ADVENTURES IN LOGIC FOR UNDERGRADUATES

by
Harvey M. Friedman
Distinguished University Professor
Mathematics, Philosophy, Computer Science
Ohio State University

Lecture 4. Gödel's Blessing and Gödel's Curse

LECTURE 1. LOGICAL CONNECTIVES. Jan. 18, 2011

LECTURE 2. LOGICAL QUANTIFIERS. Jan. 25, 2011

LECTURE 3. TURING MACHINES. Feb. 1, 2011

LECTURE 4. GÖDEL'S BLESSING AND GÖDEL'S CURSE.
Feb. 8, 2011

LECTURE 5. FOUNDATIONS OF MATHEMATICS
Feb. 15, 2011

SAME TIME - 10:30AM

SAME ROOM - Room 355 Jennings Hall

WARNING: CHALLENGES RANGE FROM EASY, TO MAJOR PARTS OF COURSES

KURT GÖDEL

Kurt Gödel (1906-1978) is responsible for the most celebrated results in mathematical logic of the 20th century.

He had a singular ability to identify and focus on the central issues in logic.

He combined powerful mathematical thinking with philosophical insights to transform logic into a highly sophisticated mathematical subject of great general intellectual interest.

His doctoral dissertation (Habilitationsschrift) was accepted by the University of Vienna in 1932.

He came to the USA in 1933 as a Visiting Professor at the Institute for Advanced Studies in Princeton, New Jersey, where he remained as a Professor until his death in 1978.

GÖDEL'S BLESSING: COMPLETENESS

Gödel's Blessing is Completeness. Gödel's Curse is Incompleteness.

We have already encountered completeness theorems in the first two lectures. We won't be relying on the first two lectures.

The most relevant completeness theorem goes back to Gödel in his dissertation, and was discussed in general terms in the second lecture.

We do need to give a high level review of Gödel's famous COMPLETENESS THEOREM FOR PREDICATE CALCULUS in order to set the stage for the main business of this lecture.

REVIEW OF PREDICATE CALCULUS - COMPLETENESS THEOREM

In predicate calculus, we have assertions (formulas) and mathematical structures (models). The assertions and the structures are required to be of a specific kind which we will review in the next two slides.

Formulas are true or false only relative to structures. Thus we speak of a given formula as being true in a given structure.

PREDICATE CALCULUS COMPLETENESS THEOREM (Gödel 1928). There is a basic finite set of axioms and rules of inference, operating on the formulas, such that the following holds. A formula is true in all structures iff it is provable in this system.

You DON'T NEED TO UNDERSTAND this brief review of predicate calculus to take advantage of the remainder of this lecture.

REVIEW OF PREDICATE CALCULUS - FORMULAS

From Lecture 2: Formulas in predicate calculus use

i. Variables. $v_1, v_2, \ldots$

ii. Connectives. $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

iii. Quantifiers. $\forall, \exists$
iv. Constant symbols. $c_1, c_2, \ldots$
v. Relation symbols. R^n_m, $n,m \geq 1$
vi. Function symbols. F^n_m, $n,m \geq 1$
vii. Equality. $=$

Example from Lecture 2 of a formula of predicate calculus:

$$(\forall x)(\neg R(x,y) \rightarrow (\exists z)(F(z,H(y),c) = d \vee \neg S(z,c)))$$

x,y,z, variables; c,d, constant symbols; R,S, 2-ary relation symbols; H, 1-ary function symbol, F, 3-ary function symbol.

REVIEW OF PREDICATE CALCULUS – STRUCTURES

From Lecture 2: Structures in the predicate calculus take the form

$$(D,=,d_n,f^n_m,r^n_m)$$

where D is a nonempty set (the domain), c_n is interpreted by the element d_n of D, F^n_m is interpreted by the function $f^n_m:D^n \to D$, and R^n_m is interpreted by the relation $r^n_m \subseteq D^n$. In particular applications, only certain constant, function, and relation symbols are actually used.

Usually = is left out, while allowing = in formulas as we do. Thus you see

$$(D,d_n,f^n_m,r^n_m)$$

COMPLETENESS THEOREM FOR PREDICATE CALCULUS - AGAIN

In predicate calculus, we have assertions (formulas) and structures. The assertions and the structures are required to be of a specific kind which we have just reviewed.

Formulas are true or false only relative to models. Thus we speak of a given formula as being true in a given model.

PREDICATE CALCULUS COMPLETENESS THEOREM (Gödel 1928). There is a basic finite set of axioms and rules of inference, operating on the formulas, such that the following holds. A formula is true in all structures if and only if it is provable in this system.

RELATIVE COMPLETENESS THEOREM FOR PREDICATE CALCULUS

PREDICATE CALCULUS COMPLETENESS THEOREM (Gödel 1928). There is a basic finite set of axioms and rules of inference, operating on the formulas, such that the following holds. A formula is true in all structures if and only if it is provable in this system.

The completeness theorem is not quite in the form that we want. Here is the really useful Corollary.

RELATIVE COMPLETENESS THEOREM. There is a basic finite set of axioms and rules of inference, operating on the formulas, such that the following holds. Let T be a set of formulas and A be a formula. A is true in all structures satisfying T if and only if A is provable from T in the system.

From now, we will use the relative completeness theorem, and speak of "the axioms and rules of predicate calculus". We don't need to be specific about them.

COMPLETE AXIOMATIZATIONS OF STRUCTURES

Modern adventures in completeness often take on the following fundamental character.

1. An important mathematical structure M is identified.

2. M predicate calculus is predicate calculus that uses only symbols for the components of M.

3. A set T of formulas in M predicate calculus are chosen that are true in M.

4. It is shown that a formula of M predicate calculus is true in M iff it can be proved from T (using the axioms and rules of M predicate calculus).

T is called a COMPLETE AXIOMATIZATION of M. In order to be interesting, T must be "nice". Finite is best, but sometimes this is impossible. Computability of T is generally a bare minimum requirement.

COMPLETE AXIOMATIZATIONS OF STRUCTURES

CHALLENGE: Prove that every structure has a complete axiomatization.

CHALLENGE: Suppose M has a complete axiomatization that is computable. Then the set of sentences true in M is computable.

SOME STRUCTURES THAT HAVE INTERESTING COMPLETE AXIOMATIZATIONS

1. An important mathematical structure M is identified.
2. M predicate calculus is predicate calculus that uses only symbols for the components of M.
3. A set T of formulas in M predicate calculus are chosen that are true in M.
4. It is shown that a formula of M predicate calculus is true in M iff it can be proved from T (using the axioms and rules of M predicate calculus).

T is called a COMPLETE AXIOMATIZATION of M.

We present simple complete axiomatizations for some structures. Here N = set of nonnegative integers, Z = set of integers, Q = set of rational numbers, R = set of real numbers, and C = set of complex numbers.

(Q,=) (Z,=) (N,=)

(Q,S) (Z,S) (N,S)

(Q,<) (Z,<) (N,<)

(Q,0,+,-) (N,0,1,+)

(R,0,1,+,-,•) (C,0,1,+,-,•)

Here S is the successor function.

(Q,=), (Z,=), (N,=)

1. Axioms and Rules of Predicate Calculus for just =.
2. $(\exists y)(y \neq x_1 \wedge \ldots \wedge y \neq x_n)$.

Note that 2) has infinitely many axioms. It is known that these structures have no finite complete axiomatization.

We give some idea of how we prove this is a complete axiomatization. The method is called ELIMINATION OF QUANTIFIERS.

All of the other complete axiomatizations that we give are verified using this method, or variants of this method.

In this case, this method shows that every formula (for =) is provably (in 1,2) equivalent to a formula without quantifiers.

This is proved by induction on formulas (for =), which leads to the key induction step - the quantifier elimination step.

(Q,=)   (Z,=)   (N,=)
quantifier elimination step

1. Axioms and Rules of Predicate Calculus for just =.
2. $(\exists y)(y \neq x_1 \wedge \ldots \wedge y \neq x_n)$.

The key step is to consider a formula $(\exists x)(A)$ for just =, where A has NO QUANTIFIERS. Then show that it is provably (in 1,2) equivalent to a formula B with no quantifiers.

This will show that every formula for just = is provably (in 1,2) equivalent to a quantifier free formula, by induction.

Quantifier free formulas are easy to analyze within 1,2. This leads quickly to: every sentence for just = is provable or refutable in 1,2. This leads quickly to: every formula true in in any of these three structures is provable in 1,2.

QUANTIFIER ELIMINATION

CHALLENGE: Let T be any set of formulas in predicate calculus. Suppose that every formula (∃x)(A) using symbols in T, where A is quantifier free, is provably (in T) equivalent to a quantifier free formula using symbols in T. Then every formula using symbols in T is provably (in T) equivalent to a quantifier free formula using symbols in T.

If the above holds, then we say that T *admits quantifier elimination*.

CHALLENGE: Suppose T is true in M, T admits quantifier elimination, and T proves or refutes every atomic formula with at most one variable. Then T is a complete axiomatization of M.

CHALLENGE: Suppose T is true in M, T admits quantifier elimination, and T proves or refutes every atomic sentence, and there is a constant symbol. Then T is a complete axiomatization of M.

(Q,=), (Z,=), (N,=) quantifier elimination step

1. Axioms and Rules of Predicate Calculus for just =.
2. $(\exists y)(y \neq x_1 \wedge \ldots \wedge y \neq x_n)$.

The real work is to understand (∃x)(A), where A is quantifier free, from the point of view of 1,2.

Put A into disjunctive normal form (lecture 1). We get a disjunction of formulas of the form (∃x)(B), where B is a conjunction of literals (lecture 1). We need only work on each of these (∃x)(B).

By other simple logical manipulations, we can assume that the conjuncts of B are of the form x = v, x ≠ v, for various variables v other than x.

It is then easy to see how (∃x)(B) is provably (in 1,2) to a quantifier free statement involving only the v's. I.e., we have eliminated the quantifier (∃x).

(Q,S), (Z,S)

1. Axioms and Rules for Logic.
2. $S(x) = S(y) \rightarrow x = y$.
3. $(\exists y)(S(y) = x)$.
4. $S...S(x) \neq x$.

(N,S)

1. Axioms and Rules for Logic.
2. $S(x) = S(y) \rightarrow x = y$.
3. $(\exists!x)(\forall y)(S(y) \neq x)$.
4. $S...S(x) \neq x$.

In both cases, we again use infinitely many axioms. This is known to be necessary here.

For the next three examples, we will be able to get finite complete axiomatizations.

$(Q,<)$

1. **Axioms and Rules of Logic.**
2. $\neg x < x$, $x < y \wedge y < z \rightarrow x < z$, $x < y \vee y < x \vee x = y$.
3. $(\exists y)(x < y)$, $(\exists y)(y < x)$.
4. $x < y \rightarrow (\exists z)(x < z < y)$.

$(Z,<)$

1. **Axioms and Rules of Logic.**
2. $\neg x < x$, $x < y \wedge y < z \rightarrow x < z$, $x < y \vee y < x \vee x = y$.
3. $(\exists y)(x < y \wedge (\forall z)(\neg(x < z < y)))$.
4. $(\exists y)(y < x \wedge (\forall z)(\neg(y < z < x)))$.

$(N,<)$

1. **Axioms and Rules of Logic.**
2. $\neg x < x$, $x < y \wedge y < z \rightarrow x < z$, $x < y \vee y < x \vee x = y$.
3. $(\exists x)(\forall y)(x < y \vee x = y)$.
4. $(\exists y)(x < y \wedge (\forall z)(\neg(x < z < y)))$.
5. $y < x \rightarrow (\exists y)(y < x \wedge (\forall z)(\neg(y < z < x)))$.

(Q,0,+,-)

1. Axioms and Rules of Logic.
2. $x+y = y+x$, $(x+y)+z = x+(y+z)$, $x+0 = x$, $x+(-x) = 0$.
3. $x+...+x = 0 \rightarrow x = 0$.
4. $(\exists y)(y+...+y = x)$.

(N,0,1,+)

1. Axioms and Rules of Logic.
2. $x+1 \neq 0$, $x+1 = y+1 \rightarrow x = y$, $x+0 = x$, $(x+y)+1 = x+(y+1)$.
3. $A[x/0] \wedge (\forall x)(A \rightarrow A[x/x+1]) \rightarrow A$, where A is a formula for 0,1,+. (This is called the induction scheme).

Both of these use infinitely many axioms, and this is necessary. The second axiomatization is of a quite different style, using the induction scheme 3).

The final two axiomatizations also necessarily use infinitely many axioms.

$(\mathbb{R}, 0, 1, +, -, \bullet)$

1. **Axioms and Rules of Logic.**
2. $x+y = y+x$, $(x+y)+z = x+(y+z)$, $x+0 = x$, $x+(-x) = 0$.
3. $x \bullet y = y \bullet x$, $(x \bullet y) \bullet z = x \bullet (y \bullet z)$, $x \bullet 1 = x$, $x \bullet (y+z) = x \bullet y + x \bullet z$.
4. $x \bullet y = 0 \rightarrow x = 0 \vee y = 0$.
5. $x_1^2 + \ldots + x_n^2 \neq -1$.
6. $(\exists y)(y^2 = x \vee y^2 = -x)$.
7. $x_n \neq 0 \rightarrow (\exists y)(x_n y^n + x_{n-1} y^{n-1} + \ldots + x_1 y + x_0 = 0)$, n odd.

$(\mathbb{C}, 0, 1, +, -, \bullet)$

1. **Axioms and Rules of Logic.**
2. $x+y = y+x$, $(x+y)+z = x+(y+z)$, $x+0 = x$, $x+(-x) = 0$.
3. $x \bullet y = y \bullet x$, $(x \bullet y) \bullet z = x \bullet (y \bullet z)$, $x \bullet 1 = x$, $x \bullet (y+z) = x \bullet y + x \bullet z$.
4. $x \bullet y = 0 \rightarrow x = 0 \vee y = 0$.
5. $1+\ldots+1 \neq 0$.
6. $x_n \neq 0 \rightarrow (\exists y)(x_n y^n + x_{n-1} y^{n-1} + \ldots + x_1 y + x_0 = 0)$, $n \geq 1$.

GÖDEL'S CURSE - INCOMPLETENESS

THEOREM. The structures $(N,+,\bullet)$, $(Z,+,\bullet)$, $(Q,+,\bullet)$ have no computable complete axiomatization.

We now present a much stronger kind of incompleteness generally called the Gödel First Incompleteness Theorem. Gödel never proved it in the great generality that we now cast it.

Let T be a set of axioms in predicate calculus. We say that T is consistent if and only if it is free of contradiction. I.e., T does not prove both a sentence and its negation.

Gödel's completeness theorem tells us that T is consistent if and only if T is true in some structure; i.e., has a model.

We say that T is *complete* if and only if every sentence using the symbols of T is either provable or refutable in T.

FIRST INCOMPLETENESS THEOREM

Robinson Arithmetic uses the symbols 0,S,+,•, and has finitely many axioms which are obviously true in (N,0,S,+,•).

STRONG FIRST INCOMPLETENESS THEOREM. Let T be a consistent extension of Robinson Arithmetic with a computable set of axioms. Then T is not complete. I.e., there is a sentence using symbols of T that is neither provable nor refutable in T.

What is this Robinson Arithmetic that is sufficient to ruin completeness? It is surprisingly primitive.

ROBINSON ARITHMETIC

STRONG FIRST INCOMPLETENESS THEOREM. Let T be a consistent extension of Robinson Arithmetic with a computable set of axioms. Then T is not complete. I.e., there is a sentence using symbols of T that is neither provable nor refutable in T.

1. Axioms and Rules for Logic using $0, S, +, \bullet$.
2. $S(x) \neq 0$, $S(x) = S(y) \rightarrow x = y$, $x \neq 0 \rightarrow (\exists y)(x = S(y))$.
3. $x+0 = x$, $x+S(y) = S(x+y)$.
4. $x \bullet 0 = 0$, $x \bullet S(y) = x \bullet y + x$.

In the Strong First Incompleteness Theorem, we even allow T to add new symbols to those of Robinson Arithmetic.

CHALLENGE: Show that Robinson Arithmetic does not prove $0+x = x$.

GÖDEL'S SECOND INCOMPLETENESS THEOREM

The Second Incompleteness Theorem focuses on the issue of how we can mathematically prove that mathematics itself is free of contradiction.

The Second Incompleteness Theorem appears to ruin any hope of accomplishing this.

Let T be a computable set of axioms that is sufficiently powerful to do a lot of standard mathematics. There are plenty of examples of such T. The restriction to finitely many axioms turns out to be rather mild for present purposes.

T will be able to handle predicate calculus, and, as a special case, handle T. Specifically, T will be able to formulate the consistency of sets of sentences. In particular, T will be able to formulate the statement that

T is consistent; i.e., free of contradiction.

GÖDEL'S SECOND INCOMPLETENESS THEOREM

THEOREM. Under very general conditions, if T is consistent, then T does not prove "T is consistent".

In Lecture 5, next week, we will present the standard foundations for mathematics, which is a system called ZFC = Zermelo Frankel set theory with the Axiom of Choice.

THEOREM. If ZFC is consistent, then ZFC does not prove "ZFC is consistent".

CHALLENGE: Assume ZFC is consistent. Find a sentence A such that ZFC + A is consistent and proves that ZFC + A is inconsistent.

We still have to be careful about the exact formulation of this Theorem. We need to have a sentence that appropriately expresses "ZFC is consistent" from the point of view of ZFC.

We can of course just formalize "ZFC is consistent" using the symbols of ZFC. This, however, is extremely tedious, and no two people would ever do it in exactly the same way.

GÖDEL'S SECOND INCOMPLETENESS THEOREM

Furthermore, we would then have the puzzling situation of one of the great theorems of mathematics being also one with one of the most complicated and ugly statement!!

So clearly we want a general sensible and relevant criteria for

a reasonable formalization, within T, of
the consistency of T.

There has been progress on this matter in several directions. One emerging approach is the criteria that

the formalization of consistency is based on
any formalization of predicate calculus
which supports the Gödel Completeness Theorem.

IS MATHEMATICS FREE OF CONTRADICTION?

As a consequence of Gödel’s Second Incompleteness Theorem, it appears that we are resigned to accepting the consistency of mathematics on faith, as we are not going to prove that mathematics is consistent without going beyond the mathematics that we are proving consistent.

But there may be a glimmer of hope for getting around this. At this point, it is only a fantasy.

Perhaps we can prove, for example, that ZFC is consistent using a portion of ZFC together with observations from the physical world.

Gödel’s Second Incompleteness Theorem does not seem to preclude this possibility - however fantastic.

CHALLENGE: Defeat Gödel’s curse.