SOME DECISION PROBLEMS RELATED TO HILBERT'S TENTH PROBLEM

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PRELIMINARY REPORT

Let Q^n be the n-th Cartesian power of Q = rationals. Q^n is viewed at a vector space over Q. An element of Q^n is nonzero if and only if it is not the zero element of Q^n.

Qn can also be viewed as a ring, where a square of x in Q^n is the vector x^2, obtained by squaring each coordinate of x.

We will consider linear subspaces of Q^n, as well as affine subspaces of Q^n. The latter are simply translates by an element of Q^n of a linear subspace of Q^n. Linear subspaces of Q^n are presented by any basis, and affine subspaces of Qn are presented by any basis together with an element of Q^n serving as the translate. Obviously there is a decision procedure in a vector and presentation, for determining membership of the vector in the affine subspace with the given presentation.

Consider the following decision problems:

1. Let G be a linear subspace of Q^n. Does G contain the square of a nonzero integral element of G?

2. Let G be a linear subspace of Qn. Does G contain $x^2 + 1$ for some integral x in G?

3. Let A be an affine subspace of Q^n. Does A contain the square of an integral element of A?

4. Let $A_1,\ldots,A_k$ be affine subspaces of Q^n. Do there exist mutually orthogonal integral elements x_1 in A_1, x_2 in A_2, ... , x_k in A_k?

THEOREM. There exists n such that problems 2 and 3 have no decision procedure. There exists k and n such that problem 4 has no decision procedure.

THEOREM. The following are equivalent:

i) problem 1 has a decision procedure that works in all dimensions at once;

ii) there is a decision procedure for determining whether or not any polynomial with rational coefficients has a rational solution (Hilbert's 10th problem on the rationals).