INCOMPLETENESS I

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Invitation to Mathematics Series

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April 25, 2012

WHAT IS INCOMPLETENESS?

The most striking results in mathematical logic have, historically, been in Incompleteness. These have been the results of the greatest general intellectual interest.

Incompleteness in mathematical logic started in the early 1930s with Kurt Gödel.

In these two lectures, I will give an account of Incompleteness, where the details of mathematical logic are black boxed.

Incompleteness refers to the fact that certain propositions can neither be proved nor refuted within the usual axiomatization for mathematics - or at least within a substantial fragment of the usual axiomatization for mathematics.

ANCIENT INCOMPLETENESS

Mathematics Before Fractions

Ordered Ring Axioms. +,-,·,<,0,1,=.

Does there exists x such that 2x = 1? Neither provable nor refutable. (?)

Mathematics Before Real Numbers

Ordered Field Axioms. +,-,·,$^{-1}$,<,0,1,=.

Does there exists x such that x^2 = 2? Neither provable nor refutable. (?).

Euclidean Geometry

Euclid's Axioms (clarified by Hilbert, and also by Tarski).

Is there at most one line parallel to a given line passing through any given point off of the given line? (Playfair's form). Neither provable nor refutable. (Beltrami 1868).

FUTURE OF INCOMPLETENESS

As mathematics has evolved, full mathematical Incompleteness has evolved.

But it is still completely unclear exactly which kinds of mathematical questions are neither provable nor refutable within the usual axiomatization of mathematics, or substantial fragments thereof.

We believe that interesting and informative information will be eventually obtained in every branch of mathematics by going beyond the usual axiomatization for mathematics - that cannot be obtained within the usual axiomatization for mathematics.

This will be a very productive form of Incompleteness.

Only time will tell.

But before we get to Incompleteness, we first need to talk about the usual foundation for mathematics.

FOUNDATIONS OF MATHEMATICS FROM 1800

By 1800, many leading mathematicians strongly believed that mathematics had a special kind of certainty beyond other subjects, but realized that there was no unified account as to what constitutes a valid mathematical proof.

Over the course of the 1800s, several mathematicians including Cauchy, Cantor, and Dedekind, and the mathematical philosopher Frege, contributed greatly to our understanding of what a valid mathematical proof is. This was followed by work of mathematicians Zermelo and Frankel in the early 1900s, culminating with the presently accepted general purpose foundation for mathematics.

KEY INGREDIENTS

1. Division of mathematics into a purely logical part, and a purely mathematical part.

2. The logic part involves $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $=$, variables, and symbols for primitive relationships between objects.

3. The mathematical part involves definite objects, normally viewed as having objective meaning.

4. Axioms and rules of inference for the purely logical part.

GÖDEL COMPLETENESS: If A cannot be refuted by the usual axioms and rules of logic, then A has a model.

5. Development of the $\in/\delta$ methodology for convergence, continuity, etc.

6. Grand Unification of the math systems (number systems, spaces, etc.), into a single kind of object with a single relationship (sets and membership).

7. Axioms about sets that, when combined with 2, are easily sufficient to support the Grand Unification.

ZFC
THE GOLD STANDARD

This series of events culminated with a symbolic system called ZFC = Zermelo Frankel set theory with the Axiom of Choice. This is the gold standard for general purpose foundations of mathematics.

ZFC consists of

A. Axioms and rules of inference of logic, with the primitive relations $\in$,=. There is no commitment as to the meaning of $\in$, or the nature of the objects represented by variables. However = is viewed as having a definite meaning.

B. Axioms about sets. These allegedly correspond to an alleged fixed reality of sets, or at least an allegedly clear mental picture of sets.

GENERAL FORMAL SYSTEMS

Let's first discuss the purely logical part. By a formal system, we commonly mean simply the axioms and rules of inference of logic, with various primitive relations, = usually included, and various nonlogical axioms.

We say that a formal system T is consistent iff there is no sentence A (in its language) such that T proves both A and ¬A.

We say that a formal system T is complete iff for all sentences A (in its language), T proves A or T proves ¬A.

Formal systems T that are consistent and complete are highly desirable.

GOOD NEWS BAD NEWS

GOOD NEWS: there are some nice consistent and complete formal systems based on important mathematical structures (Presburger, Tarski).

BAD NEWS: once a remarkably small dose of arithmetic or finite set theory is present, consistency/ completeness cannot be achieved by a reasonable formal system (Gödel, R. Robinson).

What is reasonable? Finitely many axioms is more than reasonable. We merely want the set of axioms to be computer generated.

The Good News is even better. A complete axiomatization T of a mathematical structure M is a formal system in the language of M such that a sentence is true in M iff it is provable in T.

HILBERT'S PROGRAM

THEOREM. There are computer generated complete axiomatizations for (R,Q,Z.<,+), (R,<,+,◊), and (C,R,+,◊). (Presburger, ?, Tarski).

Most people interpret Hilbert's Program as trying to

1. Give a computer generated complete axiomatization for all of mathematics.
2. Prove that mathematics is consistent using only very minimal concrete mathematical principles.

The above Good News is just the kind of thing Hilbert was looking for. But Hilbert wanted such thingss for all of mathematics.

Most people consider Hilbert's Program as having been utterly demolished by Gödel's First and Second Incompleteness Theorems.

FIRST INCOMPLETENESS THEOREM

ROBINSON ARITHMETIC. 0,S,+,$\cdot$.

$Sx \neq 0$.
$Sx = Sy \rightarrow x = y$.
$x \neq 0 \rightarrow (\exists y)(x = Sy)$.
$x+0 = x$, $x+Sy = S(x+y)$.
$x\cdot 0 = 0$, $x\cdot Sy = x\cdot y + x$.

RUDIMENTARY SET THEORY.

Empty set exists.
Every $x \cup \{y\}$ exists.

FIRST INCOMPLETENESS THEOREM. Every consistent computer generated extension of RA is incomplete. Every consistent computer generated extension of RST is incomplete. (Gödel, R. Robinson).

SECOND INCOMPLETENESS THEOREM
HILBERT DEMOLISHED BY GÖDEL

We can robustly discuss the consistency of any computer generated formal system, provided that we have a certain amount of arithmetic or finite set theory at our disposal. For this, we need somewhat more than RA or RST.

Let us call a computer generated formal system containing this modest amount of critical stuff ADEQUATE.

SECOND INCOMPLETENESS THEOREM. No consistent adequate formal system proves its own consistency.

Gödel's First and Second Incompleteness Theorems are widely considered to have utterly demolished Hilbert's Program in the foundations of mathematics.

FINITE SET THEORETIC UNIVERSE

Now to ZFC. ZFC can properly be viewed as an extrapolation of obvious facts about the FINITE set theoretic universe, to the INFINITE set theoretic universe. I.e., ZFC takes these obvious facts about the finite set theoretic universe, and adjoins the Axiom of Infinity.

The finite set theoretic universe is defined inductively as follows.

V(0) is the empty set. V(n+1) is the set of all subsets of V(n).

V(ω), the entire finite set theoretic universe, is the union of the V(n)'s.

This definition is read in the usual way as a piece of finite mathematics.

PROVABLE FACTS ABOUT THE FINITE SET THEORETIC UNIVERSE

AXIOMS USING $\in$,=
provable facts about the finite set theoretic universe

1. Extensionality. a = b iff a,b have the same elements.
2. Pairing. {a,b} exists.
3. Union. The set of all elements of elements of x exists.
4. Power set. $\{x: x \subseteq a\}$ exists.
5. Foundation. Every nonempty set has an epsilon minimal element.
6. Choice. For any set x of pairwise disjoint nonempty sets, there is a set having exactly one element in common with each element of x.
7. Separation. $\{x \in y: P(x)\}$ exists, where P is any property formulated in the language.
8. Replacement. Suppose $(\forall x \in a)(\exists! y)(P(x,y))$. Then $(\exists z)(\forall x \in a)(\exists y \in z)(Q(x,y))$, where Q is any property formulated in the language.

1-8 are PROVED to hold in $(V(\omega),\in)$ by ordinary mathematical induction.

THE ZFC AXIOMS
THE GOLD STANDARD

ZFC

1-8.
9. Infinity. There is a set x containing (as an element) the empty set, and each y ∪ {y} for y ∈ x.
10. Axioms and rules of logic for ∈,=.

ZFC is way more than enough for normal mathematical purposes. In fact, ZC = ZFC without Replacement, is quite sufficient.

Yet there are exceptions. In the second lecture, we will discuss a Borel measurable selection theorem which can be proved in ZFC, but not in ZC.

The Axiom of Choice wasn't always accepted as an axiom. Hence the historic interest in ZF = ZFC without the Axiom of Choice.

ABSTRACT SET THEORETIC INCOMPLETENESS

THEOREM. If ZF is consistent then ZF does not refute the Axiom of Choice. (Gödel 1940).

THEOREM. If ZF is consistent then ZF does not prove the Axiom of Choice. (Cohen 1963).

Cantor (founder of set theory), emphasized two open problems in set theory. One was the Axiom of Choice. The other was the Continuum Hypothesis:

EVERY INFINITE SET OF REAL NUMBERS IS IN ONE-ONE CORRESPONDENCE WITH THE SET OF ALL INTEGERS OR THE SET OF ALL REAL NUMBERS.

This is a particularly direct way to say that there is no cardinality strictly between the integers and the reals.

THE CONTINUUM HYPOTHESIS - GENERAL, AND BOREL FORMS

THEOREM. If ZF is consistent then ZFC does not refute the Continuum Hypothesis. (Gödel 1940).

THEOREM. If ZF is consistent then ZFC does not prove the Continuum Hypothesis. (Cohen 1963).

The existence, or possible existence, of pathological objects is what is behind the unprovability and unrefutability of the Continuum Hypothesis in ZFC. Witness its Borel measurable form:

EVERY BOREL MEASURABLE INFINITE SET OF REAL NUMBERS IS IN BOREL MEASURABLE ONE-ONE CORRESPONDENCE WITH THE SET OF ALL INTEGERS OF THE SET OF ALL REAL NUMBERS.

This is a well known theorem of Hausdorff.

The study of Borel measurable sets and functions in complete separable metric spaces is called Descriptive Set Theory.

GODEL AND COHEN METHODS
INNER MODELS AND OUTER MODELS

The techniques used by Gödel and Cohen for their results on the Axiom of Choice and the Continuum Hypothesis are very different.

Both start with a countable model M of ZF. This you can get from the consistency of ZF using Gödel Completeness (mentioned earlier).

Gödel manages to cut back to an inner model of M, where there is a lot of definite structure not generally present in M. Gödel uses this structure to show that the Axiom of Choice and the Continuum Hypothesis hold in his submodel - called the Constructible Sets.

Cohen starts with a countable model M of ZF, which, by Gödel, can be assumed to be a model of ZFC. Then Cohen manages to extend M to M# by adding new elements. In one form of his so called forcing construction, he obtains a model of ZF in which the Axiom of Choice fails. In another form of his forcing construction, he obtains a model of ZFC in which the Continuum Hypothesis fails. These are called generic extensions.

EXTREME FAILURE OF THE AXIOM OF CHOICE

Later developments (A. Levy) have shown just how badly the Axiom of Choice can fail in a model of ZF.

There is a model of ZF in which the entire real line can be partitioned into a countable number of countable sets. (Assuming ZF is consistent).

I.e., ZF does not suffice to prove that

the reals cannot be partitioned into a countable union of countable sets.

LEBESGUE MEASURABILITY

The Axiom of Choice has long been accepted as a legitimate axiom. Consequently, Incompleteness in ZFC has long been considered more interesting than Incompleteness in ZF.

THEOREM. The statement "all definable sets of real numbers is Lebesgue measurable" is neither provable nor refutable in ZFC (assuming ZFC is consistent). (Solovay 1970).

There is a slight problem with defining "definable sets of real numbers" within ZFC that can be corrected in several ways. For example,

THEOREM. The statement "all sets of real numbers definable using at most 1000 quantifiers is Lebesgue measurable" is neither provable nor refutable in ZFC (assuming ZFC is consistent). (Solovay 1970).

That should be enough quantifiers for anybody!

The same results hold for the Baire Property. $X \subseteq R$ has the Baire Property iff its symmetric difference from some open set is meager (countable union of nowhere dense sets). Again, Solovay 1970.