MY FORTY YEARS ON HIS SHOULDERS

by

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We wish to thank Warren Goldfarb and Hilary Putnam for help with several historical points.

1. GENERAL REMARKS

Gödel's legacy is still very much in evidence. His legacy is overwhelmingly decisive, particularly in the arena of general mathematical and philosophical inquiry.

The extent of Gödel's impact in the more restricted domain of mathematical practice is more open to question. In fact, there is an in depth assessment of this impact in Macintyre 2009. But even in this comparatively specialized domain, Gödel's impact is seen to be substantial. As indicated here, particularly in section 12, we believe that the **potential** impact of Gödel's work on mathematical practice is also overwhelming. However, the full realization of this potential impact will have to wait for some new breakthroughs. We have every confidence that these breakthroughs will materialize.

Generally speaking, current mathematical practice has now become very far removed from general mathematical and philosophical inquiry, where Gödel's legacy is most decisively overwhelming. However, there are some signs that some of our most distinguished mathematicians recognize the need for some sort of reconciliation. Here is a quote from Atiyah M. 2008b:

"Mathematicians took the role of philosophers, but I want to bring the philosophers back in. I hope someday we will be able to explain mathematics in a philosophical way using philosophical methods".

We will not attempt to properly discuss the full impact of Gödel's work and all of the ongoing important research programs that it suggests. This would require a book length manuscript. Indeed, there are several books discussing the Gödel legacy from many points of view, including, for example, (Wang 1987, 1996), (Dawson 2005), and the historically comprehensive five volume set (Gödel 1986-2003).

In sections 2-7 we briefly discuss some research projects that are suggested by some of his most famous contributions.

In sections 8-11 we discuss some highlights of a main recurrent theme in our own research, which amounts to an expansion of the Gödel incompleteness phenomena in new critical directions.

The incompleteness phenomena lie at the heart of the Gödel legacy. Some careful formulations, informed by some post Gödelian developments, are presented in sections 3,4,5.

One particular issue that arises with regard to incompleteness has been a driving force for a considerable portion of my work over the last forty years. This has been the ongoing search for necessary uses of set theoretic methods in normal mathematics.

By way of background, Gödel’s first incompleteness theorem is an existence theorem not intended to provide a mathematically intelligible example of an unprovable sentence.

Gödel’s second incompleteness theorem does provide an entirely intelligible example of an unprovable sentence - specifically, the crucially important consistency statement. (Remarkably, Gödel demonstrates by a brief semiformal argument, that the sentence he constructs for his first incompleteness theorem is demonstrably implied by the consistency statement - hence the consistency statement is not provable. It was later established that the two are in fact demonstrably equivalent.)

Nevertheless, the consistency statement is obviously of a logical nature rather than of a mathematical nature. This is a distinction that is readily noticed by the general mathematical community, which naturally resists the notion that the incompleteness theorem will have practical consequences for their own research.

Genuinely mathematical examples of incompleteness from substantial set theoretic systems had to wait until the well known work on the axiom of choice and the continuum hypothesis by Kurt Gödel and Paul Cohen. See (Gödel 1940), (Cohen 1963-64).

Here, the statement being shown to be independent of ZFC - the continuum hypothesis - is of crucial importance for abstract set theory.

However, mathematicians generally find it easy to recognize an essential difference between overtly set theoretic statements like the continuum hypothesis (CH) and "normal" mathematical statements. Again, this is a particularly useful observation for the mathematicians.

Specifically, the reference to unrestricted uncountable sets (of real numbers) in CH readily distinguishes CH from "normal" mathematics, which relies, almost exclusively, on the "essentially countable", (e.g., the continuous or piecewise continuous).

A more subtle example of an overtly set theoretic statement that requires a second look to see its overtly set theoretic character, is Kaplansky's Conjecture concerning automatic continuity. In one of its more concrete special forms, it asserts that

*) every homomorphism from the Banach algebra c_0 of infinite sequences of reals converging to 0 (under the sup norm) to any separable Banach algebra, is continuous.

Now *) was refuted using the continuum hypothesis (due independently to H.G. Dales and J. Esterle), and later shown to be not refutable without the continuum hypothesis; i.e., not refutable in the usual ZFC axioms (due to R. Solovay). See (Dales 2001) for the refutation, and (Dales, Woodin 1987) for the consistency (non refutability) result.

It is, of course, much easier for mathematicians to recognize the overtly set theoretic character after they learn that there are set theoretic difficulties. By taking the negation,

**) there exists a discontinuous homomorphism from the Banach algebra c_0 of infinite sequences of reals converging to 0 (under the sup norm) to some separable Banach algebra.

It is clear that one is asking about the existence of an object that was well known, even at the time, to necessarily have rather pathological properties. This is the case even for discontinuous group homomorphisms from $\Re$ into $\Re$ (which can be shown to exist without the continuum hypothesis). For instance, it is well known that there are no discontinuous group homomorphisms from $\Re$ into $\Re$ that are Borel measurable.

At the outer limits, normal mathematics is conducted within complete separable metric spaces. (Of course, we grant that it is sometimes convenient to use fluff - as long as it doesn't cause any trouble). Functions and sets are normally Borel measurable within such so called Polish spaces. In fact, the sets and functions normally considered in mathematics are substantially nicer than Borel measurable, generally being continuous or at least piecewise continuous - if not outright countable or even finite.[1]

We now know that the incompleteness phenomena do penetrate the barrier into the relatively concrete world of Borel measurability - and even into the countable and the finite world - with independence results of a mathematical character.

In sections 8-11 we discuss my efforts concerning such concrete incompleteness, establishing the necessary use of abstract set theoretic methods in a number of contexts, some of which go well beyond the ZFC axioms.

Yet it must be said that our results to date are very limited in scope, and demand considerable improvement. We are only at the very beginnings of being able to assess the full impact of the Gödel incompleteness phenomena.

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[1] Apparently, nonseparable arguments are being used in the proofs of certain number theoretic results such as Fermat's Last Theorem. We have been suggesting strongly that this is an area where logicians and number theorists should collaborate in order to see just how necessary such appeals to nonseparable arguments are. We have conjectured that they are not, and that EFA = $I\Sigma_0(\exp)$ = exponential function arithmetic suffices. See (Avigad 2003).

In particular, it is not yet clear how strongly and in what way the Gödel incompleteness phenomena will penetrate normal mathematical activity. Progress along these lines is steady but painfully slow. We are confident that a much clearer assessment will be possible by the end of this century - and perhaps not much earlier.

In section 12, we take the opportunity to speculate far into the future.

2. THE COMPLETENESS THEOREM.

In his Ph.D. dissertation, (Gödel 1929), Gödel proved his celebrated completeness theorem for a standard version of the axioms and rules of first order predicate calculus with equality.

This result of Gödel was anticipated, in various senses, by earlier work of T. Skolem as discussed in detail in the Introductory notes in Vol. I of (Gödel 1986-2003 44-59). These Introductory notes were written by Burton Dreben and Jean van Heijenoort.

On page 52, the following passage from a letter from Gödel to Hao Wang, is quoted (December 7, 1967):

"The completeness theorem, mathematically, is indeed an almost trivial consequence of Skolem 1923a. However, the fact is that, at the time, nobody (including Skolem himself) drew this conclusion (neither from Skolem 1923a nor, as I did, from similar considerations)."

According to these Introductory Notes, page 52, the situation is properly summarized as follows:

"Thus, according to Gödel, the only significant difference between Skolem 1923a and Gödel 1929-1930 lies in the replacement of an informal notion of "provable" by a formal one ... and the explicit recognition that there is a question to be answered."[2]

To this, we would add that Gödel himself relied on a semiformal notion of "valid" or "valid in all set theoretic structures". The appropriate fully formal treatment of the semantics of first order predicate calculus with equality is credited to Alfred Tarski. However, as discussed in detail in (Feferman 2004), surprisingly the first clear statement in Tarski's work of the formal semantics for predicate calculus did not appear until (Tarski 1952) and (Tarski, Vaught 1957).

Let us return to the fundamental setup for the completeness theorem. The notion of structure is taken in the sense most relevant to mathematics, and in particular, general algebra: a nonempty domain, together with a system of constants, relations, and functions, with equality as understood.

It is well known that the completeness proof is so robust that no analysis of the notion of structure need be given. The proof requires only that we at least admit the structures whose domain is an initial segment of the natural numbers (finite or infinite). In fact, we need only admit structures whose relations and functions are arithmetically defined; i.e., first order defined in the ring of integers.

However, the axioms and rules of logic are meant to be so generally applicable as to transcend their application in mathematics. Accordingly, it is important to interpret logic with structures that may lie outside the realm of

[2]Skolem 1923a above is (Skolem 1922) in our list of references.

ordinary mathematics. A particularly important type of structure is a structure whose domain includes *absolutely everything*.

Indeed, it can be argued that the original Fregean conception of logic demands that quantifiers range over absolutely everything. From this viewpoint, quantification over mathematical domains is a special case, as “being in a given mathematical domain” is treated as (the extensions of) a unary predicate on everything.

These general philosophical considerations were sufficient for an applied philosopher like me to begin reworking logic using structures whose domain consists of absolutely everything.

The topic of logic in the universal domain has been taken up in the philosophy community, and in particular, by T. Williamson in (Rayo, Williamson 2003), and (Williamson 2000, 2003, 2006).

We have not yet published on this topic, but unpublished reports on our results are available on the web. Specifically, in (Friedman 1999), and in (Friedman 2002a 65-99). We plan to publish a monograph on this topic in the not too distant future.

3. THE FIRST INCOMPLETENESS THEOREM.

The Gödel first incompleteness theorem is first proved in (Gödel 1931). It is proved there in detail for a specific variant of what is now known as the simple theory of types (going back to Bertrand Russell), with natural numbers at the lowest type. This is a rather strong system, nearly as strong as Zermelo set theory.

It asserts that there is a sentence that is neither provable nor refutable in this system.

In (Gödel 1932b), Gödel formulates his incompleteness theorems for extensions of a variant of what is now known as PA = Peano arithmetic.

(Gödel 1934) gives another treatment of the results in [Gödel 31], but also, most importantly, introduces the notion of recursive functions and relations.

At the end of (Gödel 1931 195), Gödel writes that "The results will be stated and proved in full generality in a sequel to be published soon." Also we find, on page 195, from Gödel:

"*Note added 28 August 1963*. In consequence of later advances, in particular of the fact that due to A.M Turing's work a precise and unquestionably adequate definition of the general notion of formal system can now be given, a completely general version of Theorems VI and XI is now possible. That is, it can be proved rigorously that in *every* consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system."

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The sequel was never published at least partly because of the prompt acceptance of his results after the publication of (Gödel 1931).

Today, Gödel is credited for quite general forms of the first incompleteness theorem. There are already claims of generality in the original paper, (Gödel 31). In modern terms: in every 1-consistent recursively enumerable formal system containing a small amount of arithmetic, there exist arithmetic sentences that are neither provable nor refutable.

(Rosser 1936) is credited for significant additional generality, using a clever modification of Gödel's original formal self referential construction. It is shown there that the hypothesis of 1-consistency can be replaced with the weaker hypothesis of consistency.

Later, methods from recursion theory were used to prove yet more general forms of first incompleteness, and where the proof avoids use of formal self reference - although even in the recursion theory, there is, arguably, a trace of self reference present in the elementary recursion theory used.

The recursion theory approach, in a powerful form, appears in (Robinson 1952), and (Tarski, Mostowski, Robinson 1953), with the use of the formal system Q.

Q is a single sorted system based on $0,S,+,\bullet,\leq,=$. In addition to the usual axioms and rules of logic for this language, we have the nonlogical axioms

1. $Sx \neq 0$.
2. $Sx = Sy \rightarrow x = y$.
3. $x \neq 0 \rightarrow (\exists y)(x = Sy)$.
4. $x + 0 = x$.
5. $x + Sy = S(x + y)$.
6. $x \bullet 0 = 0$.
7. $x \bullet Sy = (x \bullet y) + x$.
8. $x \leq y \leftrightarrow (\exists z)(z + x = y)$.

The last axiom is purely definitional, and is not needed for present purposes (in fact, we do not need $\leq$).

THEOREM 3.1. Let T be a consistent extension of Q in a relational type in many sorted predicate calculus of arbitrary cardinality. The sets of all existential sentences in L(Q), with bounded universal quantifiers allowed, that are i) provable in T, ii) refutable in T, iii) provable or refutable in T, are each not recursive.

For the proof, see (Robinson 1952), and (Tarski, Mostowski, Robinson 1953). It uses the construction of recursively inseparable recursively enumerable sets; e.g., $\{n: \varphi_n(n) = 0\}$ and $\{n: \varphi_n(n) = 1\}$.

One can obtain the following strong form of first incompleteness as an immediate Corollary.

THEOREM 3.2. Let T be a consistent extension of Q in many sorted predicate calculus whose relational type and axioms are recursively enumerable. There is an existential sentence in L(Q), with bounded universal quantifiers allowed, that is neither provable nor refutable in T.

We can use the negative solution to Hilbert's tenth problem in order to obtain other forms of first incompleteness that are stronger in certain respects. In fact, Hilbert's tenth problem is still a great source of very difficult problems on the border between logic and number theory, which we will discuss below.

Hilbert asked for a decision procedure for determining whether a given polynomial with integer coefficients in several integer variables has a zero.

The problem received a negative answer in 1970 by Y. Matiyasevich, building heavily on earlier work of J. Robinson, M. Davis, and H. Putnam. It is commonly referred to as the MRDP theorem (in reverse historical order). See

(Davis 1973), (Matiyasevich 1993). The MRDP theorem was shown to be provable in the weak fragment of arithmetic, EFA = $I\Sigma_0$(exp), in (Dimitracopoulus, Gaifman 1982).

We can use (Dimitracopoulus, Gaifman 1982) to obtain the following.

THEOREM 3.3. Let T be a consistent extension of EFA in many sorted predicate calculus whose relational type and axioms are recursively enumerable. There is a purely existential equation $(\exists x_1,\ldots,x_n)(s = t)$ in L(Q) that is neither provable nor refutable in T.

It is not clear whether EFA can be replaced by a weaker system in Theorem 3.3 such as Q.

An important issue is whether there is a "reasonable" existential equation $(\exists x_1,\ldots,x_n)(s = t)$ that can be used in Theorem 3.3 for, say, T = PA or T = ZFC. Note that $(\exists x_1,\ldots,x_n)(s = t)$ corresponds to the Diophantine problem "does the polynomial s-t with integer coefficients have a solution in the nonnegative integers?"

Let us see what can be done on the purely recursion theoretic side with regards to the complexity of polynomials with integer coefficients. The most obvious criteria are

a. The number of unknowns.

b. The degree of the polynomial.

c. The number of operations (additions and multiplications).

In 1992, Matiyasevich showed that nine unknowns over the nonnegative integers suffices for recursive unsolvability. One form of the result (not the strongest form) says that the problem of deciding whether or not a polynomial with integer coefficients in nine unknowns has a zero in the nonnegative integers, is recursively unsolvable. A detailed proof of this result (in sharper form) was given in (Jones 1982).

Also (Jones 1982) proves that, e.g., the problem of deciding whether or not a polynomial with integer coefficients defined by at most 100 operations (additions and multiplications with integer constants) has a zero in the nonnegative integers, is recursively unsolvable.

It is well known that degree 4 suffices for recursive unsolvability. In (Jones 1982), it is shown that degree 4 and 58 nonnegative integer unknowns suffice for recursive unsolvability. I.e, the problem of deciding whether or not a polynomial with integer coefficients, of degree 4 with at most 58 unknowns, is recursively unsolvable.

In fact, (Jones 1982) provides the following sufficient pairs <degree,unknowns>, where all unknowns range over nonnegative integers:

<4,58>, <8,38>, <12,32>, <16,29>,

<20,28>, <24,26>, <28,25>, <36,24>,

<96,21>, <2668,19>, <2×10^5,14>,

<6.6×10^{43},13>, <1.3×10^{44},12>,

<4.6×10^{44},11>, <8.6×10^{44},10>,

<1.6×10^{45},9>.

For degree 2 (a single quadratic) we have an algorithm (over the nonnegative integers, the integers, and the rationals), going back to (Siegel 72). See (Grunewald, Segal 1981), (Masser 1998).

For degree 3, the existence of an algorithm is wide open, even for three variables (over the integers, the nonnegative integers, or the rationals). For degree 3 in two integer variables, an algorithm is known, but it is wide open for degree 3 in two rational variables.

It is clear from this discussion that the gap between what is known and what could be the case is enormous, just in this original context of deciding whether polynomials with integer coefficients have a zero in the (nonnegative) integers. Specifically, <3,3> could conceivably be on this list of pairs.

These upper bounds on the complexity sufficient to obtain recursive unsolvability can be directly imported into Theorem 3.3, as the underlying number theory and recursion theory can be done in EFA. Although one obtains upper bounds on pairs (number of variables, degree) in this way, this does not address the question of the size of the coefficients needed in Theorem 3.3.

In particular, let us call a polynomial P a Gödel polynomial if

i. P is a polynomial in several variables with integer coefficients.

ii. The question of whether P has a solution in nonnegative integers is neither provable nor refutable in PA. (We can also use ZFC here instead of PA).

We have never seen an upper bound on the “size” of a Gödel polynomial in the literature. In particular, we have never seen a Gödel polynomial written down fully in base 10 on a small piece of paper.

One interesting theoretical issue is whether one can establish any relationship between the “size” of a Gödel polynomial using PA and the “size” of a Gödel polynomial using ZFC.

4. THE SECOND INCOMPLETENESS THEOREM.

In (Gödel 1931), Gödel only sketches a proof of his second incompleteness theorem, after proving his first incompleteness theorem in detail. His sketch depends on the fact that the proof of the first incompleteness theorem, which is conducted in normal semiformal mathematics, can be formalized and proved within (systems such as) PA.

Gödel promised a part 2 of (Gödel 1931), but this never appeared. There is some difference of opinion as to whether Gödel planned to provide detailed proofs of his second incompleteness theorem in part 2, or whether Gödel planned to let others carry out the details.

In any case, the necessary details were carried out in (Hilbert, Bernays 1934,1939), and later in (Feferman 1960), and most recently, in (Boolos 1993).

In (Hilbert, Bernays 1934,1939), the so called Hilbert Bernays derivability conditions were isolated in connection with a detailed proof of Gödel's second incompleteness theorem given in (Hilbert, Bernays 1934,1939). Later, these conditions were streamlined in (Jerosolow 1973).

We take the liberty of presenting our own particularly careful and clear version of the Hilbert Bernays conditions.

Our starting point is the usual language L = predicate calculus with equality, with infinitely many constant, relation, and function symbols. For specificity, we will use

i) variables x_n, $n \geq 1$;

ii) constant symbols c_n, $n \geq 1$;

iii) relation symbols R^n_m, $n,m \geq 1$;

iv) function symbols F^n_m, $n,m \geq 1$;

v) connectives $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$;

vi) quantifiers $\exists,\forall$.

We start with the following data:

1. A relational type RT of constant symbols, relation symbols, and function symbols.
2. A set T of sentences in (the language based on) RT.
3. A one-one function # from formulas of RT into closed terms of RT.
4. A distinguished unary function symbol NEG in RT, meaning "negation".
5. A distinguished unary function symbol SSUB in RT, meaning "self substitution".
6. A distinguished unary function symbol PR in RT, meaning "provability statement".
7. A distinguished formula PROV with at most the free variable x_1, expressing "provable in T".

We require the following. Let A be a formula of RT.

8. $\mathrm{NEG}(\#(A)) = \#(\neg A)$ is provable in T.
9. $\mathrm{SSUB}(\#(A)) = \#(A[x_1/\#(A)])$ is provable in T.
10. $\mathrm{PR}(\#(A)) = \#(\mathrm{PROV}[x_1/\#(A)])$ is provable in T.
11. $\mathrm{PROV}[x_1/\#(A)] \rightarrow \mathrm{PROV}[x_1/\mathrm{PR}(\#(A))]$ is provable in T.
12. If A is provable in T, then $\mathrm{PROV}[x_1/\#(A)]$ is provable in T.

Here #(A) is the Gödel number of the formula A, as a closed term of RT.

THEOREM 4.1. (Self reference lemma). Let A be a formula of RT. There exists a closed term t of RT such that T proves $t = \#(A[x_1/t])$.

Proof: Let $s = \#(A[x_1/SSUB(x_1)])$. Write $s = \#B$, where $B = A[x_1/SSUB(x_1)]$.

Note that

$$B[x_1/\#(B)] = B[x_1/s] = A[x_1/SSUB(s)].$$

We now apply condition 9 to B. We have that

$$SSUB(\#(B)) = \#(B[x_1/\#(B)])$$

is provable in T. Hence

$$SSUB(s) = \#(A[x_1/SSUB(s)])$$

is provable in T. Thus the closed term SSUB(s) is as required. QED

LEMMA 4.2. ("I am not provable" Lemma). There exists a closed term t such that T proves $t = \#(\neg PROV[x_1/t])$.

Proof: By Theorem 4.1, setting $A = \neg PROV$. QED

We fix a closed term t provided by Lemma 4.2.

LEMMA 4.3. Suppose T proves $\neg PROV[x_1/t]$. Then T is inconsistent.

Proof: Assume T proves $\neg PROV[x_1/t]$. By condition 12,

$$PROV[x_1/\#(\neg PROV[x_1/t])]$$

is provable in T. By Lemma 4.2, T proves $PROV[x_1/t]$. Hence T is inconsistent. QED

LEMMA 4.4. T proves $PROV[x_1/t] \rightarrow PROV[x_1/PR(t)]$. T proves $PROV[x_1/t] \rightarrow PROV[x_1/NEG(PR(t))]$.

Proof: Let $A = \neg PROV[x_1/t]$. By condition 11,

$$PROV[x_1/\#(A)] \rightarrow PROV[x_1/PR(\#(A))]$$

is provable in T. By Lemma 4.2,

$$PROV[x_1/t] \rightarrow PROV[x_1/PR(t)]$$

is provable in T. By condition 10, T proves

$$PR(t) = \#(PROV[x_1/t]).$$

By condition 8, T proves

$$NEG(\#(PROV[x_1/t])) = \#(\neg PROV[x_1/t]) .$$

By Lemma 4.2, T proves

$$NEG(PR(t)) = t.$$

The second claim follows immediately. QED

We let CON be the sentence

$$(\forall x_1)(\neg(PROV \wedge PROV[x_1/NEG(x_1)])).$$

THEOREM 4.5. (Abstract second incompleteness). Let T obey conditions 1-12.

Suppose T proves CON. Then T is inconsistent.

Proof: Suppose T is as given. By Lemma 4.4, T proves

$$PROV[x_1/t] \rightarrow PROV[x_1/PR(t)] \wedge PROV[x_1/NEG(PR(t))].$$

Since T proves CON, T proves

$$\neg(PROV[x_1/PR(t)] \wedge PROV[x_1/NEG(PR(t))]).$$

Hence T proves $\neg PROV[x_1/t]$. By Lemma 4.3, T is inconsistent. QED

Informal statements of Gödel's Second Incompleteness Theorem are simple and dramatic. However, current versions of the Formal Second Incompleteness are complicated and awkward. Even the abstract form of second incompleteness given above using derivability conditions are rather subtle and involved.

We recently addressed this problem in (Friedman 2007a), where we present new versions of Formal Second Incompleteness that are simple, and informally imply Informal Second Incompleteness.

These results rest on the isolation of simple formal properties shared by consistency statements. Here we do not address any issues concerning proofs of Second Incompleteness.

We start with the most commonly quoted form of Gödel's Second Incompleteness Theorem - for the system PA = Peano Arithmetic.

PA can be formulated in a number of languages. Of these, L(prim) is the most suitable for supporting formalizations of the consistency of Peano Arithmetic.

We write L(prim) for the language based on 0,S and all primitive recursive function symbols. We let PA(prim) be the formulation of Peano Arithmetic for the language L(prim). I.e., the nonlogical axioms of PA(prim) consist of the axioms for successor, primitive recursive defining equations, and the induction scheme applied to all formulas in L(prim).

INFORMAL SECOND INCOMPLETENESS (PA(prim)). Let A be a sentence in L(prim) that adequately formalizes the consistency of PA(prim), in the informal sense. Then PA(prim) does not prove A.

We have discovered the following result. We let PRA be the important subsystem of PA(prim), based on the same language L(prim), where we require that the induction scheme be applied only to quantifier free formulas of L(prim).

FORMAL SECOND INCOMPLETENESS (PA(prim)). Let A be a sentence in L(prim) such that every equation in L(prim) that is provable in PA(prim), is also provable in PRA + A. Then PA(prim) does not prove A.

Informal second incompleteness for PA(prim) can be derived in the usual semiformal way from the above formal second incompleteness for PA(prim).

FORMAL CRITERION THEOREM 1. Let A be a sentence in L(prim) such that every equation in L(prim) that is provable in PA(prim), is also provable in PRA + A. Then for all n, PRA + A proves the consistency of $PA(prim)_n$.

Here $PA(prim)_n$ consists of the axioms of PA(prim) in prenex form with at most n quantifiers.

The above development can be appropriately carried out for systems with full induction. However, there is a more general treatment which covers finitely axiomatized theories as well.

We use the system EFA = exponential arithmetic for this more general treatment. EFA is the system of arithmetic based on addition, multiplication and exponentiation, with induction applied only to formulas all of whose quantifiers are bounded to terms. This is the same as the system $I\Sigma_0(exp)$ in (Hajek, Pudlak 1993 p. 37).

INFORMAL SECOND INCOMPLETENESS (general many sorted, EFA). Let L be a fragment of L(many) containing L(EFA). Let T be a consistent extension of EFA in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (general many sorted, EFA). Let L be a fragment of L(many) containing L(EFA). Let T be a consistent extension of EFA in L. Let A be a sentence in L such that every universalized inequation in L(EFA) with a relativization in T, is provable in EFA + A. Then T does not prove A.

FORMAL CRITERION THEOREM II. Let L be a fragment of L(many) containing L(EFA). Let T be a consistent extension of EFA in L. Let A be a sentence in L such that every universalized inequation in L(EFA) with a relativization in T, is provable in EFA + A. Then EFA proves the consistency of every finite fragment of T.

Here, a relativization of a sentence φ of L(EFA), in T, is an interpretation of φ in T which leaves the meaning of all symbols unchanged, but where the domain is allowed to consist of only some of the nonnegative integers from the point of view of T.

Finally, we mention an interesting issue that we are somewhat unclear about, but which can be gotten around in a satisfactory way.

It can be said that Gödel's second incompleteness theorem has a defect in that one is relying on a formalization of Con(T) within T via the indirect method of Gödel numbers. Not only is the assignment of Gödel numbers to formulas (and the relevant syntactic objects) ad hoc, but one is still being indirect and not directly dealing with the objects at hand - which are syntactic and not numerical.

It would be preferable to directly formalize Con(T) within T, without use of any indirection. Thus in such an approach, one would add new sorts for the relevant syntactic objects, and introduce the various relevant relations and function symbols, together with the relevant axioms. Precisely this approach was adopted by (Quine, 1940, 1951, Chapter 7).

However, in so doing, one has expanded the language of T. Accordingly, two choices are apparent.

The first choice is to make sure that as one adds new sorts and new relevant relations and function symbols and new axioms to T, associated with syntax, one also somehow has already appropriately treated, directly, the new syntactic objects and axioms beyond T that arise when one is performing this addition to T.

The second choice is to be content with adding the new sorts and new relevant relations and function symbols and new axioms to T, associated with the syntax of T only - and not try to deal in this manner with the extended syntax that arises from this very process. This is the choice made in (Quine, 1940, 1951, Chapter 7).

We lean towards the opinion that the first choice is impossible to realize in an appropriate way. Some level of indirection will remain. Perhaps the level of indirection can be made rather weak and subtle. Thus we lean towards the opinion that it is impossible to construct extensions of, say, PA that directly and adequately formalize their entire syntax. We have not tried to prove such an impossibility result, but it seems possible to do so.

In any case, the second choice, upon reflection, turns out to be wholly adequate for casting what may be called "direct second incompleteness". This formulation asserts that for any suitable theory T, if T' is the (or any) extension of T through the addition of appropriate sorts, relations, functions, and axioms, directly formalizing the syntax of T, including a direct formalization of the consistency of T, then T′ does not prove the consistency of T (so expressed).

We can recover the usual second incompleteness theorem for T from the above direct second incompleteness, by proving that there is an interpretation of T′ in T. This was also done in (Quine, 1940, 1951, Chapter 7).

Thus under this view of second incompleteness, one does not view Con(T) as a sentence in the language of T, but instead as a sentence in the language of an extension T′ of T. Con(T) only becomes a sentence in the language of T through an interpretation (in the sense of Tarski) of T′ in T. There are many such interpretations, all of which are ad hoc. This view would then eliminate ad hoc features in the formulation of second incompleteness, while preserving the foundational implications.

5. LENGTHS OF PROOFS.

In (Gödel 1936), Gödel discusses a result which, in modern terminology, asserts the following. Let RTT be Russell's simple theory of types with the axiom of infinity. Let RTT_n be the fragment of RTT using only the first n types. Let $f:N \to N$ be a recursive function. For each $n \geq 0$ there are infinitely many sentences φ such that

$$f(n) < m$$

where n is the least Gödel number of a proof of φ in RTT_{n+1} and m is the least Gödel number of a proof of φ in RTT_n.

Gödel expressed the result in terms of lengths of proofs rather than Gödel numbers or total number of symbols. Gödel did not publish any proofs of this result or results of a similar nature. As can be surmised from the Introductory remarks by R. Parikh, it is likely that Gödel had inadvertently used lengths, and probably intended Gödel numbers or numbers of symbols.

In any case, the analogous result with Gödel numbers was proved in (Mostowski 1952). Similar results were also proved in (Ehrenfeucht, Mycielski 1971) and (Parikh 1971). Also see (Parikh 1973) for results going in the opposite direction concerning the number of lines in proofs in certain systems.

In (Friedman 79), we considered, for any reasonable system T, and positive integer n, the finite consistency statement $Con_n(T)$ expressing that "every inconsistency in T uses at least n symbols". We gave a lower bound of $n^{1/4}$ on the number of symbols required to prove in $Con_n(T)$ in T, provided n is sufficiently large. A more careful version of the argument gives the lower bound of $n^{1/2}$ for sufficiently large n. We called this "finite second incompleteness".

A much more careful analysis of finite second incompleteness is in (Pudlak 1985), which establishes an $(n(\log(n))^{-1/2})$ lower bound and an $O(n)$ upper bound, for systems T satisfying certain reasonable conditions.

It would be very interesting to extend finite second incompleteness in several directions. One direction is to give a treatment of a good lower bound for a proof of $Con_n(T)$ in T, which is along the lines of the Hilbert Bernays derivability conditions, adapted carefully for finite second incompleteness. We offer our treatment of the derivability conditions in section 4 above as a launching point. A number of issues arise as to the best way to set this up, and what level of generality is appropriate.

Another direction to take finite second incompleteness is to give some versions which are not asymptotic. I.e., they involve specific numbers of symbols that are argued to be related to actual mathematical practice.

Although the very good upper bound of O(n) is given in (Pudlak 1985) for a proof of $Con_n(T)$ in T, at least for some reasonable systems T, the situation seems quite different if we are talking about proofs in S of $Con_n(T)$, where S is significantly weaker than T. For specificity, consider how many symbols it takes to prove $Con_n(ZF)$ in PA, where n is large. It seems plausible that there is no subexponential upper bound here.

But obviously, if there is some algorithm and polynomial P that PA can prove is an algorithm for testing satisfiability of Boolean expressions whose run time is bounded by P, then PA proves $Con_n(ZF)$ using a polynomial number of symbols in n (assuming Con(ZF) is in fact true). So in order to show that there is no subexponential upper bound here, we will have to refute this strong version of $P \neq NP$. However, this appears to be as challenging as proving $P \neq NP$.

There are some other aspects of lengths of proofs that seem important. One is the issue of overhead.

Gödel established in [Gödel 1940] that any proof of an arithmetic sentence A in NBG + AxC can be converted to a proof of A in NBG. He used the method of relativization. Thus one obtains constants c,d such that if arithmetic A is provable in NBG + AxC using n symbols, then A is provable in NBG using at most cn+d symbols.

What is not at all clear here is whether c,d can be made reasonably small. There is clearly a lot of overhead involved on two counts. One is in the execution of the actual relativization, which involves relativizing to the constructible sets. The other overhead is that one must insert the proofs of various facts about the constructible sets including that they form a model of NBG.

The same remarks can be made with regard to NBG + GC + CH and NBG + GC, where GC is the global axiom of choice. Also, these remarks apply to ZFC and ZF, and also to ZFC + CH and ZFC. Also they apply equally well to the Cohen forcing method (Cohen 1963-1964), and proofs from ZF + ¬AxC, and from ZFC + ¬CH.

We close with another issue regarding lengths of proofs in a context that is often considered immune to incompleteness phenomena. Finite incompleteness phenomena is very much in evidence here.

Alfred Tarski, in (Tarski 1951), proved the completeness of the usual axioms for real closed fields using quantifier elimination. This also provides a decision procedure for recognizing the first order sentences in $(\Re,<,0,1,+,-,\bullet)$. His method applies to the following three fundamental axiom systems:

1) The language is $0,1,+,-,\bullet$. The axioms consist of the usual field axioms, together with -1 is not the sum of squares, x or -x is a square, and every polynomial of odd degree with leading coefficient 1 has a zero.
2) The language is $0,1,+,-,\bullet,<$. The axioms consist of the usual ordered field axioms, together with every positive element has a square root, and every polynomial of odd degree with leading coefficient 1 has a zero.
3) The language is $0,1,+,-,\bullet,<$. The axioms consist of the usual ordered field axioms, together with the axiom scheme asserting that if a first order property holds of something, and there is an upper bound to what it holds of, then there is a least upper bound to what it holds of.

For reworking and improvements on Tarski, see (Cohen 1969), (Renegar 1982a-c), (Basu, Pollack, Roy 2006). In terms of computational complexity, the set of true first order sentences in $(\Re,<,0,1,+,-,\bullet)$ is exponential space easy and nondeterministic exponential time hard. The gap has not been filled. Even the first order theory of $(\Re,+)$ is nondeterministic exponential time hard. See Rabin 1977.

The work just cited concerns mainly the computational complexity of the set of true sentences in the reals (sometimes with only addition). It does not directly deal with the lengths of proofs in systems 1),2),3).

What can we say about number of symbols in proofs in systems 1),2),3)? We conjecture that with the usual axioms and rules of logic, in all three cases, there is a double exponential lower and upper bound on the number of symbols required in a proof of any true sentence in each of 1),2),3).

What is the relationship between sizes of proofs of the same sentence (without <) in 1),2),3)? We conjecture that, asymptotically, there are infinitely many true sentences without < such that there is a double exponential reduction in the number of symbols needed to prove it when passing from system 1) to system 3).

These issues concerning sizes of proofs are particularly interesting when the quantifier structure of the sentence is restricted. For instance, the cases of purely universal, purely existential are particularly interesting, particularly when the matrix is particularly simple. Other cases of clear interest are $\forall\ldots\forall\exists\ldots\exists$, and $\exists\ldots\exists\forall\ldots\forall$, with the obviously related conditions of surjectivity and nonsurjectivity being of particular interest.

Another aspect of sizes of proofs comes out of strong mathematical Π^0_2 sentences. The earliest ones were presented in (Goodstein 1944) and (Paris, Harrington 1977), and are proved just beyond PA. We discovered many examples in connection with theorems of J.B. Kruskal (Kruskal 1960), and Robertson, Seymour (Roberton, Seymour 1985, 2004), which are far stronger, with no predicative proofs. See (Friedman 2002b).

None of these three references discusses the connection with sizes of proofs. This connection is discussed in (Smith 1985 132-135), and in the unpublished abstracts (Friedman 2006a-g) from the FOM Archives.[3]

The basic idea is this. There are now a number of mathematically natural Π^0_2 sentences $(\forall n)(\exists m)(R(n,m))$ which are provably equivalent to the 1-consistency of various systems T. One normally gets, as a consequence, that the Skolem function m of n grows very fast, asymptotically, so that it dominates the provably recursive functions of T.

However, we have observed that in many cases, one can essentially remove the asymptotics. I.e., in many cases, we have verified that we can fix n to be very small (numbers like 3 or 9 or 15), and consider the resulting Σ^0_1 sentence $(\exists m)(R(n,m))$. The result is that any proof in T (or certain strong fragments of T) of this Σ^0_1 sentence must have an absurd number of symbols - e.g., an exponential stack of 100 2's. Yet if we go a little beyond T, we can prove the full Π^0_2 sentence $(\forall n)(\exists m)(R(n,m))$ in a normal size mathematics manuscript, thereby yielding a proof just beyond T of the resulting Σ^0_1 sentence R(n,m) with n fixed to be a small (or remotely reasonable) number. This provides a myriad of mathematical examples of Gödel's original length of proof phenomena from (Gödel 1936).

6. THE NEGATIVE INTERPRETATION.

[3] See http://cs.nyu.edu/pipermail/fom/

Gödel wrote four fundamental papers concerning formal systems based on intuitionistic logic: (Gödel 1932a), (Gödel 1933a), (Gödel 1933b), (Gödel 1958). (Gödel 1972) is a revised version of (Gödel 1958).

In (Gödel 32a), Gödel proves that the intuitionistic propositional calculus cannot be viewed as a classical system with finitely many truth values. He shows this by constructing an infinite descending chain of logics intermediate in strength between classical propositional calculus and intuitionistic propositional calculus. For more on intermediate logics, see (Hosoi, Ono 1973) and (Minari 1983).

In (Gödel 1933a), Gödel introduces his negative interpretation in the form of an interpretation of PA = Peano arithmetic in HA = Heyting arithmetic. Here HA is the corresponding version of PA = Peano arithmetic based on intuitionistic logic. It can be axiomatized by taking the usual axioms and rules of intuitionistic predicate logic, together with the axioms of PA as usual given. Of course, one must be careful to present ordinary induction in the usual way, and not use the least number principle.

It is natural to isolate his negative interpretation in these two ways:

a. An interpretation of classical propositional calculus in intuitionistic propositional calculus.

b. An interpretation of classical predicate calculus in intuitionistic predicate calculus.

In modern terms, it is convenient to use $\perp,\neg,\vee,\wedge,\rightarrow$. The interpretation for propositional calculus inductively interprets

$\perp$ as $\perp$.

$\neg$ as $\neg$.

$\wedge$ as $\wedge$.

$\rightarrow$ as $\rightarrow$.

$\vee$ as $\neg\neg\vee$.

For predicate calculus,

$\forall$ as $\forall$.

$\exists$ as $\neg\neg\exists$.

φ as $\neg\neg\varphi$, where φ is atomic.

Now in HA, we can prove $n = m \vee \neg n = m$. It is then easy to see that the successor axioms and the defining equations of PA are sent to theorems of HA, and also each induction axiom of PA is sent to a theorem of HA.

Also the axioms of classical predicate calculus become theorems of intuitionistic predicate calculus, and the rules of classical predicate calculus become rules of intuitionsitic predicate calculus.

So under the negative interpretation, theorems of classical propositional calculus become theorems of intuitionsitic propositional calculus, theorems of classical predicate calculus become theorems of intuitionistic predicate calculus, and theorems of PA become theorems of HA.

Also, any Π^0_1 sentence $(\forall n)(F(n) = 0)$, where F is a primitive recursive function symbol of PA, is sent to a sentence that is provably equivalent to $(\forall n)(F(n) = 0)$.

It is then easy to conclude that every Π^0_1 theorem of PA is a theorem of HA.

Gödel's negative interpretation has been extended to many pairs of systems, most of them of the form T,T', where T,T' have the same nonlogical axioms, and where T is based on classical predicate calculus, whereas T' is based on intuitionistic predicate calculus. For example, see (Kreisel 68a 344), (Kreisel 68b Section 5), (Myhill 74), (Friedman 73), (Leivant 85).

A much stronger result holds for PA over HA. Every Π^0_2 sentence provable in PA is provable in HA. The first proofs of this result were from the proof theory of PA via Gentzen (see (Gentzen 1969), (Schütte 1977)), and from Gödel's so called Dialectica or functional interpretation, in (Gödel 1958), (Gödel 1972).

However, for other pairs for which the negative interpretation shows that they have the same provable Π^0_1 sentences - say classical and intuitionistic second order arithmetic - one does not have the required proof theory. In this case, the Dialectica interpretation has been extended by Spector in (Spector 1962), and the fact that these two systems have the same provable Π^0_2 sentences then follows.

Nevertheless, there are many appropriate pairs for which the negative interpretation works, yet there is no proof theory and there is no functional interpretation.

In (Friedman 1978), we broke this impasse by modifying Gödel's negative interpretation via what is now called the A translation. Also see (Dragalin 1980). We illustrate the technique for PA over HA, formulated with primitive recursive function symbols.

Let A be any formula in L(HA) = L(PA). We define the A-translation φ_A of the formula φ in L(HA), in case no free variable of A is bound in φ. Take φ^A to be the result of simultaneously replacing every atomic subformula ψ of φ by ($\psi \vee$ A). In particular, $\perp$ gets replaced by what amounts to A.

The A translation is an interpretation of HA in HA. I.e., if φ^A is defined, and HA proves A, then HA proves φ^A. Also, obviously HA proves $A \rightarrow \varphi^A$.

Now suppose $(\exists n)(F(n,m) = 0)$ is provable in PA, where F is a primitive recursive function symbol. By Gödel's negative interpretation, $\neg\neg(\exists n)(F(n,m) = 0)$ is provable in HA. Write this as $((\exists n)(F(n,m) = 0) \rightarrow \perp) \rightarrow \perp$.

By taking the A translation, with $A = (\exists n)(F(n,m) = 0)$, we obtain that HA proves

$$((\exists n)(F(n,m) = 0 \vee (\exists n)(F(n,m) = 0)) \rightarrow (\exists n)(F(n,m) = 0)) \rightarrow (\exists n)(F(n,m) = 0.$$

$$((\exists n)(F(n,m) = 0) \rightarrow (\exists n)(F(n,m) = 0)) \rightarrow$$

$$(\exists n)(F(n,m) = 0.$$

$$(\exists n)(F(n,m) = 0).$$

This method applies to a large number of pairs T/T' as indicated in (Friedman 1973) and (Leivant 1985).

(Godel 1958) and (Godel 1972) present Gödel's so called Dialectica interpretation, or functional interpretation, of HA. Here HA = Heyting arithmetic, is the corresponding version of PA = Peano arithmetic with intuitionistic logic. It can be axiomatized by taking the usual axioms and rules of intuitionistic predicate logic, together with the axioms of PA as usual given. Of course, one must be careful to present ordinary induction in the usual way, and not use the least number principle.

In Gödel's Dialectica interpretation, theorems of HA are interpreted as derivations in a quantifier free system T of primitive recursive functionals of finite type that is based on quantifier free axioms and rules, including a rule of induction.

The Dialectica interpretation has had several applications in different directions. There are applications to programming languages and category theory which we will not discuss.

To begin with, the Dialectica interpretation can be combined with Godel's negative interpretation of PA in HA to form an interpretation of PA in Gödel's quantifier free system T.

One obvious application, and motivation, is philosophical, and Gödel discusses this aspect in both papers, especially the second. The idea is that the quantifiers in HA or PA, ranging over all natural numbers, are not finitary, whereas T is arguably finitary - at least in the sense that T is quantifier free. However, the objects of T are at least prima facie infinitary, and so there is the difficult question of how to gauge this tradeoff. One idea is that the objects of T should not be construed as infinite completed totalities, but rather as rules. We refer the interested reader to the rather extensive Introductory notes to (Gödel 58) in (Gödel 1986-2003 Vol. II).

Another application is to extend the interpretation to the two sorted first order system known as second order arithmetic, or Z_2. This was carried out by Clifford Spector in (Spector 1962). Here the idea is that one may construe such a powerful extension of Gödel's Dialectica interpretation as some sort of constructive consistency proof for the rather metamathematically strong and highly impredicative system Z_2. However, in various communications, Gödel was not entirely satisfied that the quantifier free system Spector used was truly constructive.

We believe that the Spector development has not been fully exploited. In particular, it ought to give rather striking mathematically interesting characterizations of the provably recursive functions and provable ordinals of Z_2 and various fragments of Z_2.

Another fairly recent application is to use the Dialectica interpretation, and extensions of it to systems involving functions and real numbers, in order to obtain sharper uniformities in certain areas of functional analysis that had been obtained before by the specialists. This work has been pioneered by U. Kohlenbach. See the five references to Kohlenbach (and joint authors) in the list of references.

7. THE AXIOM OF CHOICE AND THE CONTINUUM HYPOTHESIS.

Gödel wrote six manuscripts directly concerned with the continuum hypothesis: Two abstracts, (Gödel 1938), (Gödel 1939a). One paper with sketches of proofs, (Gödel 1939b). One research monograph with fully detailed proofs, (Gödel 1940). One philosophical paper, (Gödel 1947,1964), in two versions.

The normal abbreviations for the axiom of choice is AxC. The normal abbreviation for the continuum hypothesis is CH.

A particularly attractive formulation of CH asserts that every set of real numbers is either in one-one correspondence with a set of natural numbers, or in one-one correspondence with the set of real numbers.

Normally, one follows Gödel in considering CH only in the presence of AxC. However, note that in this form, CH can be naturally considered without the presence of AxC. However, Solovay's model satisfying ZFCD + "all sets are

Lebesgue measurable" also satisfies CH in the strong form that every set of reals is countable or has a perfect subset (this strong form is incompatible with AxC). See (Solovay 1970).

The statement of CH is due to Cantor. Gödel also considers the generalized continuum hypothesis, GCH, whose statement is credited to Hausdorff. The GCH asserts that for all sets A, every subset of $\wp(A)$ is either in one-one correspondence with a subset of A, or in one-one correspondence with $\wp(A)$. Here $\wp$ is the power set operation.

Gödel's work establishes an interpretation of ZFC + GCH in ZF. This provides a very explicit way of converting any inconsistency in ZFC + GCH to an inconsistency in ZF.

We can attempt to quantify these results. In particular, it is clear that the interpretation given by Gödel of ZFC + GCH in ZF, by relativizing to the constructible sets, is rather large, in the sense that when fully formalized, results in a lot of symbols. It also seems to result in a lot of quantifiers. How many?

So far we have been talking about the crudest formulations in primitive notation, without the benefit of abbreviation mechanisms. But abbreviation mechanisms are essential for the actual conduct of mathematics. In fact, current proof assistants - where humans and computers interact to create verified proofs - necessarily incorporate very substantial abbreviation mechanisms. See, e.g., (Barendregt, Wiedijk 2005), (Wiedijk 2006).

So the question arises as to how simple can an interpretation be of ZFC + GCH in ZF, with abbreviations allowed in the presentation of the interpretation? This is far from clear.

P.J. Cohen proved that if ZF is consistent then so is ZF + ¬AxC and ZFC + ¬CH, thus complementing Gödel's results. See (Cohen 1963-1964). The proof does not readily give an interpretation of ZF + ¬AxC, or of ZFC + ¬CH in ZF. It can be converted into such an interpretation by a general method whereby under certain conditions (met here), if the consistency of every given finite subsystem of one system is provable in another, then the first system is interpretable in the other (see (Feferman 1960)).

Again, the question arises as to how simple can an interpretation be of ZF + ¬AxC or of ZFC + ¬CH, in ZF, with abbreviations allowed in the presentation of the interpretation? Again this is far from clear. And how does this question compare with the previous question?

There is another kind of complexity issue associated with the CH that is of interest. First some background. It is known that every 3 quantifier sentence in primitive notation ∈,=, is decided in a weak fragment of ZF. See (Gogol 1979), (Friedman 2003a). Also there is a 5 quantifier sentence in ∈,= that is not decided in ZFC (it is equivalent to the existence of a subtle cardinal over ZFC). See (Friedman 2003b). It is also known that AxC can be written with five quantifiers in ∈,=, over ZFC. See (Maes 2007).

The question is: how many quantifiers are needed to express CH over ZFC, in ∈,=? We can also ask this and related questions where abbreviations are allowed.

Most mathematicians instinctively take the view that since CH is neither provable nor refutable from the standard axioms for mathematics (ZFC), the ultimate status of CH has been settled and there is nothing left to ponder.

However, many mathematical logicians, particularly those in set theory, take a quite different view. This includes Kurt Gödel. They take the view that the continuum hypothesis is a well defined mathematical assertion with a definite truth value. The problem is to determine just what this truth value is.

The idea here is that there is a definite system of objects that exists independently of human minds, and that human minds can no more manipulate the truth value of statements of set theory than they can manipulate the truth value of statements about electrons and stars and galaxies.

This is the so called Platonist point of view that is argued so forcefully and explicitly in (Gödel 1947,1964).

The late P.J. Cohen led a panel discussion at the Gödel Centenary called On Unknowability, where he conducted a poll roughly along these lines. The question he asked was, roughly, "does the continuum hypothesis have a definite answer", or "does the continuum hypothesis have a definite truth value".

The response from the audience appeared quite divided on the issue.

Of the panelists, the ones who have expressed very clear views on this topic were most notably Cohen and Woodin. Cohen took a formalist viewpoint, whereas Woodin takes a Platonist one. See their respective contributions to this volume.

My own view is that we simply do not know enough in the foundations of mathematics to decide the truth or appropriateness of the formalist versus the Platonist viewpoint - or, for that matter, what mixture of the two is true or appropriate.

But then it is reasonable to place the burden on me to explain what kind of additional knowledge could be relevant for this issue.

My ideas are not very well developed, but I will offer at least something for people to consider.

It may be possible to develop a theory of 'fundamental mental pictures' which is so powerful and compelling that it supplants any discussion of formalism/Platonism in anything like its present terms. What may come out is a fundamental mental picture for the axioms of ZFC, even with some large cardinals, along with a theorem to the effect that there is no fundamental mental picture for CH and no fundamental mental picture for ¬CH.

8. WQO THEORY.

Wqo theory is a branch of combinatorics which has proved to be a fertile source of deep metamathematical pheneomena.

A qo (quasi order) is a reflexive transitive relation $(A,\leq)$. A wqo (well quasi order) is a qo $(A,\leq)$ such that for all infinite sequences $x_1,x_2,\ldots$ from A, $\exists\, i < j$ such that $x_i \leq x_j$.

The highlights of wqo theory are that certain qo's are wqo's, and certain operations on wqo's produce wqo's.

(Kruskal 1960), treats finite trees as finite posets, and studies the qo

there exists an inf preserving embedding from T_1 into T_2.

THEOREM 8.1. (Kruskal 1960). The above qo of finite trees as posets is a wqo.

The simplest proof of Theorem 8.1 and some extensions, is in (Nash-Williams 1963), with the introduction of minimal bad sequences.

We observed that the connection between wqo's and well orderings can be combined with known proof theory to establish independence results.

The standard formalization of "predicative mathematics" is due to Feferman/Schutte = FS. See (Feferman 1964,1968), (Feferman 1998). Poincare, Weyl, and others railed against impredicative mathematics. See (Weyl 1910), (Weyl 1987), (Feferman 1998 289-291), and (Foline 1992).

THEOREM 8.2. (Friedman 2002b). Kruskal's tree theorem cannot be proved in FS.

KT goes considerably beyond FS, and an exact measure of KT is known. See (Rathjen, Weiermann 1993).

J.B. Kruskal actually considered finite trees whose vertices are labeled from a wqo ≤*. The additional requirement on embeddings is that label(v) ≤* label(h(v)).

THEOREM 8.3. (Kruskal 1960). The qo of finite trees as posets, with vertices labeled from any given wqo, is a wqo.

Labeled KT is considerably stronger, proof theoretically, than KT, even with only 2 labels, $0 \leq 1$. We have not seen a metamathematical analysis of labeled KT.

Note that KT is a Π^1_1 sentence and labeled KT is a Π^1_2 sentence.

THEOREM 8.4. Labeled KT does not hold in the hyperarithmetic sets. In fact, RCA_0 + KT implies ATR_0.

A proof of Theorem 8.4 will appear in (Friedman, Montalban, Weiermann in preparation).

It is natural to impose a growth rate in KT in terms of the number of vertices of T_i.

COROLALRY 8.5. (Linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded sequence of finite trees. $\exists$ $i < j$ such that T_i is inf preserving embeddable into T_j.

COROLLARY 8.6. (Computational KT). Let $T_1,T_2,\ldots$ be a sequence of finite trees in a given complexity class. There exists $i < j$ such that T_i is inf preserving embeddable into T_j.

Note that Corollary 8.6 is Π^0_2.

THEOREM 8.7. Corollary 8.5 cannot be proved in FS. This holds even for linear bounds n+k with variable n and constant k.

THEOREM 8.8. Corollary 8.6 cannot be proved in FS, even for linear time, logarithmic space.

By an obvious application of weak Konig's lemma, Corollary 8.5 has very strong uniformities.

THEOREM 8.9. (Uniform linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded sequence of finite trees. There exists $i < j \leq n$ such that T_i is inf preserving embeddable into T_j, where n depends only on the given linear bound, and not on $T_1,T_2,\ldots$

With this kind of strong uniformity, we can obviously strip Theorem 8.9 of infinite sequences of trees. Using the linear bounds n+k, k fixed, we obtain:

THEOREM 8.10. (finite KT). Let n >> k. For all finite trees $T_1,...,T_n$ with each $|T_i| \leq i+k$, there exists $i < j$ such that T_i is inf preserving embeddable into T_j.

Since Theorem 8.10 → Theorem 8.9 → Corollary 8.5 (using bounds n+k, variable n, k constant), we see that Theorem 8.10 is not provable in FS.

Other Π^0_2 forms of KT involving only the internal structure of a single finite tree can be found in (Friedman 2002b).

We proved analogous results for EKT = extended Kruskal theorem, which involves a finite label set and a gap embedding condition. Only here the strength jumps up to that of Π^1_1-CA_0.

We said that the gap condition was natural (i.e., EKT was natural). Many people were unconvinced.

Soon later, EKT became a tool in the proof of the well known graph minor theorem of Robertson, Seymour (Robertson, Seymour 1985, 2004).

THEOREM 8.11. Let $G_1,G_2,...$ be finite graphs. There exists $i < j$ such that G_i is minor included in G_j.

We then asked Robertson and Seymour to prove a form of EKT that we knew implied full EKT, just from GMT. They complied, and we wrote the triple paper (Friedman, Robertson, Seymour 1987).

The upshot is that GMT is not provable in Π^1_1-CA_0. Just where GMT is provable is unclear, and recent discussions with Robertson have not stabilized. We disavow remarks in (Friedman, Robertson, Seymour 1987) about where GMT can be proved.

An extremely interesting consequence of GMT is the subcubic graph theorem. A subcubic graph is a graph where every vertex has valence ≤ 3. (Loops and multiple edges are allowed).

THEOREM 8.12. Let $G_1, G_2, \ldots$ be subcubic graphs. There exists $i < j$ such that G_i is embeddable into G_j as topological spaces (with vertices going to vertices).

Robertson, Seymour also claims to be able to use the subcubic graph theorem for linkage to EKT (see (Robertson, Seymour 1985), (Friedman, Robertson, Seymour 1987)). Therefore the subcubic graph theorem (even in the plane) is not provable in Π^1_1-CA_0.

We have discovered lengths of proof phenomena in wqo theory. We use Σ^0_1 sentences. See (Friedman 2006a-g).

*) Let $T_1, \ldots, T_n$ be a sufficiently long sequence of trees with vertices labeled from {1,2,3}, where each $|T_i| \leq i$. There exists $i < j$ such that T_i is inf and label preserving embeddable into T_j.

**) Let $T_1, \ldots, T_n$ be a sufficiently long sequence of subcubic graphs, where each $|T_i| \leq i+13$. There exists $i < j$ such that G_i is homeomorphically embeddable into G_j.

THEOREM 8.13. Every proof of *) in FS uses at least $2^{[1000]}$ symbols. Every proof of **) in Π^1_1-CA_0 uses at least $2^{[1000]}$ symbols.

9. BOREL SELECTION.

Let $S \subseteq \Re^2$ and $E \subseteq \Re$. A selection for A on E is a function $f:E \rightarrow \Re$ whose graph is contained in S.

A selection for S is a selection for S on $\mathfrak{R}$.

We say that S is symmetric if and only if $S(x,y) \leftrightarrow S(y,x)$.

THEOREM 9.1. Let $S \subseteq \mathfrak{R}^2$ be a symmetric Borel set. Then S or $\mathfrak{R}^2 \backslash S$ has a Borel selection.

My proof of Theorem 9.1 in (Friedman 1981) relied heavily on Borel determinacy, due to D.A. Martin. See (Martin 1975), (Martin 1985), and (Kechris 1994 137-148).

THEOREM 9.2. (Friedman 1981). Theorem 9.1 is provable in ZFC, but not without the axiom scheme of replacement.

There is another kind of Borel selection theorem that is implicit in work of Debs and Saint Raymond of Paris VII. They take the general form: if there is a nice selection for S on compact subsets of E, then there is a nice selection for S on E. See the five papers of Debs and Saint Raymond in the references.

THEOREM 9.3. Let $S \subseteq \mathfrak{R}^2$ be Borel and $E \subseteq \mathfrak{R}$ be Borel with empty interior. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

THEOREM 9.4. Let $S \subseteq \mathfrak{R}^2$ be Borel and $E \subseteq \mathfrak{R}$ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

THEOREM 9.5. (Friedman 2005). Theorem 9.3 is provable in ZFC but not without the axiom scheme of replacement. Theorem 9.4 is neither provable nor refutable in ZFC.

We can say more.

THEOREM 9.6. (Friedman 2005). The existence of the cumulative hierarchy up through every countable ordinal is sufficient to prove Theorems 9.1 and 9.3. However, the existence of the cumulative hierarchy up through any suitably defined countable ordinal is not sufficient to prove Theorem 9.1 or 9.3.

DOM: The $f:N \to N$ constructible in any given $x \subseteq N$ are eventually dominated by some $g:N \to N$.

THEOREM 9.7. ZFC + Theorem 9.4 implies DOM (Friedman 2005). ZFC + DOM implies Theorem 9.4 (Debs, Saint Raymond 2007).

10. BOOLEAN RELATION THEORY.

The principal reference for this section is the forthcoming book Friedman 2010.

We begin with two examples of statements in BRT of special importance for the theory.

THIN SET THEOREM. Let $k \geq 1$ and $f:N^k \to N$. There exists an infinite set $A \subseteq N$ such that $f[A^k] \neq N$.

COMPLEMENTATION THEOREM. Let $k \geq 1$ and $f:N^k \to N$. Suppose that for all $x \in N^k$, $f(x) > \max(x)$. There exists an infinite set $A \subseteq N$ such that $f[A^k] = N \setminus A$.

These two theorems are official statements in BRT. In the complementation theorem, A is unique.

We now write them in BRT form.

THIN SET THEOREM. For all $f \in MF$ there exists $A \in INF$ such that $fA \neq N$.

COMPLEMENTATION THEOREM. For all $f \in SD$ there exists $A \in INF$ such that $fA = N \setminus A$.

The thin set theorem lives in IBRT in A,fA. There are only $2^{2^{\wedge}2} = 16$ statements in IBRT in A,fA. These are easily handled.

The complementation theorem lives in EBRT in A,fA. There are only $2^{2^{\wedge}2} = 16$ statements in IBRT in A,fA. These are easily handled.

For EBRT/IBRT in A,B,C,fA,fB, fC,gA,gB,gC, we have $2^{2^{\wedge}9} = 2^{512}$ statements. This is entirely unmanageable. It would take several major new ideas to make this manageable.

DISCOVERY. There is a statement in EBRT in A,B,C,fA,fB, fC,gA,gB,gC that is independent of ZFC. It can be proved in SMAH+ but not in SMAH, even with the axiom of constructibility.

Here SMAH+ = ZFC + $(\forall n)(\exists \kappa)(\kappa$ is a strongly k-Mahlo cardinal). SMAH = ZFC + $\{(\exists \kappa)(\kappa$ is a strongly k-Mahlo cardinal$\}_k$.

The particular example is far nicer than any “typical” statement in EBRT in A,B,C,fA,fB,fC,gA,gB,gC. However, it is not nice enough to be regarded as suitably natural.

Showing that all such statements can be decided in SMAH+ seems to be too hard.

What to do? Look for a natural fragment of full EBRT in A,B,C,fA,fB,fC,gA,gB,gC that includes the example, where we can decide all statements in the fragment within SMAH+.

We also look for a bonus: a striking feature of the classification that is itself independent of ZFC. Then we have a single natural statement independent of ZFC.

In order to carry this off, we need to use the function class ELG of functions ofs expansive linear growth.

These are functions $f:N^k \to N$ such that there exist constants $c,d > 1$ such that

$$c|x| \leq f(x) \leq d|x|$$

holds for all but finitely many $x \in N^k$.

TEMPLATE. For all $f,g \in ELG$ there exist $A,B,C \in INF$ such that

$$X \cup. fY \subseteq V \cup. gW$$

$$P \cup. fR \subseteq S \cup. gT.$$

Here X,Y,V,W,P,R,S,T are among the three letters A,B,C.

Note that there are 6561 such statements. We have shown that all of these statements are provable or refutable in RCA_0, with exactly 12 exceptions.

These 12 exceptions are really exactly one exception up to the obvious symmetry: permuting A,B,C, and switching the two clauses.

The single exception is the **exotic case:**

PROPOSITION A. For all $f,g \in ELG$ there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$

$$A \cup. fB \subseteq C \cup. gC.$$

This statement is provably equivalent to the 1-consistency of SMAH, over ACA'.

If we replace "infinite" by "arbitrarily large finite" then we can carry out this second classification entirely within RCA_0.

Inspection shows that all of the non exotic cases come out with the same truth value in the two classifications, and that is of course provable in RCA_0.

Furthermore, the exotic case comes out true in the second classification.

THEOREM 10.1. The following is provable in SMAH+ but not in SMAH, even with the axiom of constructibility. An instance of the Template holds if and only if in that instance, "infinite" is replaced by "arbitrarily large finite".

11. FINITE INCOMPLETENESS.

Here we present some Fixed Point Propositions, involving operators on subsets of Q^k, where Q is the rationals. These Propositions cannot be proved in ZFC. This development leads to a finite form that is explicitly Π^0_1.

For more details, including further results involving much larger cardinals, see Friedman 2009.

We caution the reader that this is intensively ongoing research, which has not been published. We expect a submission for publication by early 2010.

We say that $R \subseteq Q^k \times Q^k$ is strictly dominating if and only if $R(x,y) \rightarrow \max(x) < \max(y)$. We say that $E \subseteq Q^k$ is order invariant if and only if membership in E depends only on the order type of $x \in Q^k$. We say that $R \subseteq Q^k \times Q^k$ is order invariant if and only if R is order invariant as a subset of Q^{2k}. For $A \subseteq Q^k$, we write R[A] for the image of A under R.

We write $SDOI(Q^k)$ for the family of all strictly dominating order invariant $R \subseteq Q^k \times Q^k$.

The upper shift of $x \in Q^k$, is defined as us(x) = x+1 if $x \geq 0$; x if $x < 0$. For $x \in Q^k$, us(x) is obtained from x by applying us coordinatewise. For $A \subseteq Q^k$, we define us(A) = {us(x): $x \in A$}.

For $A \subseteq Q^k$, write cube(A,0) for the least set V^k such that $A \subseteq V^k \wedge 0 \in V$. We use \ for set theoretic difference.

UPPER SHIFT FIXED POINT PROPOSITION. For all $R \in SDOI(Q^k)$, some A = cube(A,0)\R[A] contains us(A).

THEOREM 11.1. The Upper Shift Fixed Point Proposition is provable in SUB+ but not in any consistent fragment of SUB. It is provably equivalent to Con(SUB), in WKL_0.

Here SUB+ = ZFC + "for all k there exists a k-subtle cardinal". SUB = ZFC + $\{$there exists a k-subtle cardinal$\}_k$. For the definition of k-subtle cardinal, due to J. Baumgartner, see Friedman 2001. WKL0 is one of the principal five systems of Reverse Mathematics. See Simpson 1999.

SEQUENTIAL UPPER SHIFT FIXED POINT PROPOSITION. For all $R \in SDOI(Q^k)$, there exist finite $A_1,A_2,... \subseteq Q^k$ such that for all $i \geq 1$, $A_i \cup us(A_i) \subseteq A_{i+1} = cube(A_{i+1},0)$.

FINITE SEQUENTIAL UPPER SHIFT FIXED POINT PROPOSITION. For all $R \in SDOI(Q^k)$, there exist finite $A_1,...,A_k \subseteq Q^k$ such that for all $1 \leq i \leq k-2$, $A_i \cup us(A_i) \subseteq A_{i+1} = cube(A_{i+1},0)$.

ESTIMATED UPPER SHIFT FIXED POINT PROPOSITION. For all $R \in SDOI(Q^k)$, there exist finite $A_1,...,A_k \subseteq Q^k$ such that for all $1 \leq i \leq k-2$, $A_i \cup us(A_i) \subseteq A_{i+1} = cube(A_{i+1},0)$, where the numerators and denominators used have magnitude at most (8k)!.

Note that the second of these is explicitly Π^0_2 and the third of these is explicitly Π^0_1.

THEOREM 11.2. WKL_0 proves that all four Upper Shift Fixed Point Propositions are equivalent. In particular, Theorem 11.1 applies to both Sequential Upper Shift Fixed Point Propositions and the Estimated Upper Shift Fixed Point Proposition.

12. INCOMPLETENESS IN THE FUTURE.

The Incompleteness Phenomena, the centerpiece of Gödel's legacy, has come a long way. The same is true of the related phenomenon of recursive unsolvability, also part of the Gödel legacy. The phenomena is so deep, and rich in possibilities, that we expect the future to eclipse the past and present.

Yet continued substantial progress is expected to be painfully slow, requiring considerably more than the present investment of mathematical and conceptual power devoted to the extension and expansion of the phenomena.

This assessment also applies, if we consider the P = NP problem as part of the Gödel legacy (as is common today) on the basis of his letter of March 20, 1956, to John von Neumann (see (Gödel 1986-2003 Vol. V, letter 21, 373-377).

Also consider the recursive unsolvability phenomena. Perhaps the most striking example of this for the working mathematician is the recursive unsolvability of Diophantine problems over the integers (Hilbert's tenth problem), as discussed in section 3. We have, at present, no idea of the boundary between recursive decidability and recursive undecidability in this realm. Yet I conjecture that we will understand this in the future, and that we will find, perhaps, that recursive undecidability kicks in already for degree 4 with 4 variables. However, this would require a complete overhaul of the current solution to Hilbert's tenth problem, replete with new deep ideas. This would result in a sharp increase in the level of interest for the working mathematician who is not particularly concerned with issues in the foundations of mathematics.

In addition, we still do not know if there is an algorithm to decide whether a Diophantine problem has a solution over the rationals. I conjecture that this will be answered in the negative, and that the solution will involve some clever number theoretic constructions of independent interest for number theory.

We now come to the future of the Incompleteness Phenomena. We have seen how far this has developed thus far:

i. First Incompleteness. Some incompleteness in the presence of some arithmetic. (Gödel 1931).

ii. Second Incompleteness. Incompleteness concerning the most basic metamathematical property - consistency. (Gödel 31), (Hilbert Bernays 1934,1939), (Feferman 1960), (Boolos 93).

iii. Consistency of the AxC. Consistency of the most basic, and once controversial, early candidate for a new axiom of set theory. (Gödel 1940).

iv. Consistency of the CH. Consistency of the most basic set theoretic mathematical problem highlighted by Cantor. (Gödel 1940).

v. $\in_0$ consistency proof. Consistency proof of PA using quantifier free reasoning on the fundamental combinatorial structure, $\in_0$. (Gentzen 1969).

vi. Functional recursion consistency proof. Consistency proof of PA using higher type primitive recursion, without quantifiers. (Gödel 1958), (Gödel 1972).

vii. Independence of AxC. Independence of CH (over AxC). Complements iii,iv. (Cohen 1963-1964). Forcing.

viii. Open set theoretic problems in core areas shown independent. Starting soon after (Cohen 1963-1964), starting dramatically with R.M. Solovay (e.g., his work on Lebesgue measurability (Solovay 1970), and his independence proof of Kaplansky's Conjecture (Dales, Woodin 1987)), and continuing with many others. See the rather comprehensive (Jech 2006). Also see the many set theory papers in (Shelah, 1969-2007).

Core mathematicians have learned to avoid raising new set theoretic problems, and the area is greatly mined.

ix. Large cardinals necessarily used to prove independent set theoretic statements. Starting dramatically with measurable cardinals implies $V \neq L$ (Scott 1961). Continuing with solutions to open problems in the theory of projective sets (using large cardinals), culminating with the proof of projective determinacy, (Martin, Steel 1989).

x. Large cardinals necessarily used to prove the consistency of set theoretic statements. See (Jech 2006).

xi. Uncountably many iterations of the power set operation necessarily used to prove statements in and around Borel mathematics. See (Friedman 1971), (Martin 1975), (Friedman 2005), (Friedman 2007b). Includes Borel determinacy, and some Borel selection theorems of Debs and Saint Raymond (see section 9 above).

xii. Large cardinals necessarily used to prove statements around Borel mathematics. (Friedman 1981), (Stanley 85), (Friedman 2005), (Friedman 2007b). Includes some Borel selection theorems of Debs and Saint Raymond (see section 9 above and the references to Debs and Saint Raymond).

xiii. Independence of finite statements in or around existing combinatorics from PA and subsystems of second order arithmetic. Starting with (Goodstein 1944), (Paris, Harrington 1977), and, most recently, with (Friedman 2002b), and (Friedman 2006a-g). Uses extensions of v) above, (Gentzen, 1969), from (Buchholz, Feferman, Pohlers, Seig 1981). Includes Kruskal's theorem, the graph minor theorem of Robertson, Seymour (Robertson, Seymour, 1985, 2004), and the trivalent graph theorem of Robertson, Seymour (Robertson, Seymour, 1985).

xiv. Large cardinals necessarily used to prove sentences in discrete mathematics, as part of a wider theory (Boolean Relation Theory). (Friedman 1998), and (Friedman 2010).

xv. Large cardinals necessarily used to prove explicitly Π^0_1 sentences. See section 11 above for the current state of the art.

Yet this development of the Incompleteness Phenomena has a long way to go before it realizes its potential to dramatically penetrate core mathematics.

However, I am convinced that this is a matter of a lot of time and resources. The quality man/woman hours devoted to expansion of the incompleteness phenomena is trivial when compared with other pursuits. Even the creative (and high quality) study of U.S. tax law dwarfs the effort devoted to expansion of the incompleteness phenomena by orders of magnitude - let alone any major sector of technology, particularly the development of air travel, telecommunications, or computer software and hardware.

Through my efforts over 40 years, I can see, touch, and feel a certain combinatorial structure that keeps arising - a demonstrably indelible footprint of large cardinals. I am able to display this combinatorial structure through Borel, and discrete, and finitary statements that are increasingly compelling mathematically.

But I don't quite have the right way to express it. I likely need some richer context than the completely primitive combinatorial settings that I currently use. This difficulty will definitely be overcome in the future, and that will make a huge difference in the quality, force, and relevance of the results to mathematical practice.

In fact, I will go so far as to make the following dramatic conjecture. It's not that the incompleteness phenomena is a freak occurrence. Rather, it is everywhere.

Every interesting substantial mathematical theorem can be recast as one among a natural finite set of statements, all of which can be decided using well studied extensions of ZFC, but not all of which can be decided within ZFC itself.

Recasting of mathematical theorems as elements of natural finite sets of statements represents an inevitable general expansion of mathematical activity. This, I conjecture, will apply to any standard mathematical context.

This program has been carried out, to some very limited extent, by BRT – as can be seen in section 10 above.

This may seem like a ridiculously ambitious conjecture, which goes totally against the current conventional wisdom of mathematicians - who think that they are immune to the incompleteness phenomena.

But I submit that even fundamental features of current mathematics are not likely to bear much resemblance to the mathematics of the future.

Mathematics as a professional activity with serious numbers of workers, is quite new. Let's say 100 years old – although even that is a stretch.

Assuming the human race thrives, what is this compared to, say, 1000 more years? Probably merely a bunch of simple observations in comparison.

Of course, 1000 years is absolutely nothing in evolutionary or geological time. A more reasonable number is 1M years. And what does our present mathematics look like compared to that in 1M years time? These considerations should apply to our present understanding of the Gödel phenomena.

We can of course take this even further. 1M years time is absolutely nothing in astronomical time. This Sun has several billion good years left (although the Sun will cause a lot of global warming!).

Mathematics in 1B years time? Who can know what that will be like. But I am convinced that the Gödel legacy will remain very much alive – at least as long as there is vibrant mathematical activity.

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