Let $Q^n$ be the n-th Cartesian power of $Q = \text{rationals}$. $Q^n$ is viewed at a vector space over $Q$. An element of $Q^n$ is nonzero if and only if it is not the zero element of $Q^n$.

$Q^n$ can also be viewed as a ring, where a square of $x$ in $Q^n$ is the vector $x^2$, obtained by squaring each coordinate of $x$.

We will consider linear subspaces of $Q^n$, as well as affine subspaces of $Q^n$. The latter are simply translates by an element of $Q^n$ of a linear subspace of $Q^n$. Linear subspaces of $Q^n$ are presented by any basis, and affine subspaces of $Q^n$ are presented by any basis together with an element of $Q^n$ serving as the translate. Obviously there is a decision procedure in a vector and presentation, for determining membership of the vector in the affine subspace with the given presentation.

Consider the following decision problems:

1. Let $G$ be a linear subspace of $Q^n$. Does $G$ contain the square of a nonzero integral element of $G$?

2. Let $G$ be a linear subspace of $Q^n$. Does $G$ contain $x^2 + 1$ for some integral $x$ in $G$?

3. Let $A$ be an affine subspace of $Q^n$. Does $A$ contain the square of an integral element of $A$?

4. Let $A_1, \ldots, A_k$ be affine subspaces of $Q^n$. Do there exist mutually orthogonal integral elements $x_1$ in $A_1$, $x_2$ in $A_2$, $\ldots$, $x_k$ in $A_k$?

THEOREM. There exists $n$ such that problems 2 and 3 have no decision procedure. There exists $k$ and $n$ such that problem 4 has no decision procedure.

THEOREM. The following are equivalent:

i) problem 1 has a decision procedure that works in all dimensions at once;
ii) there is a decision procedure for determining whether or not any polynomial with rational coefficients has a rational solution (Hilbert's 10th problem on the rationals).