SOME HISTORICAL PERSPECTIVES ON CERTAIN INCOMPLETENESS PHENOMENA

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We have been particularly interested in the demonstrable unremovability of machinery, which is a theme that can be pursued systematically starting at the most elementary level - the use of binary notation to represent integers; the use of rational numbers to solve linear equations; the use of real and complex numbers to solve polynomial equations; and the use of transcendental functions to solve differential equations.

Practical situations arise such as the use of complex variables in number theory, or group theory in topology. Here there has been no demonstrable unremovability.

But it appears that when the machinery is removed, clarity and power is lost. This kind of unremovability is extremely difficult to get at rigorously.

Over the years, a growing collection of cases of demonstrable unremovability of increasing interest have been developed. Here is a brief synopsis of some of the highlights.

Around 1967, Tony Martin solved a crucial problem in infinite game theory involving Borel sets, using a massive amount of machinery, going well beyond the usual axioms for mathematics.

Around 1968, we proved that a small part of the machinery was unremovable -

uncountably many
uncountable cardinals.

In 1974, after an extended effort, Martin reduced the amount of machinery he used to

uncountably many
uncountable cardinals.

We later gave the following reinterpretation of Martin's theorem:

every Borel set in the plane that is symmetric
about the line y = x contains or is disjoint

from the graph of a Borel function,

and showed the unremovability of

uncountably many uncountable cardinals

for this statement.

Around 1974-75, we initiated a Borel measurable analysis of Cantor's theorem that there are uncountably many real numbers, which can be stated as

for every infinite sequence of reals there is a real which is not a coordinate.

It is standard that there is a Borel function F such that

for every infinite sequence of reals, x, F(x) is not a coordinate of x.

However, is there an invariant Borel function F such that

for every infinite sequence of reals, x, F(x) is not a coordinate of x?

E.g., where the value of F remains unchanged when the terms in the infinite sequence are permuted? Or when another sequence is used with the same coordinates, allowing repetitions?

We refuted this using some interesting machinery. I.e., we proved that

every invariant Borel function from infinite sequences of reals into reals sends some infinite sequence of reals to one of its coordinates,

using the machinery of the Baire category theorem on the space R^∞, where R is the reals endowed with the discrete topology. This machinery is unremovable, in that the theorem cannot be proved in so called separable mathematics.

This "Borel diagonalization theorem" was extended in various ways, including to

every invariant Borel function from and into infinite sequences of reals sends some infinite sequence of reals to an infinite subsequence,

every shift invariant Borel function from the Cantor space into itself

sends some x to its "square"
$x^{(2)} = (x_1, x_4, x_9, x_{16}, \ldots)$,

there is a continuous function on the circle group which agrees somewhere with every Borel function on the circle group which is invariant under doubling.

These involve the same demonstrably unremovable machinery.

The Borel diagonalization theorem was also extended to Borel equivalence relations and to infinite dimensional spaces such as the Hilbert cube, as well as to infinite sequences of finitely generated groups.

These infinite dimensional extensions use demonstrably unremovable machinery that goes well beyond all of the usual axioms for mathematics.

In the late 1980's Donald Martin and John Steel, building on earlier work of Solovay, Woodin, and others, used machinery going well beyond the usual axioms for mathematics to give a rather complete picture of the basic structure of the higher projective hierarchy. There are accompanying demonstrable unremovability results.

In a different direction, in 1977, Jeff Paris and Leo Harrington found a new finite form of the infinite Ramsey theorem, and established its unprovability from Peano Arithmetic.

This implies the unremovability of some machinery of significance - that of arbitrary arithmetic definitions of infinite sets, or equivalently, of arithmetic recursion over the natural numbers.

Around that time, we began a search for demonstrably unremovable uses of machinery going well beyond the usual axioms for mathematics, in discrete and finite mathematics.

In 1982 we considered the classic theorem of J.B. Kruskal which asserts that

in any infinite sequence of finite trees, one tree is homeomorphically embeddable in a later tree,

and established the demonstrably unremovable use of impredicative definitions, which allow quantification over all sets of integers in defining specific sets of integers.

Impredicative definitions can be viewed as a weak use of uncountable sets, and Poincare and Weyl objected to its use, and thought that nothing concrete could require it.

We also considered the Robertson Seymour theorem about graph minors (Wagner's conjecture),

in any infinite sequence of finite graphs,
one graph is minor included in a later graph,

and established the demonstrably unremovable use of the machinery of iterated impredicative definitions.

We provided obvious (and canoncial) finite forms for these statements about infinite sequences of trees and graphs, and also established the corresponding demonstrable unremovability.

Now all of the machinery used for the discrete and finite problems up till now lie well within small fragments of the usual axioms for mathematics. This is in contrast to the new results reported on in this talk.

The work is in a state of flux, in that we are striving towards yet simpler and more basic examples which are ever more connected with various contexts in mathematics.

What does the future hold? Will there be demonstrably unremovable uses of machinery going well beyond the usual axioms for mathematics for establishing major open problems in discrete and finite mathematics?

This is, as usual, hard to tell, but is not likely to happen for a while.

However, bear in mind that we did uncover the demonstrably unremovable use of iterated impredicative definitions in a well regarded piece of contemporary mathematics - the proof of Wagner's Conjecture (Robertson/Seymour's graph minor theorem).

We are confident in the demonstrable unremovability of machinery going well beyond the usual axioms for mathematics in order to establish discrete/finite mathematical results of growing interest.

We are beginning to understand concepts of "well behavedness" which generalize notions like "upper shift invariant." Also we have a plan to systematize conditions on operators such as "decreasing contraction."

This should culminate in a series of results of the following form:

**all mathematical statements of
a certain simple general form
can be decided, but only if
we use machinery that goes
well beyond the usual
axioms for mathematics.**

The mathematical statements might take the following form:

**all operators on $T(N^k)$
obey a certain existential condition.**