FROM RUSSELL'S PARADOX TO HIGHER SET THEORY

by

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ABSTRACT: Russell's way out of his paradox via the impredicative theory of types has roughly the same logical power as Zermelo set theory - which supplanted it as a far more flexible and workable axiomatic foundation for mathematics. We discuss some new formalisms that are conceptually close to Russell, yet simpler, and have the same logical power as higher set theory - as represented by the far more powerful Zermelo-Frankel set theory and beyond. END.

The famous Russell's paradox for sets arises out of the intuitively appealing naive principle of full comprehension written

$$(\exists x)(\forall y)(y \in x \leftrightarrow \varphi),$$

where φ is a logical formula involving $\neg, \rightarrow, \&, \vee, \leftrightarrow, \forall, \exists, \in$, and variables in which x is not free. The variables are thought of as ranging over sets. Here x,y are any distinct variables.

In particular, the simple special case

$$(\exists x)(\forall y)(y \in x \leftrightarrow y \notin y)$$

generates the inconsistency by fixing such an x and universally instantiating y by x, thus obtaining

$$x \in x \leftrightarrow x \notin x.$$

Russell's Paradox also makes perfectly good sense in other contexts besides set theory – e.g., in a theory of predicates. Here the naive comprehension axiom scheme takes the notationally similar but distinct form

$$(\exists P)(\forall Q)(P(Q) \leftrightarrow \varphi),$$

where □ is a logical formula involving □,□,&,v,□,□ ,□ ,(), and variables, where P is not free in □. This time the variables are thought of as ranging over predicates. Here P,Q are any distinct variables.

Why do I draw a distinction between the set version of Russell's Paradox and the predicate version? One big difference between the way sets and predicates are thought of is in terms of extensionality. We have pre-ferred not to use equality as primitive, and so extension-ality for sets takes on the form

(□z)(z □ x □ z □ y) □ (□z)(x □ z □ y □ z).

Extensionality is in accordance with the usual way of thinking about sets.

However, the corresponding statement about predicates,

(□R)(P(R) □ Q(R)) □ (□R)(R(P) □ R(Q)),

does not seem appropriate. For instance, one may distinguish different predicates P,Q such that P,Q fail at every argument, where P,Q can be distinguished by another predicate R. Then we have (□R)(P(R) □ Q(R)) and yet □(□R)(R(P) □ R(Q)).

Yet another important distinction between certain notions of predication and sets will be relevant later in the discussion.

Frequently people still use the notation x □ y even if x,y are predicates and we are asserting that y holds at x. We will follow this convention. Then the distinction between predicates and sets may come at the point at which we decide to include or exclude EXT.

Now after his discovery of Russell's Paradox, Russell embarked on a series of developments to repair the damage. It is customary to identify two proposals of Russell to get out from under his Paradox.

Russell's proposed ways out go under the name of various theory of types, and have been simplified and streamlined in many ways since Russell. Russell actually used a very complicated type structure.

In modern treatments of the theory of types, one usually goes to the simplest possible type structure (with infinitely many types). This is where the types are identified with nonnegative integers. Membership x y is considered well formed only if the type of y is exactly one higher than the type of x. These types appear explicitly as superscripts on the variables.

Regardless of the type structure chosen, Russell's original proposal is now identified with the predicative theory of types, which we shall call PTT. As we shall see, the predicativity here refers to the decision to avoid any semblance of the circularity in the inconsistent full comprehension axiom scheme.

More specifically, Russell's first proposal is identified with PTT with extensionality (EXT) and infinity (INF), presented below. And this is normally cast in the simple theory of types. However, Russell realized that this predicative approach proved to be too draconian, and allowed no way of formalizing, say, the least upper bound principle for the real numbers (every nonempty bounded set of real numbers has a least upper bound).

(In connection with this point, there are modern formalizations of mathematics that avoid such things as the least upper bound principle for the real numbers. But a system such as PTT + EXT + INF is still woefully inadequate; e.g., with respect to Cantor's theory of closed subsets of the real line.)

Now under this version of events, Russell addressed this lack of power by moving to the impredicative theory of types, IPTT. Thus it is customary to identify Russell's second proposal with IPTT + EXT + INF.

However, in actuality, Russell explicitly introduced a principle of reducibility (RED) to augment the predicative theory of types in his first proposal. In Russell's context, one then derives the impredicative theory of types from the predicative theory of types and RED.

Granted, RED is equivalent to IPTT over PTT. But that does not mean that there isn't something to learn by first formulating and dwelling on PTT + EXT + INF + RED before passing to IPPT + EXT + INF.

However, RED doesn't seem to have a natural formulation in the simple theory of types independently of just going directly to IPTT. So RED gets put into the invisible back-ground if the simple theory of types is used.

But in the context of a more complicated type structure than the simple theory of types, RED has an appropriate formalization. In particular, we use the cumulative type structure.

In the cumulative type structure, or cumulative theory of types, $x \in y$ is well defined exactly when the type of y is strictly greater than the type of x; not just exactly 1 higher than the type of x.

In this context of the cumulative theory of types, we can identify Russell's second proposal with the system PTT + EXT + INF + RED, based on the cumulative theory of types.

In fact, Russell's idea of Reducibility receded even further into the background upon the next major development in axiomatic foundations of mathematics.

This was the move by Zermelo to abolish all type theory in favor of a single sorted theory where all variables range over sets (no types). Zermelo set theory supports a much more elegant and flex-ible formalization of mathematics than any form of type theory.

Zermelo's set theory was further extended by Frankel to include the axiom scheme of Replacement. With some further additions, ZFC emerged (Zermelo-Frankel set theory with the axiom of Choice), which is now considered to be the standard vehicle for the formalization of mathematics.

As you can see, post Russellian events as well as streamlined history pushes us further away from the conceptions of Russell with his type theory and axiom of Reducibility. In addition, there is the general feeling that one loses philosophical and conceptual purity in the process. So maybe there is something to be gained by trying to go back to Rus-sell's way of looking at things. This seems particularly tantalizing in light of the fact that the philosophical underpinnings of higher set theory are so murky – often disguised in an overly cluttered technical frame-work.

We have now laid the ground-work for the presentation of our new formalisms. However before doing this, we fulfill our promise of presenting PTT, IPTT, EXT, INF, and RED in the context of simple and cumulative type theory.

Simple type theory is based on infinitely many variables of each type, where the types are identified with the non-negative integers. Variables of sort $n \geq 0$ are written x^n.

PTT in simple type theory is the following scheme:

$$(\Box x^{n+1})(\Box y^n)(y^n \Box x^{n+1} \Box \Box),$$

where all variables occurring in $\Box$ are of type $\Box$ n.

Actually, it is natural to relax this condition somewhat by insisting only that all bound variables occurring in $\Box$ are of sort $\Box$ n, and all free variables occurring in $\Box$ are of sort $\Box$ n+1.

If $\Box$ is unrestricted (except we always need that x^{n+1} is not free in $\Box$), then we get IPTT.

Extensionality (EXT) is formulated in the simple theory of types as follows:

$$(\Box z^n)(z^n \Box x^{n+1} \Box z^n \Box y^{n+1}) \Box (\Box z^{n+2})(x^{n+1} \Box z^{n+2} \Box y^{n+1} \Box z^{n+2})$$

Infinity (INF) is formulated in the simple theory of types as follows:

$$(\Box x^2)(\Box y^1)(y^1 \Box x^2 \ \& \ (\Box z^1 \Box x^2)(\Box w^1 \Box x^2)(z^1 \Box w^1)).$$

Here $\Box$ denotes proper inclusion.

We now give the formulation in the cumulative theory of types. The cumulative theory of types is also based on infinitely many variables of each sort, where the sorts are identified with the nonnegative integers. Also variables of sort $n \geq 0$ are written x^n. However, the atomic formulas are more comprehensive; they are of the form $x^n \Box y^m$, where $n < m$.

PTT in the cumulative theory of types takes on the form

$$(\Box x^n)((\Box y^0)(y^0 \Box\ x^n \Box\ \Box_0)\ \&\ldots\ \&\ (\Box y^{n-1})(y^{n-1} \Box\ x^n \Box\ \Box_{n-1})),$$

where n > 0 and where all variables occurring in $\Box$ are of type < n.

Actually, it is natural to relax this condition somewhat by insisting only that all bound variables occurring in $\Box$ are of type < n, and all free variables occurring in $\Box$ are of type $\Box$ n.

EXT in the cumulative theory of types is:

$$(\Box z^0)(z^0 \Box\ x^n \Box\ z^0 \Box\ y^n)\ \&\ \ldots\ \&\ (\Box z^{n-1})(\Box z^{n-1} \Box\ x^n \Box\ z^{n-1} \Box\ y^n)\ \Box\ (\Box z^m)(x^n \Box\ z^m \Box\ y^n \Box\ z^m),$$

where 0 < n < m.

INF in the cumulative theory of types is exactly the same as in the simple theory of types.

And now for a punch line. The axiom of Reducibility, RED, in the cumulative theory of types is:

$$(\Box x^n)(\Box y^m)((\Box z^0)(z^0 \Box\ x^n \Box\ z^0 \Box\ y^m)\ \&\ \ldots\ \&\ (\Box z^{m-1})(z^{m-1} \Box\ x^n \Box\ z^{m-1} \Box\ y^m)),$$

where n > m.

Thus RED asserts that by pas-sing to higher types, one gets nothing really new about lower types. E.g., given x of type 8, there is always a y of type 4 such that x and y have the same elements of each of types 0,1,2, and 3.

As indicated earlier, one easily proves that PTT + RED = IPTT. For this reason, people focus on the simpler IPTT as Russell’s main proposal, and also use the simple theory of types rather the cumulative theory of types.

We have now set the stage for the discussion of one of our new formalisms, K(W). The language is that of first order predicate calculus based on $\Box$ and the constant symbol W. W represents a first set theoretic universe, whereas the variables in K(W) range over the elements of a second, larger, set theoretic universe.

The rationale for having two set theoretic universes, and the conceptual relationship between these two universes is particularly crucial in the formulation of the third axiom of K(W), and we post-pone the discussion of this until after we have presented the first two axioms of K(W).

The first axiom of K(W) is just the usual axiom of extensionality:

EXT. (□z)(z □ x □ z □ y) □ (□z)(x □ z □ y □ z).

The second axiom (scheme) of K(W) is a strong formulation of the characteristic move of Zermelo adapted to this context of two set theoretic universes. We call this axiom scheme Subworld Separation:

SS. (□x □ W)(□y □ W)(□z)(z □ y □ (z □ x & □)), where □ is a formula in L(□,W) without y free.

Informally, SS says that for any x □ W, we can form the set {z □ x: □(z)}, and know that it is an element of W. Here □ may mention W as well as arbitrary side parameters. If we were to restrict □ to be a formula in L(□), then we would arrive at an equivalent scheme, since W can be used as a parameter.

So far, the two EXT and SS, consitute the major characteristic move of Zermelo that lead from Russell's theory of types to axiomatic set theory, naturally adapted to the context of two set theoretic universes. And EXT does not even need to be adapted.

In fact, EXT + SS do not even suffice to prove the existence of more than two objects. It has a plethora of finite models.

So at this point you must be skeptical that we are going to get something very far reaching by simply adding one more axiom (scheme)!

The third and final axiom (scheme) of K(W) requires more discussion, and is based on a conceptualization of the relationship between the first set theoretic universe W, and the second set theoretic universe over which the variables of the theory range.

The general philosophy is that at any time, we can attempt to conceive of the entire set theoretic universe. But according to Russell's Paradox, as we later conceive of the entire set theoretic universe, reflecting on the previous conception, we obtain a yet larger set theoretic universe. However, according to this general philosophy, nothing really new is obtained – the first set theoretic universe is, for all intents and purposes, the same as the second set theoretic universe, even though it is not literally the same.

An important way to think of this sameness is geometric-ally. We can view both of these set theoretic universes as extending indefinitely far out into the horizon. But the second set theoretic universe is longer than the first one. Nevertheless the effect of the two horizons is that both set theoretic universes look the same in isolation.

In fact, we can go even further. If we stand at any place in the first set theoretic universe and look look at the entire first set theoretic universe (up and down), what we see is the same thing as if we were to look at the second set theoretic universe (up and down).

This a very familiar idea in geometry. For example, consider the first interval [0,1) and the second interval [0,2). Clearly from the point of view of any $x \in [0,1)$, [0,1) and [0,2) look the same.

This is normally expressed mathematically by the existence of an order isomorphism from [0,1) to [0,2) that sends x to x. This can be expressed formally by introducing a unary function symbol representing an isomormorphism. We can do this for all $x \in [0,1)$ by adding a binary function symbol, with the obvious axiom that for any choice of first argument in the first interval, the cross section represents an order isomorphism between the first and second interval that fixes the chosen first argument.

This line of investigation leads to a whole series of new geometrically motivated axiomatic set theories involving the addition of virtual (cross sectional) isomorphisms, which now appears to me to interpret not only ZFC but its extensions by the higher axioms of infinity (large cardinals axioms) such as the existence of a measurable cardinal.

Let me stop discussion of this line of investigation by calling this “geometric set theory” and “geometric foundations of set theory.” Just as there were very surprising and very significant completeness theorems in geometry due to Tarski, we expect completeness theorems in geometric set theory (for statements of limited complexity) that establish the unique position of certain axiomatic set theories in the logical universe.

Now let’s get back to the presentation of the third axiom (scheme) of K(W). Here we will not expand the language past L(□,W). Going back to the interval example with [0,1) and [0,2), we can formulate the geometric idea we have been discussing more awkwardly as follows. Let x □ [0,1). We can assert that, given any true statement internal to [0,1) mentioning x and <, the exactly corresponding statement internal to [0,2) mentioning x and < is also true. And we might as well state this for any $x_1,...,x_k$ □ [0,1).

This somewhat awkward way of stating this geometric idea immediately leads to a corresponding formulation of our long awaited third axiom (scheme) of K(W). However, the final formulation of the third axiom (scheme) will be simpler and more direct. This intermediate formulation reads as follows:

ES. (□$x_1,...,x_k$ □ W)(□^{W} □ □), where $k \geq 0$ and □ is a formula in L(□) with at most the free variables $x_1,...,x_k$.

Here □^{W} is the result of relativizing all quantifiers in □ to W.

ES stands for “elementary submodel” which makes perfectly good sense to mathematical logicians, as it is a fundamental notion in model theory.

Now ES is outright logically equivalent to the following:

RED. (□$x_1,...,x_k$ □ W)((□y)(□) □ (□y □ W)(□)), where $k \geq 0$ and □ is a formula in L(□) with at most the free variables $x_1,...,x_k$.

But this has the very same flavor as Russell Reducibility (hence the name RED). We have come full circle right back to Russell!

Let us recapitulate. The axioms of K(W) are:

1. EXT. $(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow (\forall z)(x \in z \rightarrow y \in z)$.
2. SS. $(\forall x \in W)(\exists y \in W)(\forall z)(z \in y \leftrightarrow (z \in x \;\&\; \varphi))$, where φ is a formula in $L(\in,W)$ without y free.
3. RED. $(\forall x_1,\ldots,x_k \in W)((\exists y)(\varphi) \rightarrow (\exists y \in W)(\varphi))$, where $k \geq 0$ and φ is a formula in $L(\in)$ with at most the free variables $x_1,\ldots,x_k$.

K(W) is a very strong system.

The surprise is that K(W) interprets ZFC (Zermelo Frankel set theory with the axiom of Choice). In fact, K(W) proves outright a significant fragment of ZFC that is well known to be sufficient to interpret ZFC. Specifically, K(W) proves all of ZFC except for the axioms of Foundation and Choice. And every theorem of K(W) that doesn't mention W can be proved in ZFC – in fact, without the axioms of Foundation and Choice.

What is particularly remarkable here is the combined effect of SS and RED, in the presence of EXT. Zermelo's SS pushed Russell's RED into the background. Each alone does not really go very far. SS can't prove the existence of more than two objects, and has plenty of finite models. And RED has basic geometric models such as [0,2) with W = [0,1), and $\in$ as <. (One can even restrict to rational points only). Yet when com-bined, they generate a system which interprets (and cor-responds very closely to) the currently accepted generous formalization for the whole of mathematics!

Now let us go back to the deep well of geometry. Recall the discussion of [0,1) and [0,2) relative to any $x_1,\ldots, x_k \in [0,1)$. We said that given any true statement internal to [0,1) mentioning $x_1,\ldots,x_k$, the exactly corresponding statement internal to [0,2) mentioning $x_1,\ldots,x_k$ is also true.

A geometer might say some-thing much stronger. ***Given any statement about [0,1) mentioning $x_1,\ldots,x_k \in [0,1)$ and <, the exactly corresponding statement about [0,2) mentioning $x_1,\ldots,x_k$ is also true.***

Now you might object to this by saying that "the right endpoint of [0,1) is 1" and "the right endpoint of [0,2) is 2." But the geometer thinks of [0,1) and [0,2) as abstract geometric objects.

In fact, both intervals are included in [0,3), and there is an order automorphism of [0,3) that sends [0,1) onto [0,2) and fixes any given finite list of $x_1,\ldots,x_k$ □ [0,3).

This geometric idea leads directly to various. One of them is particularly elegant.

The axioms of $T(W_1,W_2)$ are:

1. EXT.
2. COMP. (□x □ W_2)(□y □ W_1)(y □ x □ □), where □ is a formula in L(□,W_1,W_2) in which x is not free.

RES. (□$x_1,\ldots,x_k$ □ W_1)(□(W_2) □ □(W_1)), where k ≥ 0 and □ is a formula in L(□,W_2) with at most the free variables $x_1,\ldots,x_k$.

Here COMP means “comprehension” and RES means “resemblance.” We have deliberately formulated RES so that it again looks like a form of Russell Reducibility! And now COMP looks like Russell’s IPTT (with only two or three types)!

And what is the big surprise? $T(W_1,W_2)$ goes far beyond even ZFC, and into the depths of the mysterious large cardinal hierarchy. Roughly at the level of strong forms of indescribable cardinals. And it’s all surprisingly close to Russell!!

We anticipate a whole new chapter in axiomatic set theory where geometric ideas are transferred into the set theoretic context, thereby generating corresponding axiomatic set theories. We conjecture that all large cardinal axioms are so geometrically generated. And the geometrically generated set theories might have a natural limit determined by fundamental geometric considerations.

We now formulate a basic principle of mathematics and philosophy.

AXIOM. For every theorem there is a better theorem. For every analysis there is a better analysis.

Recall K(W) = EXT + SS + RED. Now consider SS + RED.

We have now been able to show that even this system can interpret all of ZFC! SS is Zermelo whereas RED is Russell.

Also recall $K(W_1,W_2)$ = EXT + COMP + RES. This is particularly Russellian, as is $K(W_1,W_2,...)$. Again, we can show that EXT can be dropped, while preserving the strength of these systems, which is well beyond ZFC and into the depths of the large cardinal hierarchy.

We also have alluded to a reason to be interested in dropping extensionality - it is inappropriate for the theory of predicates. Yet SS + RED seem to make sense in the context of predicates as well as in the context of sets. In a sense, one is con-sidering limited domains of applicability when interpreting these remaining two axioms set theoretically.

We can take these ideas further by distinguishing two forms of predication. The first form is where one does not require that a predicate be fully defined; it may reference a finite list of objects as parameters. This can be referred as mathematical predication.

A second kind of predication is pure predication, where one is not allowed to reference objects as parameters. Any parameters referenced would have to be removed in favor of a description.

Under pure predication, the Subworld Separation axiom scheme has to restricted so that □ has at most the free variable x. However, an additional restriction on SS is warranted – that x be given by an explicit definition without parameters. In addition, some restrictions on RED may also be appropriate, although it is less clear how this is to be determined.

We now conjecture that SS and RED alone, even under such restrictions, is sufficiently strong to provide an inter-pretation of ZFC.

We already now that under such restrictions on parameters, the system $T(W_1,W_2)$ can interpret ZFC + indescribable cardinals.

A natural extension of these ideas is to consider $T(W_1,W_2,...)$ based on the language $L(□,W_1,W_2,...)$, which uses □ and the constant symbols $W_1,W_2,...$. The axioms are:

1. Extensionality.
2. COMP.

RES. (□x_1,...,x_n □ W_r) □ (□(W_{r+1}) □ □(W_r)), where n ≥ 0 and □ in L(□,W_{r+1}).

The geometry can be given by the intervals [0,1),[0,2), ..., all sitting inside [0,).

This system corresponds to subtle cardinals of finite order in ZFC, which goes even deeper into the mysterious large cardinal hierarchy. But look! A type structure has reemerged – W_1,W_2,..., - and moreover we are using Russel-lian Comprehension, and RES is clearly a form of Reducibility. So this is Russell all over again!

Thank you.