

EXPANDING MIND THEORY

by

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Abstract. We present a theory EM (expanding minds) of a Younger and Older Mind that contemplates objects and unary/binary/ternary relations on objects. This inserts a subjective element into a small fragment of the usual theory of types pioneered by Bertrand Russell. The resulting formal systems interpret the usual ZFC axioms for mathematics, and are interpretable in ZFC augmented with a certain large cardinal hypothesis (1-extendibility). The results can be viewed as a consistency proof for mathematics relative to this Expanding Mind Theory EM. The thesis is that EM embodies principles that have a plausibility that is independent of that of ZFC. EM can be naturally strengthened in various ways so that it interprets certain large cardinal extensions of ZFC. This development suggests a new kind of Philosophy of Mind with deep interactions with Philosophy and Foundations of Mathematics.

1. The Expanding Mind - Informal.
2. EM in ZF + 1-extendible.
3. ZFC in EM.
 - 3.1. EMX in EM.
 - 3.2. ZFC in EMX.
4. Extensions.

1. THE EXPANDING MIND - INFORMAL

The plan is to interpret the usual ZFC axioms for mathematics - and more - in formalizations that represent some arguably compelling extra mathematical intuition. It may not be more compelling than ZFC. But the thesis is that it represents an alternative kind of intuition with a prima facie legitimacy. We will be interested in minimalism - i.e., we strive for limiting the level of commitment needed to interpret the formalizations.

A particular aspect of what our minds grasp are the objects and the unary/binary/ternary relations on objects. We will simply refer to these as unary/binary/ternary relations, where it is understood that they apply onto to objects and not to relations. There is a strict separation between objects and relations as in a simple theory of types going back to Bertrand Russell.

Of course, there are many other aspects of what our minds grasp, and maybe these other aspects will also be subject to related developments. Here we are referring to the objects and unary/binary/ternary relations on objects grasped by our minds.

Now hopefully, as we grow older, our minds become more powerful, and we grasp more. We now take a snapshot of our minds at two points of time, call them the Younger Mind and the Older Mind.

The Older Mind will grasp more objects than the Younger Mind. The Older Mind will also grasp more unary/binary/ternary relations than the Younger Mind. Any unary/binary/ternary relation grasped by the Younger Mind will remain grasped by the Older Mind, but objects grasped by the Older Mind and not grasped by the Younger Mind might fall under the relation. In fact this happens for any unary relation R which holds of all objects, as R is grasped by the Younger Mind and R holds of all objects grasped by the Older Mind.

IMAGINATION asserts that any unary/binary/ternary relation that can be defined by the Older Mind, with references allowed to specific objects and unary/binary/ternary relations grasped by the Older Mind, can actually be grasped by the Older mind. Definitions are allowed to use all of the primitives being discussed here. Having such a definition is a particularly clear form of imagination. We need formulate IMAGINATION only for the Older Mind, in light of the STUBBORNNESS discussed below.

We are STUBBORN as we age. We don't change our opinions (we don't change our minds). STUBBORNNESS asserts the following. Any statement that refers to any particular objects and unary/binary/ternary relations grasped by the Younger Mind (parameters), and quantifying over all objects and unary/binary/ternary relations grasped by the Younger

Mind, remains true for the Older Mind, referring to the same particular objects and unary/binary/ternary relations grasped by the Younger Mind (parameters), and quantifying over all objects and unary/binary/ternary relations grasped by the Older Mind.

COMPLETENESS asserts that for every unary/binary/ternary relation grasped by the Older Mind, there is a unary/binary/ternary relation grasped by the Younger Mind, which is equivalent as far as objects grasped by the Younger Mind are concerned. In this sense, the Younger Mind and the Older Mind do not differ with respect to the objects that the Younger Mind grasps.

We can view COMPLETENESS as another form of STUBBORNNESS where we don't change our minds about the unary/binary/ternary relations seen by the Younger Mind, as far as objects seen by the Younger Mind are concerned.

There is an alternative view of COMPLETENESS based on an analogy between COMPLETENESS and the usual completeness of the real number system in mathematics. We formed the real number system through Dedekind or Cauchy Completeness relatively early in the modern history of mathematics, and relatively early in mathematics education (at the University level), and we later go on to build more and more sophisticated mathematical systems. This doesn't affect the completeness of the original real number system, which survives.

TRANSCENDENCE asserts that the Older Mind is transcendentally more powerful than the Younger Mind. This is formulated in the following way. There is a binary/ternary relation R seen by the Older Mind, such that every unary/binary relation seen by the Younger Mind is (extensionally equal to) a section of R (obtained by fixing the first argument to be an object grasped by the Older Mind). We shall see that TRANSCENDENCE implies that there is an object grasped by the Older Mind but not grasped by the Younger Mind. We can also think of the R in TRANSCENDANCE as a kind of naming relation, where each object x is viewed as "naming" the cross section of R at x (multiple names allowed). Thus there is a language theoretic interpretation of TRANSCENDENCE.

The above describes the informal basis for the system EM.

Obviously it is premature at this stage to go more deeply into various philosophical issues that are raised by these informal presentations. E.g., the challenge of further developing a theory of grasping and defining.

2. EM IN ZF + 1-EXTENDIBLE

We present the formal system EM corresponding to the presentation in section 1. EM is sufficient to interpret ZFC, as shown in section 3. Thus we have a consistency proof for mathematics, as formalized by the usual ZFC axioms, relative to that of EM.

The language of EM, $L(EM)$, has four sorts, sorts $0,1,2,3$. Sort i is for i -ary relations. 0 -ary relations are objects. We use variables $v[i,n]$, $0 \leq i \leq 3$, $n \geq 0$, over sort $i = 0,1,2,3$.

We use the $(i+1)$ -ary relation of application between one i -ary relation and i objects, $i = 1,2,3$. We use the unary predicate Y on each sort $0,1,2,3$. No confusion will result here in using the same letter Y for all four sorts.

The atomic formulas are $v(w_1, \dots, w_i)$, $Y(w)$, where $i = 1,2,3$, v is a variable of sort i , w_1, \dots, w_i are variables of sort 0 , and w is a variable of any of the four sorts. We do not use equality.

Formulas are defined as usual. We follow the convention that \wedge has higher precedence than $\rightarrow, \leftrightarrow$. The axioms of EM are the formulas in $L(EM)$ included below.

1. LOGIC. The usual axioms and rules for this four sorted predicate calculus without identity.
2. IMAGINATION. $(\exists v) (\forall x_1, \dots, x_i) (v(x_1, \dots, x_i) \leftrightarrow \varphi)$, v not free in φ .
3. COMPLETENESS. $(\exists w) (Y(w) \wedge (\forall x_1, \dots, x_i) (Y(x_1) \wedge \dots \wedge Y(x_i) \rightarrow (w(x_1, \dots, x_i) \leftrightarrow v(x_1, \dots, x_i))))$.
4. STUBBORNNESS. Let φ be a formula of EM with free variables x_1, \dots, x_k , $k \geq 0$, and no Y . $Y(x_1) \wedge \dots \wedge Y(x_k) \wedge \varphi \rightarrow \varphi^{(Y)}$. Here $\varphi^{(Y)}$ is the result of relativizing all quantifiers in φ to (relations falling under) Y .
5. TRANSCENDENCE. $(\exists v) (\forall w) (Y(w) \rightarrow (\exists x) (\forall y_1, \dots, y_i) (w(y_1, \dots, y_i) \leftrightarrow v(x, y_1, \dots, y_i)))$.

There is a simpler equivalent form of Stubbornness.

4'. STUBBORNNESS'. Let φ be a formula of EM with free variables x_1, \dots, x_k , $k \geq 0$, and no Y . $Y(x_1) \wedge \dots \wedge Y(x_k) \wedge \varphi \rightarrow (\exists y)(Y(y) \wedge \varphi)$.

THEOREM 2.1. EM is equivalent to 1,2,3,4',5. In fact, 4,4' are equivalent over LOGIC for $L(EM)$.

Proof: This corresponds to a well known criteria for elementary substructure in elementary model theory. QED

DEFINITION 2.1. α is 1-extendible if and only if there exists an elementary embedding from $(V(\alpha+1), \in)$ into some $(V(\beta+1), \in)$, where α is the first ordinal moved.

THEOREM 2.1. Every 1-extendible α is a measurable cardinal, as well as any β in Definition 2.1.

Proof: This is well known. QED

THEOREM 2.2. $ZF + (\exists \kappa)(\kappa \text{ is a 1-extendible cardinal})$ proves $\text{Con}(EM)$.

Proof: Let $j:V(\kappa+1) \rightarrow V(\lambda+1)$ be an elementary embedding with critical point κ . The (Older Mind's) objects are the elements of $V(\lambda)$. The Younger Mind's objects are the elements of $V(\kappa)$. The (Older Mind's) unary, binary, ternary relations are those that lie in $V(\lambda+1)$. The Younger Mind's unary/binary/ternary relations are the $j(U), j(B), j(T)$, where $U, B, T \in V(\kappa+1)$. Membership is interpreted as membership. IMAGINATION is through the use of the cumulative hierarchy. STUBBORNNESS comes from j being an elementary embedding. COMPLETENESS is by applying j and using that j is the identity on $V(\kappa)$. TRANSCENDENCE follows from $|V(\kappa+1)| \leq |V(\lambda)|$. QED

We will use the abbreviations IMAG, COMP, STUB, TRANS for Imagination, Completeness, Stubbornness, Transcendence. It is often more convenient to use Stubbornness' than Stubbornness.

3. ZFC IN EM

In section 3.1, we expand EM to include equality and extensionality. We interpret this expansion EMX in EM. In section 3.2, we interpret ZFC in EMX.

3.1. EMX IN EM

The language of EMX, $L(\text{EMX})$ is the same as $L(\text{EM})$ except that we have $=$ on each of the four sorts.

The atomic formulas are $x = y$, $v(w_1, \dots, w_i)$, $Y(w)$, where $i = 1, 2, 3$, v is a variable of sort i , and x, y, w are variables of any of the four sorts, with x, y of the same sort.

The axioms of EMX are the formulas of $L(\text{EMX})$ below.

1. LOGIC. The usual axioms and rules for this four sorted predicate calculus with identity in each sort.
2. EXTENSIONALITY. $v = w \leftrightarrow (\forall x_1, \dots, x_i) (v(x_1, \dots, x_i) \leftrightarrow w(x_1, \dots, x_i))$.
3. IMAGINATION. Same as in EM but with $L(\text{EMX})$.
4. COMPLETENESS. Same as in EM.
5. STUBBORNNESS. Same as in EM but with $L(\text{EMX})$.
6. TRANSCENDENCE. Same as in EM.

There is a simplified alternative to Stubbornness.

5'. STUBBORNNESS'. Same as in section 2 but for $L(\text{EMX})$.

THEOREM 3.1.1. EMX is equivalent to 1-4, 5', 6. In fact, 5, 5' are equivalent over LOGIC for $L(\text{EMX})$. $\text{ZF} + (\exists \kappa) (\kappa \text{ is a 1-extendible cardinal})$ proves $\text{Con}(\text{EMX})$.

Proof: As for Theorems 2.1 and 2.2. QED

We use the following abbreviations. EXT, IMAG, COMP, STUB, TRANS. It is often more convenient to use Stubbornness' than Stubbornness.

We now interpret EMX in EM. The relations of EM and EMX are the same. $v(x_1, \dots, x_i)$ in EM is interpreted as $v(x_1, \dots, x_i)$.

For sort 0, $x = y$ in EM is interpreted as $(\forall v) (v(x) \leftrightarrow v(y))$. For sorts $i = 1, 2, 3$, $v = w$ in EM is defined as $(\forall x_1, \dots, x_i) (v(x_1, \dots, x_i) \leftrightarrow w(x_1, \dots, x_i))$.

For sort 0, $Y(x)$ in EM is interpreted as $Y(x)$. For sorts i

$= 1, 2, 3$, $Y(v)$ in EM is interpreted as
 $(\exists w) ((\forall x_1, \dots, x_i) (w(x_1, \dots, x_i) \leftrightarrow v(x_1, \dots, x_i) \wedge Y(w)))$. I.e.,
 v is extensionally equal to some w falling under Y .

The interpretations of the equality axioms in EMX are provable in EM because the syntax is such that sorts 1, 2, 3 apply only to sort 0 and we have IMAG in EM. First note that the interpretations of the three equality relations are equivalence relations, provably in EM. Here is a convenient presentation of the equality axioms.

1. $x = x$, $x = y \rightarrow y = x$, $x = y \wedge y = z \rightarrow x = z$, x, y, z of the same sort 0, 1, 2, 3.
2. $x_j = x \wedge v(x_1, \dots, x_i) \rightarrow v(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_i)$.
3. $v = w \wedge Y(v) \rightarrow Y(w)$.
4. $v = w \wedge v(x_1, \dots, x_i) \rightarrow w(x_1, \dots, x_i)$.

The interpretations of 1 are obviously provable in EM.

For the interpretations of 2, note that x_j, x are of sort 0. Assume $(x_j = x)^* = (\forall w) (w(x_j) \leftrightarrow w(x))$. By IMAG, apply this for w where $(\forall y) (w(y) \leftrightarrow v(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_i))$.

For the interpretations of 3, first assume v, w are of sort 0. Assume $(v = w)^* = (\forall u) (u(v) \leftrightarrow u(w))$. By IMAG, apply this for u where $(\forall y) (u(y) \leftrightarrow Y(y))$. Now assume v, w are of sort 1, 2, 3. The interpretation reads v, w are extensionally equal $\wedge (\exists v') (v', v$ are extensionally equal $\wedge Y(v')) \rightarrow (\exists w') (w', w$ are extensionally equal $\wedge Y(w'))$, which is obviously provable in Logic in EM.

The interpretations of Extensionality are obviously provable in Logic in EM.

The interpretations of Imagination read $(\exists v) (\forall x_1, \dots, x_i) (v(x_1, \dots, x_i) \leftrightarrow \varphi^*)$, which are instances of Imagination in EM.

The interpretations of Completeness read $(\exists w) (Y(w)^* \wedge (\forall x_1, \dots, x_i) (Y(x_1) \wedge \dots \wedge Y(x_i) \rightarrow (w(x_1, \dots, x_i) \leftrightarrow v(x_1, \dots, x_i))))$ which follows immediately from Completeness using Logic in EM.

The interpretations of 5, 6 are trivially provable in EM.

For the interpretation of 7, assume $(A = B)^* \wedge Y(A)^*$. Then A, B are extensionally equal and A is extensionally equal to some C falling under Y . Hence B is extensionally equal to some C (the same C) falling under Y . Handle 8 analogously.

Below we use φ^* for the interpretation of φ .

The interpretations of EXT are trivially provable in EM.

The interpretations of IMAG in EMX are instances of IMAG in EM.

The interpretations of COMP are $(\exists B)(Y(B)^* \wedge (\forall x)(Y(x) \rightarrow (A(x) \leftrightarrow B(x, Y))))$. $(\exists S)(Y(S)^* \wedge (\forall x, Y)(Y(x) \wedge Y(Y) \rightarrow (R(x, Y) \leftrightarrow S(x, Y))))$. These are provable in EM since $Y(B)^*$ and $Y(S)^*$ follow from $Y(B)$ and $Y(S)$, respectively.

The interpretations of TRANS are $(\exists v)(\forall w)(Y(w)^* \rightarrow (\exists x)(\forall y_1, \dots, y_i)(w(y_1, \dots, y_i) \leftrightarrow v(x, y_1, \dots, y_i)))$. By TRANS in EM, let v be such that $(\forall w)(Y(w) \rightarrow (\exists x)(\forall y_1, \dots, y_i)(w(y_1, \dots, y_i) \leftrightarrow v(x, y_1, \dots, y_i)))$. Let $Y(w)^*$. Let w', w be extensionally equal, $Y(w')$. Then $(\exists x)(\forall y_1, \dots, y_i)(w'(y_1, \dots, y_i) \leftrightarrow v(x, y_1, \dots, y_i))$. Hence $(\exists x)(\forall y_1, \dots, y_i)(w(y_1, \dots, y_i) \leftrightarrow v(x, y_1, \dots, y_i))$.

LEMMA 3.1.2. Each instance of the following is provable in Logic in EM. The truth value of φ^* remains the same if the parameters are replaced by extensionally equivalent parameters, assuming Y is not in φ .

Proof: By induction on φ . QED

By Theorem 3.1.1, we can interpret Stubbornness' in EMX instead of Stubbornness in EMX. The interpretations of Stubbornness' are $Y(x_1)^* \wedge \dots \wedge Y(x_k)^* \wedge \varphi^* \rightarrow (\exists y)(Y(y)^* \wedge \varphi^*)$, assuming Y is not in φ . We need the following lemma.

Let x_1', \dots, x_k' be extensionally equal to x_1, \dots, x_k , respectively, $Y(x_1'), \dots, Y(x_k')$, and φ^* . By Lemma 3.1.2, $\varphi^*[x_1, \dots, x_i/x_1', \dots, x_k']$, and so by Stubbornness' in EM, $(\exists y)(Y(y) \wedge \varphi^*[x_1, \dots, x_i/x_1', \dots, x_k'])$. By Lemma 3.1.2, $(\exists y)(Y(y) \wedge \varphi^*)$. Hence $(\exists y)(Y(y)^* \wedge \varphi^*)$.

LEMMA 3.1.3. EFX is interpretable in EF.

Proof: By the above. QED

3.2. ZFC IN EMX

DEFINITION 3.2.1. An object is a relation of sort 0. A set is a relation of sort 1 (unary). A relation is a relation of sort 2 (binary). Sort 3 (ternary) will seldom be used, and will be highlighted. For sets, we use \in and abstraction notation. We generally use lower case letters for objects, early upper case letters for sets, and later upper case letters for relations. $\text{fld}(R) = \{x: (\exists y) (R(x,y) \vee R(y,x))\}$. $A \subseteq B \Leftrightarrow (\forall x \in A) (x \in B)$. $R \subseteq S \Leftrightarrow (\forall x,y) (R(x,y) \rightarrow S(x,y))$. A pre well ordering (pwo) is a relation \leq which is reflexive, transitive, connected, and well founded in the sense that every nonempty subset of $\text{fld}(\leq)$ has a \leq least element. Let \leq be a pwo. $x < y \Leftrightarrow x \leq y \wedge \neg y \leq x$. $x \leq\!\!\!\leq y \Leftrightarrow x \leq y \wedge y \leq x$. $\leq x = \{y: y \leq x\}$. $<x = \{y: y < x\}$. $\leq|A = \leq \cap A^2$.

LEMMA 3.2.1. Some $x \notin Y$.

Proof: Suppose every object is in Y . Let A be a set. By Completeness let $Y(B)$, where A, B have the same elements from Y . Then A, B have the same elements, and so $A = B$, $Y(A)$. Hence every set lies in Y . By Transcendence, we have an enumeration of all sets by objects. This is impossible. QED

DEFINITION 3.2.2. Let \leq, \leq' be pwo's. A set A or relation R is $\leq\!\!\!\leq$ respecting if and only if $A \subseteq \text{fld}(\leq)$ or $\text{fld}(R) \subseteq \text{fld}(\leq)$ and for all $x \leq\!\!\!\leq z$, $y \leq\!\!\!\leq w$, we have $x \in A \Leftrightarrow z \in A$ and $R(x,y) \Leftrightarrow R(z,w)$. An isomorphism between \leq, \leq' is a relation $R \subseteq \text{fld}(\leq) \times \text{fld}(\leq')$ which is an isomorphism in the usual sense if we factor out by $\leq\!\!\!\leq$ in $\text{dom}(R)$ and $\leq'\!\!\!\geq$ in $\text{rng}(R)$. A comparison relation between \leq, \leq' is an isomorphism between \leq, \leq' or an isomorphism between \leq and some $\leq'|\!<'x'$ or an isomorphism between some $\leq|\!<x$ and \leq' .

LEMMA 3.2.2. Any two pwo \leq, \leq' have a unique comparison relation. The x or x' (if used) is unique.

Proof: Standard using IMAG and EXT. QED

DEFINITION 3.2.3. Let \leq, \leq' be pwo's. \leq is the same length as \leq' if and only if they are isomorphic. \leq is shorter than \leq' if \leq is isomorphic to some $\leq'|\langle x$. \leq is longer than \leq' if and only if some $\leq|\langle x$ is isomorphic to \leq' . A limit point in a pwo is a point with a predecessor but no immediate predecessor. A finitary pwo is a pwo which is either empty or (has a greatest element and no limit point). A limit pwo is a nonempty pwo with no greatest element.

DEFINITION 3.2.4. A tight pwo is a pwo $\leq \in Y$ where $(\forall x \in \text{fld}(\leq)) (\exists y \leq x) (Y(y))$. A is \leq respecting if and only if $A \subseteq \text{fld}(\leq) \wedge (\forall x, y) (x \leq y \rightarrow (x \in A \leftrightarrow y \in A))$. Binary R is \leq respecting if and only if R is on $\text{fld}(\leq) \wedge (\forall x, y, z, w) (x \leq z \wedge y \leq w \rightarrow (R(x, y) \leftrightarrow R(z, w)))$.

LEMMA 3.2.3. Let \leq be a tight pwo. Every \leq respecting set and binary relation lies in Y .

Proof: Let \leq be a tight pwo. Let $A \subseteq \text{fld}(\leq)$ be \leq respecting. By COMP, let $B \in Y$, where A, B have the same intersection with Y . $B \subseteq \text{fld}(\leq)$, for if otherwise, then by STUB, B would have an element outside $\text{fld}(\leq)$ lying in Y , and this element outside $\text{fld}(\leq)$ must also be in A . Also B is \leq respecting, for if otherwise, then by STUB, there would be a counterexample to B being \leq respecting lying in Y , which is impossible since A is \leq respecting. Now A, B must meet the same equivalence classes under \leq because \leq is tight. Since A, B are both \leq respecting, $A = B$ and $Y(A)$. The binary relation case is handled in the same way. QED

LEMMA 3.2.4. Let \leq be a tight pwo. There is a tight pwo longer than \leq . Every $\leq|\leq x$ is tight.

Proof: Let \leq be a tight pwo. If $\text{fld}(\leq) \subseteq Y$ then by Lemma 3.2.1, let $x \notin Y$, and put x on top of \leq . If $y \in \text{fld}(\leq) \setminus Y$, then remove y and put it on top of \leq .

So we have shown that \leq has a single point extension. By STUB, \leq has a single point extension lying in Y , and this top point must lie in Y . Therefore \leq has a single point extension which is tight.

For the second claim, it suffices to show that $\leq|\leq x$ is tight

where $x \in \text{fld}(\leq) \cap Y$. By STUB, $\leq|_{\leq x} \in Y$, and therefore $\leq|_{\leq x}$ is tight. QED

LEMMA 3.2.5. There is a limit pwo longer than all tight pwo's. For every tight pwo there is a longer tight limit pwo.

Proof: We use a ternary R from TRANS. Define $x \leq y$ if and only if R_x, R_y are tight pwo's and the length of x is at most the length of y . By IMAG + EXT, \leq is a pwo. Because of the tightness and Lemma 3.2.4, \leq is at least as long as all tight pwo's. By Lemma 3.2.4, \leq is longer than all tight pwo's. Clearly there is an initial segment \leq' , possibly \leq itself, which is a limit pwo longer than all tight pwo's.

For the second claim, let \leq be a tight pwo. By the first claim, there is a longer limit pwo. By STUB, there is a longer limit pwo \leq' in Y . Let \leq and $\leq'|_{<x}$ have the same length. By STUBB, $x \in Y$.

case 1. There is a limit point in \leq' above x . By STUB let y be the least limit point above x . Then $y \in Y$. By STUB, every point in the interval $[x, y)$ lying in Y has an immediate successor in $[x, y)$ lying in Y . Hence $\leq'|_{<y}$ is a tight limit pwo longer than \leq .

case 2. There is no limit point in \leq' above x . Argue as in case 1 that \leq' is a tight limit pwo longer than \leq .

QED

We now leverage off of well known details concerning the constructible hierarchy which are well supported by IMAG + EXT. Leveraging in this way reduces the technicalities that we need to discuss.

In the usual set theoretic context, let λ be a limit ordinal. We can construct a binary relation R on λ such that (λ, R) is isomorphic to $(L(\lambda), \in)$. Furthermore, R can be given by a uniform second order definition over (λ, \in) without parameters.

Now let \leq be a limit pwo. We can analogously construct a $\leq\geq$ respecting binary relation R such that $(\text{fld}(\leq), \leq\geq, R)$ is

isomorphic to $(L(\lambda), =, \in)$ when we factor out by \simeq . Here the order type of $\leq \text{ mod } \simeq$ is λ . Furthermore, R can be given by a uniform second order definition over $(\text{fld}(\leq), \leq)$ without parameters.

DEFINITION 3.2.5. Let \leq be a limit pwo. $L[\leq]$ is the structure $(\text{fld}(\leq), \simeq, R)$, where R is constructed in the previous paragraph. The language of set theory, LST, is based on $\in, =$. $L[\leq]$ provides an interpretations of LST with \in interpreted as R and $=$ interpreted as \simeq .

LEMMA 3.2.6. Each $L[\leq]$ satisfies Equality, Extensionality, Pairing, Union, Bounded Separation, Foundation. If \leq has a limit point then $L[\leq]$ also satisfies Infinity.

Proof: Left to the reader. QED

LEMMA 3.2.7. Let \leq, \leq' be limit pwo's. There is a unique comparison relation between $L[\leq]$ and $L[\leq']$. If \leq, \leq' are of the same length then the comparison relation between $L[\leq]$ and $L[\leq']$ is the same as those between \leq and \leq' , and provides an isomorphism between $L[\leq]$ and $L[\leq']$. Suppose \leq is shorter than \leq' . Then the range of the comparison relation between $L[\leq]$ and $L[\leq']$ is the initial segment of \leq' that is of the same length as \leq , and provides an isomorphism between $L[\leq]$ and the corresponding initial segment of $L[\leq']$.

Proof: Left to the reader. QED

At this point we break the development into two alternatives. We argue differently under these two alternatives.

- A1. Every pwo in Y is tight.
- A2. Not every pwo in Y is tight.

Until further notice, we assume A1.

DEFINITION 3.2.6. We define the virtual structure (D, \equiv^*, \in^*) as follows. D consists of the $(L[\leq], x)$, where \leq is a limit pwo and $x \in \text{fld}(\leq)$. Let $(L[\leq], x), (L[\leq'], x') \in D$. These are related by \equiv^* if and only if the comparison relation between $L[\leq]$ and $L[\leq']$ relates x and x' . They are related by R if

and only if the comparison relation between $L[\leq]$ and $L[\leq']$ relates x to an R' predecessor of x' , where R' is as in $L[\leq']$.

LEMMA 3.2.8. The virtual structure (D, \equiv^*, \in^*) satisfies Equality, Extensionality, Pairing, Union, Bounded Separation, Foundation, and Infinity.

Proof: Left to the reader. These are exactly the axioms of LST cited in Lemma 3.2.6, together with Infinity. For Infinity, use Lemma 3.2.5. QED

DEFINITION 3.2.7. $(D, \equiv^*, \in^*)^Y$ is the virtual structure (D, \equiv^*, \in^*) from the perspective of Y . I.e., Definition 3.2.6 is relativized to Y .

LEMMA 3.2.9. $(D, \equiv^*, \in^*)^Y$ is a proper initial elementary substructure of (D, \equiv^*, \in^*) in the following strong sense.

- i. $(D, \equiv^*, \in^*)^Y$ is a virtual substructure of (D, \equiv^*, \in^*) , with virtual domain $D^Y \subseteq D$.
- ii. Every \in^* predecessor of an (element of D that is \equiv^* to an element of D^Y) is \equiv^* to an element of D^Y .
- iii. There is a point in (D, \equiv^*, \in^*) whose \in^* predecessors are exactly the elements of D which are \equiv^* to an element of D^Y .
- iv. Every formula in LST with parameters from D^Y holds in (D, \equiv^*, \in^*) if and only if it holds in $(D, \equiv^*, \in^*)^Y$.

Proof: Left to the reader. ii,iii depend very much on A1, the assumption that every pwo in Y is tight. ii,iii use Lemmas 3.2.4, 3.2.5. iv is by STUB. QED

LEMMA 3.2.10. (D, \equiv^*, \in^*) satisfies ZFC without the power set axiom.

Proof: By Lemma 3.2.8, we need only verify Separation, Replacement, and Choice. We first verify Collection in $(D, \equiv^*, \in^*)^Y$ as follows. Assume $(\forall y \in x) (\exists z) (\varphi)$ hold in $(D, \equiv^*, \in^*)^Y$ with x and parameters from D^Y . By Lemma 3.2.16, $(\forall y \in x) (\exists z) (\varphi)$ holds in (D, \equiv^*, \in^*) . By Lemma 3.2.9, these z can be taken to be \equiv^* to elements of D^Y . Hence we can use the point w in Lemma 3.2.9, iii to obtain $(\forall y \in x) (\exists z \in w) (\varphi)$ in (D, \equiv^*, \in^*) . Hence $(\exists w) (\forall y \in x) (\exists z \in w) (\varphi)$ holds in (D, \equiv^*, \in^*) . Hence $(\exists w) (\forall y \in x) (\exists z \in w) (\varphi)$ holds in

$(D, \equiv^*, \in^*)^Y$.

We can now verify Separation in (D, \equiv^*, \in^*) . We want $(\exists x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$ in (D, \equiv^*, \in^*) with z and parameters from D . Apply Collection with Infinity to obtain an element of D which forms an elementary submodel of the universe (D, \equiv^*, \in^*) with respect to subformulas of φ in the usual familiar way from set theory. Of course, Collection will generally overshoot and get an element of D that contains all of the relevant witnesses. But we can arrange things so that we can cut it back to a limit level in the coded constructible hierarchy that is exact. We then apply Bounded Separation (or argue directly) to obtain the x in D such that $(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$ holds in (D, \equiv^*, \in^*) .

From Collection and Separation we immediately obtain Replacement.

Choice is verified in (D, \equiv^*, \in^*) the same way it is normally verified in set theory in the constructible hierarchy. QED

LEMMA 3.2.11. (D, \equiv^*, \in^*) satisfies the power set axiom.

Proof: It suffices to show that Power Set holds in $(D, \equiv^*, \in^*)^Y$. Let (\leq, x) be in D^Y . So \leq is a tight pwo and $x \in \text{fld}(\leq)$. It suffices to show that according to (D, \equiv^*, \in^*) , the Power Set of (\leq, x) exists.

(\leq', y) is a subset of (\leq, x) according to (D, \equiv^*, \in^*) if and only if every \in^* predecessor of (\leq', y) is an \in^* predecessor of (\leq, x) . This is equivalent to: every \in^* predecessor (\leq', y^*) of (\leq', y) is \equiv^* to an \in^* predecessor (\leq, x^*) of (\leq, x) . Because \leq is critical, this is equivalent to: every \in^* predecessor (\leq', y^*) of (\leq', y) is \equiv^* to an \in^* predecessor (\leq, x^*) of (\leq, x) with $x^* \in Y$.

These (\leq', y) above are determined, up to \equiv^* , by the $\leq \geq$ respecting set $\gamma(\leq', y) = \{x^* \in \text{fld}(\leq) : (\leq, x^*) \in^* (\leq', y)\}$. We claim that for any $A \subseteq \text{fld}(\leq)$, if A is some $\gamma(\leq', y)$ then A is some $\gamma(\leq', y)$, \leq' a pwo in Y (and hence critical).

To see this, let $A = \gamma(\leq', y)$, \leq' a pwo, $y \in \text{fld}(\leq')$. Then A is $\leq \geq$ respecting. By Lemma 3.2.3, $A \in Y$. By Lemma 3.2.9, iii, we have, according to (D, \equiv^*, \in^*) , a set whose \in^*

predecessors include all subsets of (\leq, x) . We make this exact using Bounded Separation. QED

LEMMA 3.2.12. ZFC is interpretable in EMX + Alternative 1 (every pwo in Y is critical).

Proof: By Lemmas 3.2.10 and 2.1.11. QED

We now assume Alternative 2.

LEMMA 3.2.13. There is a limit pwo $\leq \in Y$ with a limit point Θ such that

- i. No $x \leq \Theta$ lies in Y .
- ii. $(\forall x < \Theta) (\exists y \leq x) (y \in Y)$.

Proof: By A2, Let $\leq \in Y$ be a pwo which is not tight. Let Θ be \leq least such that no $x \leq \Theta$ lies in Y . If Θ is a \leq least element then by STUB, there is an $x \leq \Theta$ lying in Y . Therefore Θ is not a \leq least element. If Θ is not a limit point in \leq then let x be an immediate predecessor of Θ . Then $x \in Y$, and so by STUB, x has an immediate successor lying in Y . This is a contradiction. Hence Θ is a limit point with i,ii. We are done except that \leq may not be a limit pwo. Assume \leq is not a limit pwo, and let x be the greatest limit point in \leq . By STUB, we can assume that $x \in Y$. Hence $x > \Theta$. Instead of using \leq , we can use $\leq | < x$, which is a limit pwo in Y with all of the desired properties. QED

LEMMA 3.2.14. \leq has a limit point $< \Theta$. $L[\leq]$, $L[\leq | < \Theta]$ satisfies extensionality, pairing, union, bounded separation, foundation, infinity.

Proof: By STUB, there is a least limit point in \leq lying in Y . This least limit point in \leq must be $< \Theta$ by Lemma 3.2.13, i. This establishes infinity in $L[\leq]$, $L[\leq | < \Theta]$, and the other axioms are immediate. QED

LEMMA 3.2.15. Let S be a \leq respecting binary relation on $\text{fld}(\leq)$ and $x < \Theta$, where $(\forall y < x) (\exists z < \Theta) (S(y, z))$. Then $(\exists w < \Theta) (\forall y < x) (\exists z < w) (S(y, z))$.

Proof: Let S be as given. By COMP, let $T \in Y$ where S, T agree on Y . We claim that T is \leq respecting. If not, then

by STUB, we can find a counterexample to \leq respecting from Y which would constitute a counterexample to \leq respecting for S .

Now we claim that $(\exists w \in \text{fld}(\leq)) (\forall y < x) (\exists z < w) (T(y, z))$, with witness $w = \Theta$. To begin with, we have $(\exists w \in \text{fld}(\leq)) (\forall y < x \text{ with } Y) (\exists z < w \text{ with } Y) (S(y, z))$, with witness $w = \Theta$, since S is \leq respecting. Since S, T agree on Y , we have $(\exists w \in \text{fld}(\leq)) (\forall y < x \text{ with } Y) (\exists z < w \text{ with } Y) (T(y, z))$, with witness $w = \Theta$. Since T is \leq respecting, we have $(\exists w \in \text{fld}(\leq)) (\forall y < x) (\exists z < w) (T(y, z))$, with witness $w = \Theta$. By STUB, there is a least w with $(\exists w \in \text{fld}(\leq)) (\forall y < x) (\exists z < w) (T(y, z))$, with $Y(w) \wedge w < \Theta$. Fix $w < \Theta$, $Y(w)$, $(\forall y < x) (\exists z < w) (T(y, z))$. Since T is \leq respecting, $(\forall y < x \text{ with } Y) (\exists z < w \text{ with } Y) (T(y, z))$, $(\forall y < x \text{ with } Y) (\exists z < w \text{ with } Y) (S(y, z))$. Since S is \leq respecting, $(\forall y < x) (\exists z < w) (S(y, z))$. QED

LEMMA 3.2.16. $L[\leq | < \Theta]$ satisfies ZFC without power set.

Proof: By Lemma 3.2.15. Separation is proved in the usual way using Lemma 3.2.15 by an elementary substructure type argument, which also establishes Replacement. Choice is verified in the usual way in the constructible hierarchy. QED

LEMMA 3.2.17. $L[\leq | < \Theta]$ satisfies power set.

Proof: Let $x < \Theta$ where $L[\leq | < \Theta]$ satisfies that there are internal subsets of x whose levels are arbitrarily high up $< \Theta$. By Lemma 3.2.13, we can assume that $Y(x)$. From the point of view of $L[\leq]$, look at the levels in $L[\leq]$ at which the various internal subsets of x appear. If these levels have a sup within \leq then by STUB, they have a sup u in Y , and clearly $u > \Theta$. In any case, there is an $A \subseteq \leq x$ in the sense of $L[\leq]$ which first appears in $L[\leq]$ at some level $w > \Theta$.

Fix A, w . According to this setup, A is \leq respecting. By COMP, let $B \in Y$, where A, B agree on Y . By STUB, $B \subseteq \leq x$ and also B is \leq respecting, and $A = B$. Therefore $A \in Y$, and by STUB, w is \leq to an element of Y . By standard L technology adapted to this framework, we obtain we obtain a relation from $\leq w$ into $\leq x$ which is \leq respecting, and is a one-one

function modulo \leq , and by STUB, lies in Y . From $w > \Theta$ and STUB, we now see that Θ is \leq an element of Y , which is a contradiction. QED

LEMMA 3.2.18. ZFC is interpretable in EMX + Alternative 2 ((not every pwo in Y is critical)).

Proof: By Lemmas 3.2.16 and 3.2.17 QED

THEOREM 3.2.19. ZFC is interpretable in EM.

Proof: By Lemmas 3.1.3, 3.2.12, and 3.2.18. QED

4. EXTENSIONS

There is a series of extensions of EM that interpret a range of large cardinal hypotheses and are more or less in the same spirit as EM. They are still based a Young and Older Mind. A natural high point for this is $ZFC + j:V(\kappa+1) \rightarrow V(\kappa+1)$.

To be continued...

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