order invariant graph on $Q[0,k]^k$, where max(v) < min(w) → v,w are adjacent. Let E be the edge set of G. If E = ∅ then obviously {(0,...,0)} is a stable maximal clique. So we assume E ≠ ∅.

We first construct a nonempty finite clique C in G such that every element of E is order equivalent to an element of C^2. Let $(x_1,y_1),(x_2,y_2),...,(x_n,y_n)$ enumerate a finite number of elements of E, without repetition, such that every element of E is order equivalent to some (x_i,y_i). Make copies of each $\{x_i,y_i\}$ in $Q[1,k]^k$ so that for all $1 \leq i < k$, the largest rational appearing in $\{x_i,y_i\}$ is less than the smallest rational appearing in $\{x_{i+1},y_{i+1}\}$. This results in the desired clique C in G.

Let S be a stable maximal emulator of C. We show that S is a stable maximal clique in G. Clearly S ≠ ∅.
1. S is a clique in G. Let x,y ∈ S, x ≠ y. Let (x,y) be order equivalent to $(z,w) \in C^2$, z ≠ w. Then (x,y) is an edge in G.

2. S is a maximal clique in G. Let $S \cup. \{x\}$ be a clique in G. We claim that $S \cup. \{x\}$ is an emulator of C, which is a contradiction. Let $y,z \in S \cup. \{x\}$.

2.1. Assume $y,z \in S$. Since S is an emulator of C, (y,z) is order equivalent to an element of C^2.

2.2. Assume $y = x \neq z$. Then $z \in S$ and since $S \cup \{x\}$ is a clique in G, (y,z) is an edge in G, and therefore order equivalent to an element of C^2.

2.3. Assume $y = z = x$. Since $S \neq \emptyset$, let $w \in S$. Hence (x,w) is an edge in G, and so (x,w) is order equivalent to an element of C^2. In particular, (x,x) is order equivalent to an element of C^2.

QED

It is not clear how to give an elementary proof that MCS(res) → MES.

THEOREM 2.3.4. MES, NDS, MCS, MCS(res) are implicitly Π^0_1 via Gödel's Completeness Theorem. (MDS ∨ MCS) → MES → MCS(res) → Con(SRP) → (MDS ∧ MCS) is provable in WKL_0, establishing equivalence of MES, MDS, MCS, MCS(res) over WKL_0. All of these implications except the last are provable in RCA_0. The last one is provable in WKL_0.

Proof: All of these implications except MCS(res) → Con(SRP) are established in this manuscript. QED

2.4. MES(ext), MDS(ext), MCS(ext).

MAXIMAL EMULATION STABILITY(extended). MES. Let $A \subseteq Q[0,1)^k$ be a finite emulator of finite $B \subseteq Q[0,k]^k$. Some stable maximal emulator of B contains A.

MAXIMAL DUPLICATION STABILITY(extended). MDS. Let $A \subseteq Q[0,1)^k$ be a finite duplicator of finite $B \subseteq Q[0,k]^k$. Some stable maximal duplicator of B contains A.

MAXIMAL CLIQUE STABILITY(extended). MCS. Let $A \subseteq Q[0,1)^k$ be a finite clique in the order invariant graph G on $Q[0,k]^k$. Some stable maximal clique in G contains A.

LEMMA 2.4.1. RCA_0 proves MES(ext) → MDS(ext).

Proof: Assume MES(ext). Let A ⊆ $Q[0,1)^k$ be a finite duplicator of finite B ⊆ $Q[0,k]^k$. Let S be a maximal emulator of B ⊆ $Q[0,k]^k$, where S ⊇ B. Then S is a duplicator of B ⊆ $[0,k]^k$. S must be a maximal duplicator of B ⊆ $[0,k]^k$ since every duplicator is an emulator. QED

LEMMA 2.4.2. Let A ⊆ $Q[0,1)^k$ be a finite clique in the order invariant graph G on $Q[0,k]^k$. There is a finite clique A ⊆ C ⊆ $Q[0,1)^k$ such that every clique C ∪ {x} is an emulator of C.

LEMMA 2.4.3. RCA_0 proves MES(ext) → MCS(ext).

Proof: Assume MDS(ext). Let A ⊆ $Q[0,1)^k$ be a finite emulator of finite B ⊆ $Q[0,k]^k$.

For i → iii, assume MES(ext). Let G be an order invariant graph on $Q[0,k]^k$ and A ⊆ $Q[0,1)^k$ be a finite clique in G. By Lemma 2.4.2, let A ⊆ C, where C is a finite clique such that every clique C ∪ {x} is an emulator of C.

Let S be a stable maximal emulator of C, where S ⊇ C. We claim that S is a maximal clique in G. We show the following.

1. S is a clique in G. Let x,y ∈ S, x ≠ y. Then (x,y) is order equivalent to some element of C2, and therefore order equivalent to some edge.

2. S is a maximal clique in G. Let S ∪. {x} be a clique in G. Since S ∪. {x} ⊇ C, S ∪. {x} is an emulator of C. This contradicts that S is a maximal emulator of C.

QED

LEMMA 2.4.4. RCA_0 proves MCS(ext) → MES(ext).

Proof: Assume MCS, and let B ⊆ $Q[0,1)^k$ be a finite emulator of finite A ⊆ $Q[0,k]^k$. (We have switched A,B in order to easily refer to the proof of Lemma 2.3.1). We construct a stable maximal emulator of B ⊆ $Q[1,k]^k$, with B ⊇ A. As is the proof of Lemma 2.3.1, we assume |A| ≥ 2.

Let G,G' be the order invariant graph on $Q[0,k]^k$ defined in the proof of Lemma 2.3.1, using the same X.

From the proof of Lemma 2.3.1, we have:

1) if S is a clique in G' then S\X is a clique in G.
2) if S is a clique in G then S ∪ X is a clique in G'.
3) if S is a maximal clique in G' then X ⊆≠ S and S\X is a maximal clique in G.
4) if S ⊆ $Q[0,k]^k$ be a stable maximal clique in G', then S\X is a stable maximal emulator of A ⊆ $Q[0,k]^k$.

We now bring in the finite emulator B ⊆ $Q[0,1)^k$ of A ⊆ $Q[0,k]^k$. Then B is a clique in G and hence in G'. By MCS(ext), let S ⊆ $Q[0,k]^k$ be a stable maximal clique in G', S ⊇ B. Then S\X is a stable maximal emulator of A ⊆ $Q[0,k]^k$, and S ⊇ B. But by the definition of X, B ∩ X = ∅. Hence S\X ⊇ B. QED

THEOREM 2.4.5. MES(ext), MDS(ext), MCS(ext) are implicitly Π^0_1 via Gödel's Completeness Theorem. RCA_0 proves MES(ext) ↔ MCS(ext) → MDS(ext). MES(ext), MDS(ext), MCS(ext) are provably equivalent to Con(SRP) over WKL_0.

Proof: The second claim is from Lemmas 2.4.1, 2.4.3, 2.4.4. For the last claim, the derivation from Con(SRP) is a straightforward adaptation of the proof of Theorem 2.1.2. The derivation of Con(SRP) follows from the derivation of Con(SRP) from MES (this will appear in [Fr19]).

3. MAXIMALITY NOTIONS

DEFINITION 3.1. For E ⊆ Q^k, $E|\leq p$ = {x ∈ E: max(x) ≤ p}.

DEFINITION 3.2. S is an r-emulator of E ⊆ $Q[0,k]^k$ if and only if S ⊆ $Q[0,k]^k$ and every element of S^r is order equivalent to an element of E^r. S is a maximal r-emulator of E ⊆ $Q[0,k]^k$ if and only if S is an r-emulator of E ⊆ $Q[0,k]^k$, and S is not a proper subset of any r-emulator of E ⊆ $Q[0,k]^k$.

DEFINITION 3.3. S is a step maximal r-emulator of E ⊆ $Q[0,k]^k$ if and only if S is an r-emulator of E ⊆ $Q[0,k]^k$ such that for all i ∈ Z^+, $S|\leq i$ is not a proper subset of any r-emulator of E ⊆ $Q[0,k]^k$ contained in $Q[0,i]^k$.

DEFINITION 3.4. S is an r-duplicator of $E \subseteq Q[0,k]^k$ if and only if $S \subseteq Q[0,k]^k$, every element of S^r is order equivalent to an element of E^r, and every element of E^r is order equivalent to an element of S^r. S is a maximal r-duplicator of $E \subseteq Q[0,k]^k$ if and only if S is an r-duplicator of $E \subseteq Q[0,k]^k$, and S is not a proper subset of any r-duplicator of $E \subseteq Q[0,k]^k$.

DEFINITION 3.5. S is a step maximal r-duplicator of $E \subseteq Q[0,k]^k$ if and only if S is an r-duplicator of $E \subseteq Q[0,k]^k$ such that for all $i \in Z^+$, $S|\leq i$ is not a proper subset of any r-duplicator of $E \subseteq Q[0,k]^k$ contained in $Q[0,i]^k$.

The proper formulation here for graphs requires hypergraphs.

DEFINITION 3.6. An r-hypergraph is an $H = (V,E)$, where V is the vertex set and $E \subseteq V^r$ is the edge set. It is required that V is symmetric and no two coordinates of any element of V are equal. A clique in H is a subset of V where every r-tuple of distinct elements lies in the edge set.

DEFINITION 3.7. Let $H = (Q[0,k]^k,E)$ be an r-hypergraph on $Q[0,k]^k$. S is a step maximal clique in H if and only if S is a clique in H such that for all $i \in Z^+$, $S|\leq i$ is not a proper subset of any clique in H contained in $Q[0,i]^k$.

In the above, 2-emulators and 2-duplicators are emulators and duplicators (even when modified by maximal or step maximal). Evidently 2-hypergraphs are graphs.

4. STABILITY NOTIONS

Stability notions for the $S \subseteq Q[0,k]^k$ are within the realm of complete invariance.

DEFINITION 4.1. Let $R \subseteq Q[0,k]^k \times Q[0,k]^k$. $S \subseteq Q[0,k]^k$ is completely invariant under R if and only if $x R y \rightarrow (x \in S \leftrightarrow y \in S)$.
However, we are only interested in using rather special relations R on $Q[0,k]^k$.

DEFINITION 4.2. $A \subseteq Q^k$ is order theoretic if and only if $A = \{x \in Q^k: \varphi\}$ where φ is a finite propositional combination of inequalities $v_i < v_j$, $v_i < p$, $p < v_i$, where $1 < i,j \leq k$, and $p \in Q$. These p's are the parameters in the definition. $A \subseteq Q^k$ is order theoretic over $P \subseteq Q$ if and only if A can be defined in this way with all parameters drawn from P. $A \subseteq Q[0,k]^k$ is order theoretic over P if and only if $A = \{x \in Q[0,k]^k: \varphi\}$, where φ is as above with all parameters drawn from P.

DEFINITION 4.3. The notions of stability for $S \subseteq Q[0,k]^k$ are given by the $R \subseteq Q[0,k]^k \times Q[0,k]^k$ which are order theoretic over $\{0,...,k\}$. They are the $S \subseteq Q[0,k]^k$ that are completely invariant under R.

Perhaps the weakest notion of stability relevant to MES is this:

1) for all $p < 1$, $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k)$

We had made claims to the effect that MES with 1) as the notion of stability is enough for independence from ZFC (in fact equivalence with Con(SRP) over WKL_0). However, 1) does not seem to strong enough for our reversals. We have fixed on the simplest and most natural stability notion for the reversal. We have been using the following notion. $S \subseteq Q[1,k]^k$ is stable if and only if

2) for all $p_1,...,p_i < i$,
$(p_1,...,p_i,i,...,k-1) \in S \leftrightarrow (p_1,...,p_i,i+1,...,k) \in S)$

Obviously 1) is just 2) with only $i = 1$. We have also been using $S \subseteq Q[1,k]^k$ is stable↑ if and only if

3) no matter how you extend any given length $0 \leq i \leq k$ sequence of rationals by k-i strictly greater strictly increasing integers from $\{1,...,k\}$, membership in S is equivalent.

Again, obviously 2) is a special case of 3).

We now want to address what we should mean by fully stable, as the natural stopping point for such notions relevant to MES. The idea is to develop an appropriate natural notion "critically equivalent" and take fully stable to be invariance under critical equivalence on $Q[0,k]^k$.

DEFINITION 4.4. Let $x,y \in Q[0,k]^k$. x,y are critically equivalent if and only if x,y are order equivalent and their respective strictly increasing enumerations lie in {1,...,k} at or past any position at which they differ.

DEFINITION 4.5. $S \subseteq Q[0,k]^k$ is fully stable if and only if for all critically equivalent $x,y \in Q[0,k]^k$, $x \in S \leftrightarrow y \in S$.

We now want to derive MES with fully stable from the MESU/2 of [Fr17], uniformly in dimension $k \geq 2$. As in section 2.1, we need only link critical equivalence with the $R_k(A)^*$.

THEOREM 4.1. (EFA) $x,y \in Q[0,k]^k$ are critically equivalent if and only if $R_k(A)^*(x,y)$, where A = {1,...,k}. I.e., critical equivalence on $Q[0,k]^k$ and $R_k(\{1,...,k\})^*$ are the same.

Proof: Let A = {1,...,k}. We use Definition 2.1.6 of $R_k(A)^*$. It is clear that $R_k(A)^*(x,y)$ if and only if x,y are order equivalent and for the respective strictly increasing enumerations x',y' of x,y, $R_k(A)^*(x',y')$. But evidently $R_k(A)^*(x',y')$ is the same as saying that x',y' lie in {1,...,k} at or past any position at which they differ. QED

THEOREM 4.2. (RCA_0) MESU/2 of [Fr17] implies MES with fully stable. For all $k \geq 2$, MESU/2 of [Fr17] in dimension k implies MES with fully stable in dimension k. It follows that

i. MES with fully stable is equivalent to Con(SRP) over WKL_0.

ii. MES with fully stable in dimension k = 2 is provable in Z_3.

iii. MES with fully stable in dimension k = 3 is provable in ZFC + "there exists a subtle cardinal".

Proof: Immediate from Theorems 3.5.9, 3.5.13 of [Fr17], and Theorem 4.1. For MES with fully stable → Con(SRP) in i, we are relying on this reversal, to appear in [Fr19]. In fact, MES → Con(SRP) is to be shown there. QED

1) has an interesting geometric interpretation. It asserts complete invariance under the unique isometry between the two left closed right open line segments of length 1,

[(0,1,...,k-1),(1,1,...,k-1)), [(0,2,...,k),(1,2,...,k)) in $Q[1,k]^k$. This suggests investigating which isometries between line segments in $Q[0,k]^k$ with integral endpoints can be used in MES with complete invariance, and what logical strength do we obtain. More generally, we can consider any pair of line segments in $Q[0,k]^k$ with integral endpoints (with the same of the four endpoint possibilities), using their one or possibly two linear bijections, and consider whether we can use them in MES with complete invariance.

There is also an equivalence between fully stable and complete invariance under certain natural partially defined functions.

DEFINITION 4.6. $\gamma[k;i]$ is the partial function from $Q[0,k]$ into $Q[0,k]$ given by p if $0 \leq p < i$; $p+1$ if $p \in \{i,...,k-1\}$. $\gamma^k[k;i]$ is $\gamma[k;i]$ acting coordinatewise, thereby mapping $Q[0,k]^k$ partially into $Q[1,k]^k$.

THEOREM 4.3. $S \subseteq Q[0,k]^k$ is fully stable if and only if $S \subseteq Q[0,k]^k$ is completely invariant under all $\gamma^k[k;i]$, $i = 1,...,k-1$.

As promised in section 2.1, we calculate full stability for $S \subseteq Q[0,2]^2$.

LEMMA 4.4. $x,y \in Q[0,2]^2$ are critically equivalent if and only if
i. $x = y$; or
ii. $x = (1,1)$, $y = (2,2)$, or vice versa; or
iii. $x = (p,1)$, $y = (p,2)$, $p < 1$, or vice versa; or
iv. $x = (1,p)$, $y = (2,p)$, $p < 1$, or vice versa.

Proof: It is clear by inspection that any x,y with any of i-iv are critically equivalent. Now Let $x,y \in Q[0,2]^2$ critically equivalent, and x',y' be as in Definition 4.4.

If $x'1 \neq y'1$ then all coordinates of x',y' are positive integers, and so $x'_1 = 1$, $y'_1 = 2$ (or vice versa), and so $y'_2 = 2$, $x'_2 = 1$ (or vice versa).

If $x'_1 = y'_1$ and $x'_2 \neq y'_2$, then $x'_2 = 1$, $y'_2 = 2$ (or vice versa), and $x'_1 = y'_1 < 1$.

So for x',y' we must have only i,ii,iii. Now since x,y is either x',y' or the result of interchanging first and second coordinates, we must have only i,ii,iii,iv. QED

THEOREM 4.5. $S \subseteq Q[0,2]^2$ is fully stable if and only if the following hold.
i. $(1,1) \in S \leftrightarrow (2,2) \in S$.
ii. For all $p < 1$, $(p,1) \in S \leftrightarrow (p,2) \in S$.
iii. For all $p < 1$, $(1,p) \in S \leftrightarrow (2,p) \in S$.

Proof: By definition, S is fully stable if and only if for all critically equivalent $x,y \in Q[0,2]^2$, $x \in S \leftrightarrow y \in S$. It is clear that this latter condition is equivalent to this i,ii,iii in light of Lemma 4.4. QED

5. GENERAL MES, MDS, MHS

GENERAL MAXIMAL EMULATION STABILITY. GMES. Let $A \subseteq Q[0,1)^k$ be a finite r-emulator of finite $B \subseteq Q[0,k]^k$. Some fully stable step maximal r-emulator of B contains A.
GENERAL MAXIMAL DUPLICATOR STABILITY. GMDS. Let $A \subseteq Q[0,1)^k$ be a finite r-duplicator of finite $B \subseteq Q[0,k]^k$. Some fully stable step maximal r-duplicator of B contains A.

GENERAL MAXIMAL HYPERGRAPH STABILITY. GMHS. Let $A \subseteq Q[0,1)^k$ be a finite clique in the order invariant r-hypergraph H on $Q[0,k]^k$. Some fully stable step maximal clique of H contains A.

THEOREM 5.1. GMES, GMDS, GMHS are implicitly Π^0_1 via Gödel's Completeness Theorem. RCA_0 proves GMES $\leftrightarrow$ GMDS $\leftrightarrow$ GMHS via elementary proofs. GMES for r = 1 or k = 1 is provable in RCA_0. GMES for k = 2 is provable in Z_3. GMES, GMDS, GMHS are equivalent to Con(SRP) over WKL_0.

Proof: Straightforward, and will be discussed in [Fr19]. QED

We use GMES[k,r] for GMES with k,r fixed. We use *'s in one or two places here to indicate that that letter is to vary.

It follows from [Fr17] that GMES[2,*] is provable in Z_3 and GMES[3,*] is provable in ZFC + "there exists a subtle cardinal". We conjecture that the former is not provable in Z_2, and the latter is not provable in ZFC.

6. FINITE TOWERS

DEFINITION 6.1. $[n] = \{0,1,...,n\}$. For $A \subseteq N$, $nA = \{nm: m \in A\}$.

We work in $[kn]^k$, where $n[k]$ is the natural arithmetic progression $\{0,n,2n,...,kn\} \subseteq [kn]$.

DEFINITION 6.2. S is an emulator of $E \subseteq [kn]^k$ if and only if $n[k] \subseteq D$ and $S \subseteq D^k$ and every element of S^2 is order isomorphic to an element of E^2. $S \subseteq [kn]^k$ is stable if and only if for all $0 \leq p < n$, $(p,n,2n,...,(k-1)n) \in S \leftrightarrow (p,2n,3n,...,kn) \in S)$. $(D,S) \subseteq (D',S')$ if and only if $D \subseteq D'$ and $S \subseteq S'$. However, we do not require that $S' \cap D^k = S$.

So far, everything is entirely expected, as the obvious analog of 1,...,k in Q[0,k] is n,2n,...,kn in [kn].

DEFINITION 6.3. A towered emulator of $E \subseteq [kn]^k$ is a nonempty finite length series $(D_1,S_1) \subseteq ... \subseteq (D_r,S_r)$ of emulators of $E \subseteq [kn]^k$. It is called internally maximal if and only if no $(D,S_i \cup. \{x\})$, $x \in D_{i-1}{}^k$, is an emulator of $E \subseteq [kn]^k$. It is called stable if and only if each S_i is stable.

FINITE MAXIMAL EMULATION STABILITY. FMES. Let $n > (8k)!$. Every subset of $[kn]^k$ has a stable internally maximal towered emulator.

It is natural to sharpen FMES with r-emulators and full stability.

FINITE MAXIMAL EMULATION STABILITY(full). FMES(full). Let $n > (8kr)!$. Every subset of $[kn]^k$ has a fully stable internally maximal towered emulator.

In this finite tower context, we can actually obtain a very strong form of internal maximality.

DEFINITION 6.4. A towered emulator $(D_1,S_1) \subseteq ... \subseteq (D_r,S_r)$ of $E \subseteq [kn]^k$ is called initially maximal if and only if for all $x \in D_{i-1}{}^k\backslash S_i$, $(S_i \cup \{x\})|_{\leq x}$ is not an emulator of $E \subseteq [kn]^k$.

FINITE INITIAL MAXIMAL EMULATION STABILITY. FIMES. Let n > (8kr)!. Every subset of $[kn]^k$ has a fully stable initially maximal towered emulator.

THEOREM 6.4. FMES, FMES(full), FIMES are equivalent to Con(SRP) over EFA.

These statements are obviously explicitly Π^0_1.

7. FURTHER INVESTIGATIONS

We are currently engaging in several further directions.

A. Nondeterministic algorithms for building stable maximal cliques in stages. These are being fine tuned for actual implementation in order to confirm (or possibly refute) the consistency of ZFC and fragments of SRP. Implementation is always at the level of modest finite lengths, producing finite cliques with stability properties. See [Fr18c].

B. Nondeterministic algorithms for building stable maximal cliques in stages. These are also being fine tuned for a different purpose. Namely for intellectual implementation, where the mathematical simplicity and naturalness of the procedures is paramount. This give a plethora of explicitly Π^0_1 sentences of various logical strengths ranging from weak fragments of arithmetic to various large cardinals. [Fr18d].

C. The various projects discussed in section 4.

8. FORMAL SYSTEMS USED

EFA Exponential function arithmetic. Based on 0, successor, addition, multiplication, exponentiation and bounded induction. Same as $I\Sigma_0(\exp)$, [HP93], p. 37, 405.

RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].

WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].

Z_2 Second order arithmetic as a two sorted first order theory. [Si99,09].

Z_3 Third order arithmetic as a three sorted first order theory. Extends Z_2 with a new sort for sets of subsets of ω.

ZF(C) Zermelo Frankel set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Ka94].

SRP ZFC + (∃λ)(λ has the k-SRP), as a scheme in k. [Fr01].

SRP^+ ZFC + (∀k)(∃λ)(λ has the k-SRP). [Fr01].

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