# EXTREMELY LARGE CARDINALS IN THE RATIONALS

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In 1995 we gave a new simple principle of combinatorial set theory and showed that it implies the existence of a nontrivial elementary embedding from a rank into itself, and follows from the existence of a nontrivial elementary embedding from V into M, where M contains the rank at the first fixed point above the critical point. We then gave a “diamondization” of this principle, and proved its relative consistency by means of a standard forcing argument.

We have recently discovered how to pull this diamondization down into the rationals in a natural and simple way using the concept of first order definability. This results in a $\Pi$-0-1 sentence which implies the consistency of ZFC + the existence of a nontrivial elementary embedding of a rank into a rank, and which follows from the consistency of ZFC + the existence of a nontrivial elementary embedding from V into M, where M contains the rank at the first fixed point above the critical point. Here are the details.

First we state the extremely large cardinal hypotheses commonly called I1, I2, and I3:

I1. There is a nontrivial elementary embedding from some $V(\alpha+1)$ into $V(\alpha+1)$.

I2. There is a nontrivial elementary embedding from V into a transitive class M such that $V(\lambda) \subseteq M$, where $\lambda$ is the first fixed point after the critical point.

I3. There is a nontrivial elementary embedding from some $V(\alpha)$ into $V(\alpha)$.

See The Higher Infinite, Aki Kanamori, Perspectives in Mathematical Logic, Springer-Verlag, 1994, p. 325 for a discussion, where it is shown that I1 implies I2 implies I3, and in fact I1 implies the consistency of I2, and I2 implies the consistency of I3.

Now let $\kappa$ be a cardinal and M be a relational structure in a finite similarity type. For $\alpha < \kappa$ we write $M|\alpha$ for the restriction of M to domain $\alpha$. Now if M has interpretations of function symbols then $M|\alpha$ may be undefined because the functions may not map $\alpha$ into $\alpha$.

An elementary embedding from $M|\alpha$ into $M|\alpha$ is required to have domain $\alpha$.

The first new principle that we state is more model theoretic than combinatorial:

P1. There is a cardinal $\kappa$ such that the following holds. Let M be a relational structure on $\kappa$ in a finite similarity type. Then there exists a nontrivial elementary embedding from some $M|\alpha$ into $M|\alpha$.

It turns out that we only need to consider a single binary relation R, and a mapping from $R|\alpha$ into $R|\alpha$. Elementary is not needed, nor is embedding, nor any additional structure. Specifically, let $R \subseteq \kappa$ and $\alpha < \kappa$. Let $R|\alpha$ simply be the intersection of R with $\alpha$ x $\alpha$.

A mapping from $R|\alpha$ into $R|\alpha$ is a function $f:\alpha \to \alpha$ such that for all $x,y < \alpha$, R(x,y) implies R(f(x),f(y)). Notice the slight abuse of notation that requires the domain of f to be $\alpha$, whereas $\alpha$ may not be recoverable from $R|\alpha$.

A function is said to be nontrivial if and only if it is not the identity function on its domain.

P2. There is a cardinal k such that the following holds. For all $R \subseteq \kappa$ x $\kappa$, there exists a nontrivial mapping from some $R|\alpha$ into $R|\alpha$.

We now pause to state a result we obtained in 1995.

THEOREM 1. P1 and P2 are provably equivalent in ZFC, and are both provable in I2. P1 and P2 each prove I3. Moreover, the consistency of P1 and P2 follow from I2, and P1 and P2 each prove the consistency of I3.

(We are indebted to Donald Martin for suggesting the possible use of an absoluteness/tree argument in connection with the proofs from I2. Such arguments had already been used in connection with such axioms).

**In 1995 we also gave diamondizations of P1 and P2 and proved their relative consistency by forcing:**

**D1. There is a cardinal $\kappa$ and a partial function $j:\kappa \times \kappa \to \kappa$ such that the following holds. Let M be a relational structure on $\kappa$ in a finite similarity type. Then there exists a cardinal $\alpha < \kappa$ such that $j_\alpha$ is a nontrivial elementary embedding from $M|\alpha$ into $M|\alpha$.**

**D2. There is a cardinal $\kappa$ and a partial function $j:\kappa \times \kappa \to \kappa$ such that the following holds. For all $R \subseteq \kappa \times \kappa$ there exists a cardinal $\alpha < \kappa$ such that $j_\alpha$ is a nontrivial mapping from $R|\alpha$ into $R|\alpha$.**

**THEOREM 2. D1 and D2 are equivalent over ZFC. P1,P2,D1,D2 are equiconsistent over ZFC.**

**Now before we are ready to pull these statements down into the rationals, we still need a little more massaging.**

**We say that a mapping f from $R|\alpha$ into $R|\alpha$ is sharp if and only if the domain of f is exactly the set of coordinates used in $R|\alpha$.**

**Note that in P2, we cannot require that the nontrivial mapping be a nontrivial sharp mapping. This is because R be, e.g., the empty relation. However, we have the following:**

**P3. There is a cardinal $\kappa$ such that the following holds. For all unbounded $R \subseteq \kappa \times \kappa$, there exists a nontrivial sharp mapping from some $R|\alpha$ into $R|\alpha$.**

**Here unbounded means that the set of coordinates used is unbounded in $\kappa$.**

**We can also state the following:**

**D3. There is a cardinal $\kappa$ and a partial function $j:\kappa \times \kappa \to \kappa$ such that the following holds. For all unbounded $R \subseteq \kappa \times \kappa$ there exists a cardinal $\alpha < \kappa$ such that $j_\alpha$ is a nontrivial sharp mapping from $R|\alpha$ into $R|\alpha$.**

**We also make some further minor modifications. An R-limit is simply a limit ordinal that is a limit of ordinals used as coordinates of R.**

P4. There is a cardinal $\kappa$ such that the following holds. For all unbounded $R \subseteq \kappa \times \kappa$ there exists an R-limit $\alpha < \kappa$ and a nontrivial sharp mapping from $R|\alpha$ into $R|\alpha$.

D4. There is a cardinal $\kappa$ and a partial function $j:\kappa \times \kappa \to \kappa$ such that the following holds. For all unbounded $R \subseteq \kappa \times \kappa$ there exists an R-limit $\alpha < \kappa$ such that $j_\alpha$ is a nontrivial sharp mapping from $R|\alpha$ into $R|\alpha$.

We say that $R \subseteq \kappa \times \kappa$ is sharply unbounded if and only if for all $\alpha < \kappa$ there exists $\beta,\gamma > \alpha$ such that $R(\beta,\gamma)$. We say that $\alpha$ is a sharp R-limit if and only if for all $x < \alpha$ there exists $\beta,\gamma$ in the open interval $(x,\alpha)$ such that $R(\beta,\gamma)$.

P5. There is a cardinal $\kappa$ such that the following holds. For all sharply unbounded $R \subseteq \kappa \times \kappa$ there exists a sharp R-limit $\alpha < \kappa$ and a nontrivial sharp mapping from $R|\alpha$ into $R|\alpha$.

D5. There is a cardinal $\kappa$ and a partial function $j:\kappa \times \kappa \to \kappa$ such that the following holds. For all sharply unbounded $R \subseteq \kappa \times \kappa$ there exists a sharp R-limit $\alpha < \kappa$ such that $j_\alpha$ is a nontrivial sharp mapping from $R|\alpha$ into $R|\alpha$.

THEOREM 3. In ZFC, P1-P5 are equivalent; D1-D5 are equivalent. All ten statements are equiconsistent over ZFC.

We now pull D5 down into the rationals. All of the terminology used in D5 makes perfectly good sense in Q.

Q1. There is a partial function $j:Q \times Q \to Q$ such that the following holds. Let $R \subseteq Q \times Q$ be sharply unbounded and $(Q,<,j)$-definable. Then there exists a sharp R-limit x such that $j_x$ is a nontrivial sharp mapping from $R|x$ into $R|x$.

THEOREM 4. Q1 is independent of ZFC. In ACA, Q1 is provably equivalent to a $\Pi$-0-1 sentence. In ACA, Q1 implies the consistency of ZFC + I3. In ACA, the consistency of ZFC + I2 proves Q1.