IMPOSSIBLE COUNTING

by
Harvey M. Friedman
Distinguished University Professor of Mathematics, Philosophy, and Computer Science Emeritus
The Ohio State University
September 2, 2017

Abstract. The number of $f:N^4 \to N$, up to isomorphism, is the same as the number of real numbers. The number of $f:N^4 \to N$, up to 7-isomorphism (i.e., having the same 7 element restrictions up to isomorphism), is, crudely, less than $2^{245^\wedge 2408}$. The exact count is the same if N is replaced by any other infinite set. We show that an exact count cannot be determined within the usual ZFC axioms for mathematics (assuming ZFC does not prove its own inconsistency). This precludes giving an algorithm and establishing within ZFC that the algorithm returns the exact count (assuming ZFC is does not prove its own inconsistency). This result is extended to show that this Impossible Counting holds for the extremely strong system ZFC + I1 (assuming ZFC + I1 does not prove its own inconsistency, as is generally believed by the set theory community).

1. INTRODUCTION AND PRELIMINARIES

DEFINITION 1.1. N is the set of all nonnegative integers. The letters k,n,m,r,s,t,i,j represent positive integers unless indicated otherwise. A k-ary operation is an f:Ek → E, where E is a set. A function is in N if and only if its arguments and values all lie in N.

How many k-ary operations are there, up to isomorphism? On a given set, up to isomorphism?

THEOREM 1.1. (ZC) For infinite E, there are exactly $2^{|E|}$ operations f:E → E up to isomorphism. The same result holds in higher dimensions. For any k and finite E, the number of operations $f:E^k \to E$ up to isomorphism can be determined in ZFC.

Proof: We will discuss the exact meaning of the third claim below (see Definition 1.4), but this is not needed to be convinced of the third claim. Namely, there is an obvious algorithm for determining this number by enumerating all $f:\{1,...,|E|\}^k \to \{1,...,|E|\}$, determining which pairs are isomorphic, and then making the count. This algorithm is obviously provably correct in ZC and much less. For ZC and all formal systems used in the paper, see section 7.

For the first two claims, the upper bound of $2^{|E|^2} = 2^{|E|}$ is immediate. We now focus on the lower bound. Fix infinite E and take a well ordering < of E. We will assume k = 1. For each x ∈ E, look at the tree T[<x] of strictly decreasing finite sequences starting with x, under extension, where the sequence of length 1 consisting of x, <x>, serves as the root. Clearly the "ordinal" of T[<x] is x. Define f_x:T[<x] → T[<x] by chopping off the last term, but with f(<x>) = <x>. Note that the various T[<x] are mutually disjoint. For each S ⊆ E of the same cardinality as E, let h_S be the union of the f_x, x ∈ S. Since we can recover T[<x] from f_x, and then the "ordinal" of T[<x] which is x, all up to isomorphism, we have recovered S from h_S. We arrive at S in this way even if we start with any isomorphic copy of f_S. So use isomorphic copies of the fS chosen so that they are maps from E into E. The number of S ⊆ E with |S| = |E| is of course $2^{|E|}$. QED

For the second claim, looking at the $f:E^k \to E$ such that $f(x_1,...,x_k)$ depends only on x_1 gives the lower bound $2^{|E|}$. QED

Note that we have put (ZC) in front of Theorem 1.1 indicating its provability in ZC. Theorem 1.1 cannot be proved in BSEP, but almost all of our results are provable in BSEP. The axioms of BSEP are given in section 7, and BSEP is equiconsistent with ACA_0.

Our results are based on a natural weakening of the isomorphism relation between operations, where we ask only that "small pieces" of the operations are isomorphic. This results in only finitely many functions up to such a weak isormorphism, regardless of the size of the operations.

One natural notion of "piece" of an operation is simply an operation that it extends, or suboperation. A second natural notion of "piece" of an operation is a restriction of the form $f:A^k \to B$, even if it does not form an operation. For example, the usual addition table for the numbers 0,1,2,3,4,5,6,7,8,9 forms such a piece of the addition function on the nonnegative integers, and does not form a suboperation. The deep Incompleteness issues arise regarding the second and not the first of these two notions of "piece". Here are the details.

DEFINITION 1.2. A k-ary function is an $f:A^k \to B$, where A,B are sets, which are allowed to be empty. An r-element k-ary function is an $f:A^k \to B$ such that $|A| = r$. A ≤r-element k-ary function is an $f:A^k \to B$ such that $|A| \leq r$. Let $f:A^k \to B$, $g:C^k \to D$. f,g are isomorphic if and only if there exists a bijection $h:A \cup rng(f) \to C \cup rng(g)$ such that $(\forall x_1,\ldots,x_k,y \in A \cup rng(f))(f(x_1,\ldots,x_k) = y \leftrightarrow g(h(x_1),\ldots,h(x_k)) = h(y))$.

The use of r-element here is an abuse of notation as the actual size of f in this case is r^k. The hyphen after "r" should make this abuse more harmless.

THEOREM 1.2. (BSEP) Every r-element k-ary function is isomorphic to some $f:\{1,\ldots,r\}^k \to \{1,\ldots,r^k+r\}$. There are finitely many r-element k-ary functions up to isomorphism.

Proof: Let f be an r-element k-ary function. Obviously f is isomorphic to some $g:\{1,\ldots,r\}^k \to N$. The number of values outside $\{1,\ldots,r\}$ is at most r^k, and so they can be moved to $\{r+1,\ldots,r^k+r\}$ and adjusting the isomorphism accordingly. QED

DEFINITION 1.3. Let $f:A^k \to B$, $g:C^k \to D$ be k-ary functions. A restriction of f is a k-ary function contained in f. A suboperation of f is a restriction of f that is an operation. f,g are operationally r-isomorphic if and only if f,g have the same r element suboperations up to isomorphism. I.e., every r element suboperation of f is isomorphic to some (r element) suboperation of g and vice versa. f,g are operationally ≤r-isomorphic if and only if f,g have the same ≤r element subsoperations up to isomorphism. f,g are r-isomorphic if and only if f,g have the same r element restrictions up to isomorphism. f,g are ≤r-isomoprhic if and only if f,g have the same ≤r element restrictions up to isomorphism.

THEOREM 1.3. (BSEP) r-isomorphism, ≤r-isomorphism, operational r-isomorphism, operational ≤r-isomorphism, on the k-ary operations, form four equivalence relations with finitely many equivalence classes. The strongest of these is ≤r-isomorphism, and the weakest of these is operational r-isomorphism.

Proof: For all four equivalence relations, we associate a unique set of functions $f:\{1,\ldots,i\}^k \to \{1,\ldots,i^k+i\}$, $0 \leq i \leq r$, to each equivalence class via Theorem 1.2. QED

We now make a fundamental connection between these notions and the usual first order predicate calculus with equality, written PC(=).

THEOREM 1.4. (BSEP) Let $f:A^k \to B$ be finite. There is a sentence φ in PC(=) such that the following holds. A nonempty k-ary operation g has a restriction isomorphic to f if and only if g satisfies φ. I.e., (B,g) satisfies φ, where $g:B^k \to B$.

Proof: Let $f:A^k \to B$ be onto, where $x_1,\ldots,x_r$ enumerates A without repetition. Set $\varphi = (\exists v_1,\ldots,v_r)(\psi)$, where $v_1,\ldots,v_r$ are distinct variables and ψ is the conjunction of all formulas $v_i = v_j$, $v_i \neq v_j$, $g(z_1,\ldots,z_k) = z$, $g(z_1,\ldots,z_k) \neq z$, $g(z_1,\ldots,z_k) = g(w_1,\ldots,w_k)$, $g(z_1,\ldots,z_k) \neq g(w_1,\ldots,w_k)$, where $z_1,\ldots,z_k,z,w_1,\ldots,w_k,w$ are among $v_1,\ldots,v_r$, which are true when we interpret $v_1,\ldots,v_r$ as $x_1,\ldots,x_r$. Suppose (B,g) satisfies φ. Let $v_1,\ldots,v_r$ witness ψ. Then clearly $v_1,\ldots,v_r$ are distinct and the restriction of g to $\{v_1,\ldots,v_r\}^k$ is isomorphic to f via the following map. Send $v_1,\ldots,v_r$ onto

$x_1,\ldots,x_r$ and each $g(w_1,\ldots,w_k)$ to the corresponding $f(w_1',\ldots,w_k')$, where $w_1,\ldots,w_k \in \{v_1,\ldots,v_r\}$ and $w_1',\ldots,w_k' \in \{x_1,\ldots,x_r\}$. This map is well defined and is the required isomorphism by the construction of ψ. Conversely, Suppose f is isomorphic to $g|\{v_1,\ldots,v_r\}^k$, where we have listed $v_1,\ldots,v_r$ so that we have an isomorphism mapping $x_1,\ldots,x_r$ onto $v_1,\ldots,v_r$. Then $v_1,\ldots,v_r$ witnesses φ by the definition of isomorphism. QED

The form of the sentences φ constructed in Theorem 1.4 will be of major importance throughout the paper. In particular, note that φ is an existential sentence with no nesting, in a single k-ary function symbol, where the number of variables is the same as the cardinality of A.

THEOREM 1.5. (BSEP) Let f be a k-ary operation and $r \geq 1$. There is a sentence φ in PC(=) such that the following holds. A nonempty k-ary operation g is r-isomorphic to f if and only if g satisfies φ. This is also true with operationally r-isomorphic, and also with $\leq$r, for a total of four sentences $\varphi_1,\varphi_2,\varphi_3,\varphi_4$.

Proof: Let f,r be s given. Let $h_1,\ldots,h_n$ list all $f:\{1,\ldots,r\}^k \to \{1,\ldots,r^k+r\}$ without repetition. By Theorem 1.2, every r-element function is isomorphic to exactly one h_i. Now let g be a k-ary operation. Assert of g that "g has a restriction isomorphic to h_i" and "g has no restriction isomorphic to h_j", where f has a restriction isomorphic to h_i and f has no restriction isomorphic to h_j. These assertions are formalized by Theorem 1.4. If we are using $\leq$r-isomorphism, then we list all $f:\{1,\ldots,i\}^k \to \{1,\ldots,i^k+i\}$, $0 \leq i \leq r$, without repetition. If we use operational r-isomorphism, $\leq$r-isomorphism, then we use only those hi that are k-ary operations. QED

The form of the sentences φ constructed in Theorem 1.5 will also be of major importance throughout the paper.

We now introduce eight fundamental counting problems.

DEFINITION 1.4. Let $k,r \geq 1$ and E be a set. $\Omega(k,r,E)$ is the number of k-ary operations $f:E^k \to E$ up to operational r-isomorphism. $\Omega(k,\leq r,E)$ is the number of k-ary operations $f:E^k \to E$ up to operational $\leq$r-isomomorphism. $\Omega(k,r,\subseteq E)$ is the number of k-ary operations $f:E'^k \to E'$, $E' \subseteq E$, up to operational r-isomorphism. $\Omega(k,\leq r,\subseteq E)$ is the number of k-ary operations f:E'k → E', E' ⊆ E, up to operational ≤r-isomorphism.

DEFINITION 1.5. Let $k,r \geq 1$ and E be a set. $\Theta(k,r,E)$ is the number of k-ary operations $f:E^k \to E$ up to operational r-isomorphism. $\Theta(k,\leq r,E)$ is the number of k-ary operations $f:E^k \to E$ up to operational ≤r-isomomorphism. $\Theta(k,r,\subseteq E)$ is the number of k-ary operations $f:E'^k \to E'$, $E' \subseteq E$, up to operational r-isomorphism. $\Theta(k,\leq r,\subseteq E)$ is the number of k-ary operations $f:E'^k \to E'$, $E' \subseteq E$, up to operational ≤r-isomorphism.

Note that by Theorem 1.3, all eight of these quantities exist as positive integers.

THEOREM 1.6. (BSEP) Let $k,r,t \geq 1$. There are sentences $\varphi_1,\ldots,\varphi_n$ of PC(=), depending only on k,r, such that the following holds for all nonempty sets E. $\Omega(k,r,E) = t$ if and only if the number of i such that φ_i has a model with domain E is t. This is also true for the other seven expressions using different sentences.

Proof: Let k,r,t and E be given. Let φ1,...,φn list the sentences that are constructed in the proof of Theorem 1.5, without repetition. There are a total of four versions from Theorem 1.5, and they depend only on k,r, and the choice of $\Omega(k,r,E)$, $\Omega(k,\leq r,E)$, $\Theta(k,r,E)$, $\Theta(k,\leq r,E)$. For $\Omega(k,r,\subseteq E)$, $\Omega(k,\leq r,\subseteq E)$, $\Theta(k,r,\subseteq E)$, $\Theta(k,\leq r,\subseteq E)$, we need to adjust the sentences by introducing a unary predicate symbol to carve out a subdomain corresponding to subsets of E. QED

THEOREM 1.7. (ZC) The eight quantities are independent of the choice of infinite set E.

Proof: This is an immediate consequence of Theorem 1.6 and the upward and downward Skolem Lowenheim theorems. These together assert that any sentence in predicate calculus has a model in some infinite domain if and only if it has a model in every infinite domain. QED

In light of Theorem 1.7, we turn our attention on the eight quantities $\Omega(k,r,N)$, $\Omega(k,\leq r,N)$, $\Omega(k,r,\subseteq N)$, $\Omega(k,\leq r,\subseteq N)$,

Θ(k,r,N), Θ(k,≤r,N), Θ(k,r,⊆N), Θ(k,≤r,⊆N).

DEFINITION 1.6. α(k,r) is the number of ≤r-element k-ary functions up to isomorphism. B[n] is the n-th Bell number, which is the number of equivalence relations on an n element set.

THEOREM 1.8. (BSEP) $\alpha(k,r) \leq B[r^k+r]$. All of our eight quantities are at most $2^{\alpha(k,r)}$. $B[n] < (.792n/\ln(n+1))^n$.

Proof: Let f be a ≤r-element k-ary function. Its isomorphism type is determined by an equivalence relation on the set of terms $F(y_1,\ldots,y_k)$,y where $y,y_1,\ldots,y_k$ are among the variables $x_1,\ldots,x_r$. Hence α(k,r) is bounded by the number of equivalence relations on $\{1,\ldots,r^k+r\}$, which is the (r^k+r)-th Bell number. The upper bound on B[n] is from [BT10]. QED

The first two upper bounds in Theorem 1.8 are extremely crude, and it would be interesting to obtain serious bounds or relevant recurrence relations. Obtaining exact computations for k,r small for α(k,r) seem challenging, and for the Ω,Θ values would seem extremely challenging.

THEOREM 1.9. (BSEP) Every k-ary operation is ≤r-isomorphic to some ≤rα(k,r)-element restriction.

Proof: Let f be a k-ary operation. Let $f_1,\ldots,f_n$ be ≤r-element restrictions of f such that every ≤r-element restriction of f is isomorphic to exactly one f_i. Clearly $n \leq \alpha(k,r)$. Let A be the set of all coordinates of elements of the domains of $f_1,\ldots,f_n$. Then f and f|A are ≤r-isomorphic. Note that $|A| \leq r\alpha(k,r)$. QED

THEOREM 1.10. (ZC) Let $\lambda \geq \omega$ and $k \geq 1$. Every k-ary operation on a finite set extends to a k-ary operation $g:\lambda^k \to \lambda$ with the same finite suboperations.

Proof: Let $f:\{1,\ldots,s\}^k \to \{1,\ldots,s\}$. Extend f by $g(\alpha,\ldots,\alpha) = \alpha+1$ for $s < \alpha < \lambda$, with the default value 0 elsewhere on λ^k. Let $h:E^k \to E$ be a finite suboperation of g. If $E \subseteq \{0,\ldots,s\}$ then h is a suboperation of f. If $\alpha \in E\backslash\{0,\ldots,s\}$ then E contains $\alpha,\alpha+1,\alpha+2,\ldots$, and is not finite. QED

THEROEM 1.11. (ZC) Let E be infinite. $\Omega(k,r,E)$, $\Omega(k,\leq r,E)$, $\Omega(k,r,\subseteq E)$, $\Omega(k,\leq r,\subseteq E)$ are the same as $\Omega(k,r,\{1,...,r\alpha(k,r))$, $\Omega(k,\leq r,\{1,...,r\alpha(k,r))$, $\Omega(k,r,\subseteq\{1,...,r\alpha(k,r)\})$, $\Omega(k,\leq r,\subseteq\{1,...,r\alpha(k,r)\})$, respectively. $\Omega(k,r,E)$, $\Omega(k,\leq r,E)$, $\Omega(k,r,\subseteq E)$, $\Omega(k,\leq r,\subseteq E)$ are elementary recursive functions of k,r.

Proof: By Theorem 1.7, these four quanatities depend only on k,r, and not infinite E. It suffices to show that every k-ary operation on E is operationally $\leq$r-isomorphic to an operation on a subset of $\{1,...,r\alpha(k,r)\}$, and every k-ary operation on a subset of $\{1,...,r\alpha(k,r)\}$ is operationally $\leq$r-isomorphic to a k-ary operation on E. The former is from Theorem 1.9, and the latter is from Theorem 1.10. QED

Although it makes sense to continue a combinatorial investigation of the four Ω functions, their computability precludes their role in the present Impossible Counting development.

However, Θ behaves very differently from Ω, as we no longer have anything like Theorem 1.10.

To simplify the exposition, we focus on the one function $\Theta(k,r,N)$ and postpone any discussion of the other three, $\Theta(k,\leq r,N)$, $\Theta(k,r,\subseteq N)$, $\Theta(k,\leq r,\subseteq N)$, until section 6. In section 6, we show that all of these functions are closely related, and we obtain the same results for all four, as well as four other variants.

In section 2, we prove that for any fixed $k \geq 2$, $\Theta(k,r,N)$ is not recursive, and in fact has Turing degree 0' (Theorem 2.1.7).

In section 3 and beyond, we focus on the evaluation of certain values of $\Theta(k,r,N)$ within certain systems. In section 3, we develop some general theory, where the evaluations take place in extensions of BSEP in the following sense.

DEFINITION 1.7. T is an extension of BSEP if and only if T is a system in many sorted logic with a sort for sets, with at least $\in,=$, whose axioms include BSEP. T is a 1-consistent extension of BSEP if and only if T is an extension of BSEP where every Σ^0_1 sentence (formulated in

$\in$,= via the usual interpretation of arithmetic in the hereditarily finite sets) provable in T is true, respectively.

THEOREM 1.13. (BSEP) $\alpha(4,7) \leq B(2408) < 245^{2408}$. $\Theta(4,7,N) < 2^{245\hat{}2408}$.

Proof: By Theorem 1.8 and calculation. QED

The estimates in Theorem 1.13 are extremely crude. $\alpha(4,7)$ should be subject to exact calculation with a displayed base ten number, with man/machine interaction.

We have to be briefly pedantic about some of the relevant logical issues since ZFC cannot talk directly about specific integers, but only about their names, which are customarily bit strings.

DEFINITION 1.8. For any integer $k \geq 0$, let k^* be the standard name of k in the language of ZFC via its base 2 representation. For any algorithm γ (say in the Turing Machine model), γ^* is the standard definition of γ in the language of ZFC via strings.

DEFINITION 1.9. Let T be an extension of BSEP. Let φ be a formula in the language of ZFC with at most the free variable x. T evaluates φ if and only if there exists an integer $n \geq 0$ such that T proves "$(\forall x)(\varphi(x) \leftrightarrow x = n^*)$". T evaluates φ to be n if and only if T proves "$\varphi(x) \leftrightarrow x = n^*$". T correctly evaluates φ if and only if there exists an integer $n \geq 0$ with $\varphi(n)$, where T proves "$\varphi(x) \leftrightarrow x = n^*$".

THEOREM 1.14. (BSEP) Let φ be a formula in the language of ZFC with at most the free variable x. Let T be an inconsistent extension of BSEP. T evaluates φ. For all $n \geq 0$, T evaluates φ to be n. T correctly evaluates φ if and only if φ holds of some nonnegative integer. Alternatively, let T be a consistent extension of BSEP. There is at most $n \geq 0$ such that T evaluates φ to be n. If T correctly evaluates φ then T evaluates φ.

Proof: Obvious. QED

Theorem 1.14 warns us that the notion of evaluation in T is only interesting if T is consistent.

We apply Definition 1.9 to $\Theta(k,r,N)$ and related expressions. In the spirit of pedantry, we are actually applying Definition 1.9 to the φ which takes the form: $\Theta(k^*,r^*,N) = x$. Fix $k,r \geq 1$. If T is inconsistent then T evaluates $\Theta(k,r,N)$ at every $n \geq 0$ and also T correctly evaluates $\Theta(k,r,N)$.

Of course, it is not realistic that a mathematician could actually work directly with expressions n* where n has anything like $2^{245\text{^}2408}$ base 2 digits. However, working with algorithms that return precisely such numbers is realistic. Hence the following observation is relevant.

THEOREM 1.15. (BSEP) Let T be a consistent extension of BSEP and φ be a formula in the language of ZFC with at most the free variable x. Let $k,r \geq 1$. The following are equivalent.
i. T evaluates $\Theta(k,r,N)$.
ii. There is an algorithm γ that returns an integer, and T proves "γ returns $\Theta(k,r,N)$".
Furthermore, the following are equivalent.
iii. T correctly evaluates $\Theta(k,r,N)$.
iv. There is an algorithm γ that returns $\Theta(k,r,N)$, and T proves "γ returns $\Theta(k,r,N)$".

Proof: For i $\rightarrow$ ii, let $n \geq 0$ where T proves $\Theta(k,r,N) = n$. Let γ be the algorithm designed to return n. For ii $\rightarrow$ i, let γ return $n \geq 0$, where T proves γ returns $\Theta(k,r,N)$. Then T proves that γ returns n. Hence T proves that γ returns $\Theta(k,r,N)$. For iii $\rightarrow$ iv, let $n = \Theta(k,r,N)$ and T proves $\Theta(k,r,N) = n$. Let γ be the algorithm designed to return n. For iv $\rightarrow$ iii, let γ return $\Theta(k,r,N)$, and T prove γ returns $\Theta(k,r,N)$. Let $n = \Theta(k,r,N)$. Then T proves $\Theta(k,r,N) = n$. QED

A major lesson from Theorem 1.15 is that our concepts of T evaluates and T evaluates correctly are rather robust.

In section 4.3, we show that ZFC does not evaluate $\Theta(4,7,N)$ correctly, assuming ZFC is consistent. Also ZFC does not evaluate $\Theta(4,7,N)$, assuming ZFC does not prove its own inconsistency. This precludes giving an algorithm that returns the correct value, and establishing within ZFC that the algorithm returns the correct count (assuming ZFC is consistent). This also precludes giving an algorithm that returns, and establishing within ZFC that the algorithm returns the correct count (assuming ZFC does not proves its own inconsistency). These results are extended in section 5.2 to the extremely strong system ZFC + I1. See section 7 for ZFC + I1.

These results rely on controlling the complexity of axiomatizations of these two set theories - where they are axiomatized up to equiconsistency. In sections 4.2 and 5.2, we construct systems of low complexity in predicate calculus with equality, associated with the small parameters 4,7, that is equiconsistent with ZFC and ZFC + I1, respectively. By the general theory developed in section 3.2, this suffices to derive the Impossible Counting described in the preceding paragraph.

2. NONRECURSIVENESS OF Θ(k,r,N)

Here we investigate the the unary function Θ(k,r,N) with k fixed. We show that it is recursive for k = 1, and not recursive for any fixed k ≥ 2.

DEFINITION 2.1.1. Let k,r ≥ 1. W[k,r] is the set of all $f:\{1,...,r\}^k \to \{1,...,r^k+r\}$, $0 \le i \le r$. Let F be a k-ary operation. $\alpha_r(F)$ is the set of all f ∈ W[k,r] which are isomorphic to some r-element restriction of F. (r,S) is (k,N)-special if and only if there exists $G:N^k \to N$ such that $S = \alpha_r(G)$. W[k] is the union of the W[k,r], r ≥ 1. FS(E) is the set of all finite subsets of E.

LEMMA 2.1.1. (BSEP) Let k,r ≥ 1. The number of (k,N)-special (r,S) is Θ(k,r,N).

Proof: $f,g:N^k \to N$ are r-isomorphic if and only if $\alpha_r(f) = \alpha_r(g)$. Furthermore, the range of α_r is the set of S such that (r,S) is (k,N)-special. QED

LEMMA 2.1.2. (BSEP) Let k ≥ 1. The set of all (k,N)-special (r,S) is a Π^0_1 subset of $Z^+ \times$ FS(W[k]).

Proof: Let k ≥ 1. We can effectively associate a sentence β(r,S) in predicate calculus to any given (r,S) ∈ $Z^+ \times$ W[k] such that (r,S) is (k,N)-special if and only if $\beta(r,S)$ has an infinite model. Thus puts the set of all (k,⊆N)-special in Π^0_1 by Gödel's Completeness Theorem.

LEMMA 2.1.3. (BSEP) Let $k \geq 1$. The set of all (k,N)-special $(r,S) \in Z^+ \times FS(W[k])$ is recursive in the function $\Theta(k,r,N)$.

Proof: Let $k \geq 1$. For $r \geq 1$, let h(k,r) be the number of subsets of W[k,r]. We use Lemmas 2.1.1., 2.1.2. Given $r \geq 1$, enumerate the $(r,S) \in Z^+ \times FS(W[k])$ which are not (k,N)-special, until we reach a total of $h(k,r)-\Theta(k,r,N)$ (r,S)'s. We will get there, and what remains are the (k,N)-special elements in $Z^+ \times FS(W[k])$. QED

DEFINITION 2.1.2. The variables are the x_i, $i \geq 1$. A k-term is a variable or a term $G(y_1,...,y_k)$, where $y_1,...,y_k$ are variables and G is a k-ary function symbol. A k-equation is an equation $\alpha = \beta$ where α,β are k-terms. A k-atom is a k-equation or negated k-equation. A $(\forall,k)$-sentence is a sentence of the form $(\forall x_1 \neq ... \neq x_r)(\varphi)$, $r \geq 1$, where φ is a disjunction of k-atoms. A sentence in predicate calculus is N satisfiable if and only if it is true in some structure in its language with domain N.

LEMMA 2.1.4. (BSEP) Let $k \geq 1$. The set of all N satisfiable $(\forall,k)$-sentences is recursive in the set of all (k,N)-special $(r,S) \in Z^+ \times FS(W[k])$.

Proof: Let $k \geq 1$. Let $\varphi = (\forall x_1 \neq ... \neq x_r)(\psi)$ be a given sentence in the k-ary function symbol G, where φ is a disjunction of k-atoms. N satisfiability of φ means that there exists $G:N^k \rightarrow N$ such that certain particular complete sets of k-atoms in variables $x_1,...,x_r$ cannot be realized in G. (Here a complete set of k-atoms in $x_1,...,x_r$ is a set of such k-atoms which, for each k-equation, includes it or its negation but not both). Each complete set of k-atoms in variables $x_1,...,x_r$ corresponds to at most one $\leq$r-element function up to isomorphism. So G realizing a given complete set of k-atoms is equivalent to asserting that G has a $\leq$r-element restriction of a certain isomorphism type. So G satisfying φ is equivalent to saying that the set of all isomorphism types of the r-element restrictions of φ are among some particular list of such sets. (The $\leq r$ are changed to r because of the $\neq$ in φ). Hence the N satisfiability of φ is equivalent to asserting that there exists $G:N^k \to N$ whose set of r-element restrictions are among a list of sets of r-element functions up to isomorphism. This clearly reduces to the problem of determining (k,N)-specialness. QED

We now work with universal sentences. Here the terms and the connectives may compound.

LEMMA 2.1.5. (BSEP) The set of all N satisfiable universal sentences in a single binary function symbol has Turing degree 0'.

Proof: This is well known, and we give a self contained proof of this. We start with a Turing Machine with an infinite one way tape, two tape symbols 0,1, k states, and which is initialized by all 0's. We effectively associate a universal sentence φ in the single binary function symbol so that all models of φ are infinite, and the TM halts if and only if φ is N satisfiable.

We use the binary function symbol F only. We need to create out of F, a binary function G to represent the action of the TM in the sense that G(t,n) encodes the triple consisting of the state of the TM at time t, the tape symbol being read at time t, and 1,0 according to whether the reading head at time t is at n. There are $k \times 2 \times 2 = 4k$ possible values of this virtual G.

We need to manage G with its range of 4k values. We use 0,1,...,4k-1 for these values, a successor function S for navigation, along with G. These must all be built from F.

To get 0,1,...,4k-1 and S, we need to use up part of F. We are forced to tease out G nicely from the remainder of F left untouched. We first develop a strict linear ordering with 0 and successor function. Then we define G(x,y) = $F(\alpha x,\beta y)$, where α,β are certain explicitly defined unary functions, arranged so that there is no commitment thus far for F applied to any pair $(\alpha x,\beta y)$.

Accordingly, we start by asserting that < is a strict linear ordering, with x < y abbreviating F(x,y) = x. We that Sx is the immediate successor of x in < where Sx abbreviates F(x,x). This already forces all models to be infinite. We assert that F(Sx,x) is constantly the < least point, and define 0 = F(Sx,x). We define i* = S...S0 and $S^i x$ = S...Sx.

We still need to define binary G. The idea is that G(x,y) = F(αx,βy) if αx > βy; F(βy,αx) otherwise, where α,β are chosen so that (∀x,y)(αx ≠ βy ∧ αx ≠ Sβy ∧ βx ≠ Sαy). For this purpose, we need the quadrupling axioms Q0 = 0, QSx = SSSSQx, where Qx abbreviates F(SSx,x). We assert that Qx,SSQx are one-one, and Qx ≠ SQx,SSQx,SSSQx.

We use $0,S0,S^2 0,\ldots,S^{4k-1}0$ for the 4k values of G. We then assert the standard initialization and transition axioms for G(x,y), and also that G(x,y) never corresponds to a halting state. Here G(x,y) abbreviates F(Qx,SSQy) if Qx > SSQy; F(SSQy,Qx) if SSQy ≥ Qx.

We have defined a finite set of universal sentences in F, which we can combine to form a single one, φ. Let M be a model of φ. Then < is in fact a strict linear ordering. S is in fact the strict successor function for <. F(Sx,x) is constantly the < least point 0. G(x,y), for standard x,y, correctly defines the action of TM. Because of the no halting state axiom, it is clear that if φ is satisfiable then TM does not halt.

Now suppose TM does not halt. We claim that φ is N satisfiable. This is because we have an obvious standard model of φ. Namely, the domain is N, F(x,y) = x if x < y, F(x,x) = x+1, F(x+1,x) = 0, F(x+2,x) = 4x. Also the binary function F(4x,4y+2) if 4x > 2y+1; F(4y+2,4x) if 4y+2 ≥ 4x. At other pairs of arguments, set F to be 0.

We have thus shown that the two problems have Turing degree ≥ 0' by reduction to the halting problem. Degree ≤ 0' is from the Gödel Completeness Theorem. QED

LEMMA 2.1.6. (BSEP) Let k ≥ 2. The set of all N satisfiable (∀,k)-sentences has Turing degree 0'.

Proof: Inductively replace terms by new variables to get rid of the nesting in Lemma 2.1.6. This verifies the claims with k = 2. Then replace all subterms F(x,y) by G(x,y,...,y), where G is k-ary. QED

THEOREM 2.1.7. (BSEP) For fixed $k \geq 2$, the unary function $\Theta(k,r,N)$ has Turing degree 0'. The function $\Theta(1,r,N)$ is primitive recursive.

Proof: Fix $k \geq 2$. By Lemmas 2.1.6 and 2.1.4, the special pair problem for k is $\geq 0'$. By Lemma 2.1.3, the unary function $\Theta(k,r,N) \geq 0'$. By Lemma 2.1.2, the special pair problem is $\leq 0'$. By Lemma 2.1.1, $\Theta(k,r,N) \leq 0'$ by counting. The second claim is by the well known decision procedure for PC(=) with one unary function symbol, [BGG01], p. 315. QED

3. EVALUATING $\Theta(k,r,N)$

In this section, we discuss the evaluation of $\Theta(k,r,N)$ in systems T. We show that if T is a reasonable extension of BSEP that is equiconsistent with a theory in Y(k,r) (a certain fragement of predicate calculus with =), then T cannot evaluate $\Theta(k,r,N)$.

A key well known point here is that ACA_0 proves the Gödel Completeness Theorem for countable sets of sentences in countable languages, formulated for countable models with complete diagrams, where we include Soundness in the Completeness Theorem. Another key well known point is that if we have a model in the usual sense for a theory presented with finitely many axioms (maybe without a complete diagram), then we obtain consistency (i.e., soundness) and therefore a model with a complete diagram. This argument uses cut elimination (for the soundness) and the Gödel Completeness Theorem. Since BSEP is an extension of ACA_0, these remarks apply to BSEP as well.

DEFINITION 3.1. Let T,T' be recursively presented systems in many sorted predicate calculus with = in zero or more of the sorts. T,T' are equiconsistent if and only if ACA_0 proves "T has a countable model with complete diagram if and only if T' has a countable model with complete diagram".

We can replace ACA_0 in Definition 3.1 by BSEP without change as BSEP is a conservative extension of ACA_0 in the appropriate sense.

3.1. PREDICATE CALCULUS FRAGMENTS

DEFINITION 3.1.1. We use F^k for a preferred k-ary function symbol, $k \geq 1$. A (k,r)-term is an x_i, $1 \leq i \leq k$, or F^k(y1,...,yk), where $y_1,\ldots,y_k$ are among $x_1,\ldots,x_r$. A (k,r)-equation is an equality between (k,r)-terms. A (k,r)-atom is a (k,r)-equation or a negated (k,r)-equation. A k-term is a (k,r)-term for some $r \geq 1$. A k-equation is a (k,r)-equation for some $r \geq 1$.

DEFINITION 3.1.2. A (k,r)-complete formula is a conjunction $+-A_1 \wedge \ldots \wedge +-A_n$, where $A_1,\ldots,A_n$ is a listing of the (k,r)-equations in lexicographic order based on the order of symbols: F^k , ( ) = $x_1 \ldots x_r$. An (∃,k,r)-complete sentence is a sentence $(\exists x_1 \neq \ldots \neq x_r)(\varphi)$, where φ is a (k,r)-complete formula.

DEFINITION 3.1.3. A (k,r)-layout is a conjunction $+-B_1 \wedge \ldots \wedge +-B_m$, where $B_1,\ldots,B_m$ is a listing of the (∃,k,r)-complete sentences in lexicographic order based on the order of symbols: F^k , ( ) = $x_1 \ldots x_r$ ¬ ∧ ∃.

DEFINITION 3.1.4. A (∀,k,r)-sentence is a sentence $(\forall x_1,\ldots,x_r)(\varphi)$ where φ is a propositional combination of (k,r)-equations. Y(k,r) is the set of all propositional combinations of (∀,k,r)-sentences.

We will be making a lot of use of the crucial complexity class Y(k,r) throughout the rest of the paper.

3.2. EVALUATION

In this section, we develop some general theory of the evaluation of values of Θ. For this purpose, we use the extensions of BSEP as the evaluators, as defined in Definition 1.7.

THEOREM 3.2.1. (BSEP) BSEP correctly evaluates Θ(1,r,N).

Proof: This follows from the usual procedure for N satisfiability of sentences in PC(=) with one unary function symbol, since these decision procedures can be treated within BSEP. See [BGG01], p. 315.

QED

THEOREM 3.2.2. (BSEP) Let T be a consistent extension of BSEP and $k,r \geq 1$. T evaluates $\Theta(k,r,N)$ as the number of (k,N)-special (r,S). The following are equivalent.
i. Every (k,N)-special (r,S) is provably (k,N)-special in T.
ii. T correctly evaluates $\Theta(k,r,N)$.

Proof: Let T,k,r be as given. The first claim is immediate from the provability of Lemma 2.1.1 in BSEP. For the second claim, assume i. T correctly identifies all of the non (k,N)-special $(r,S) \in W[k,r]$ as such, since this is a Σ^0_1 property. By i, T correctly identifies all $(r,S) \in W[k,r]$ as (k,N)-speical or not, and so T's count is correct. Thus we have established i $\rightarrow$ ii. Assume ii. Since the non (k,N)-special $(r,S) \in W[k,r]$ are recognized by T as such, the (k,N)-special $(r,s) \in W[k,r]$ must also be recognized as such in order for the count to be correct. Hence i. QED

THEOREM 3.2.3. (BSEP) Let $k,r \geq 1$. Every sentence "(r,S) is (k,N)-special", $(r,S) \in W[k,r]$, is BSEP equivalent to some sentence "φ is N satisfiable", $\varphi \in Y(k,r)$. Every sentence "φ is N satisfiable", $\varphi \in Y(k,r)$, is BSEP equivalent to a disjunction of sentences "(r,S) is (k,N)-special", $(r,S) \in W[k,r]$.

Proof: Let $(r,S) \in W[k,r]$. "(r,S) is (k,N)-special" asserts that there exists $F:N^k \rightarrow N$ such that $S = \alpha_r(F)$. For each $f \in W[k,r]$, $f \in \alpha_r(F)$ can be expressed as a $(\exists,k,r)$-complete sentence in F. Therefore "(r,S) is (k,N)-special" is expressed as the N satisfiability of a (k,r)-layout, (Definition 3.1.3).

Now let $\varphi \in Y(n,r)$, and look at "φ is N satisfiable." Put φ in disjunctive normal form, $\varphi_1 \vee \ldots \vee \varphi_m$, where the φ's are conjunctions of (k,r)-layouts and their negations. Then "φ is N satisfiable" is BSEP equivalent to "some φ_i is N satisfiable". Let $F:N^k \rightarrow N$. Each (k,r)-layout assigns a value to $\alpha_r(F)$, which may or may not be N satisfiable (it may even be patently absurd). These values are subsets of W[k,r]. So each φ_i is either absurd or assigns one particular value to $\alpha_r(F)$ or asserts merely that $\alpha_r(F)$ is not among some list of values. This can be rewritten as asserting that $\alpha_r(F)$ is among a certain list of possible values (values being subsets of W[k,r]). Hence φ ssserts that $\alpha_r(F)$ is among a combined list of possible values. Therefore the N satisfiability of φ takes the form "$\alpha_r(F) = S_1$ is N satisfiable" ∨ ... ∨ "$\alpha_r(F) = S_n$ is N satisfiable", which is equivalent to "(r,S_1) is (k,N)-special" ∨ ... ∨ "(r,S_n) is (k,N)-special". QED

THEOREM 3.2.4. (BSEP) Let T be a consistent extension of BSEP and $k,r \geq 1$. The following are equivalent.
i. Every N satisfiable sentence in Y(k,r) is provably N satisfiable in T.
ii. T correctly evaluates Θ(k,r,N).

Proof: Let T,k,r be as given. Assume i. We claim that i of Theorem 3.2.2 holds. To see this, let (r,S) be (k,N) special. By Theorem 3.2.3, let "(r,S) is (k,N)-speical" be BSEP equivalent to "φ is N satisfiable", φ ∈ Y(k,r). Then φ is N satisfiable, and so BSEP proves "φ is N satisfiable", "(r,S) is (k,N)-special". By Theorem 3.2.2, T correctly evaluates Θ(k,r,N). So we have shown that i → ii. Now suppose ii. By Theorem 3.2.2, every (k,N)-special (r,S) is provably (k,N)-special in T. We now claim i above. Let φ ∈ Y(k,r) be satisfiable. By Theorem 3.2.3, "φ is N satisfiable" is BSEP equivalent to a disjunction of sentences "(r,S) is (k,N)-special". Hence this disjunction is N satisfiable, and therefore by Theorem 3.2.2, this disjunction is provable in T. Hence T proves "φ is N satisfiable". QED

THEOREM 3.2.5. (BSEP) Let T be a consistent recursively presented extension of BSEP, and $k,r \geq 1$. Suppose T is equiconsistent with some sentence in Y(k,r). Then
i. T does not correctly evaluate Θ(k,r,N).
ii. If T does not prove its own inconsistency then T does not evaluate Θ(k,r,N).

Proof: Let T,k,r be as given, where T is equiconsistent with φ ∈ Y(k,r). Then φ is consistent. If φ has a finite model then BSEP proves that T has a model with complete diagram, and in particular BSEP proves that BSEP has a model with complete diagram. Therefore BSEP proves that BSEP has a model with complete diagram, which violates Gödel's Second Incompleteness Theorem. Hence φ has no finite model. Therefore φ is N satisfiable.

For i, suppose T correctly evaluates $\Theta(k,r,N)$. By Theorem 3.2.4, every N satisfiable sentence in Y(k,r) is provably N satisfiable in T. Since $\varphi \in Y(k,r)$ is N satisfiable, φ is provably N satisfiable in T. Hence T proves T is satisfiable. Hence T proves Con(T), because soundness is provable in ACA_0. By Godel's Second Incompleteness theorem, T is inconsistent. This establishes i.

For ii, suppose T does not prove its own inconsistency, and T evaluates $\Theta(k,r,N)$. Let M be a model of T + Con(T). By Gödel's Second Incompleteness Theorem, M satisfies Con(T + ¬Con(T)). So inside M we can use Gödel's Completeness Theorem to construct a model M' of T + ¬Con(T) with complete diagram. The integers in M' will form a proper end extension of the integers in M. All statements of the form "ψ is not N satisfiable" that hold in M, $\psi \in Y(k,r)$, still hold in M'. But "φ is N satisfiable" holds in M, yet fails in M'. We now convert over to (k,N)-special (r,S) via Theorem 3.2.3. All statements of the form "(r,S) is not (k,N)-special" that hold in M, $S \subseteq W[k,r]$, still hold in M'. But some disjunction of sentences "(r,S) is (k,N)-special" holds in M, yet fails in M', this disjunction being associated with φ via Theorem 3.2.3. Hence some sentence "(r,S) is (k,N)-special" holds inM yet fails in M'. Hence M,M' have different counts on $\Theta(k,r,N)$, by Theorem 3.2.2 (first claim). QED

4. EVALUATING $\Theta(4,7,N)$.

ZFC involves axiom schemes. For our results, we have to rid ourselves of these schemes, maintaining equiconsistency, on the way to giving equiconsistent purely universal systems with few variables, low arity, and no nesting.

In section 4.1, we present a general tool for ridding ourselves of unary schemes (as in separation). The extra complexity of working with the often encountered binary schemes (as in Replacement or Collection), is worth avoiding.

In section 4.2, we apply section 4.1 to a convenient axiomatization of ZFC without foundation and choice (but with an innocuous constant 0). We could try to use pared down fragments here that still correspond to ZFC in logical strength but that will not improve the results in this section.

4.1. AXIOMATIZING DEFINABILITY

We start with a system K_1 with finitely many axioms which we use to give an equiconsistent form of ZFC without schemes, in section 4.2.

DEFINITION 4.1. In PC(=), we use $v_1, v_2, \ldots$ for the variables. We write "M satisfies $\varphi[x_1,\ldots,x_n]$" if and only if φ is in the language of M, the free variables of φ are among $v_1,\ldots,v_n$, and M satisfies φ under any (some) assignment that assigns $x_1,\ldots,x_n \in \mathrm{dom}(M)$ to $v_1,\ldots,v_n$.

The language of K_1 is as follows.

1. Binary relation symbol $\in$. This is used in the axiomatization of ZF in section 4.2. Here $\in$ representa a binary relation which genrally is not actual membership.
2. Binary function symbol V (value). The idea is that V(x,y) is the value of x at y. Thus every x codes a function from objects to objects. In the intended models, V is internal to $(D,\in)$, and these coded functions are trivial off of some finite set, in the sense of $(D,\in)$. So if we fix any standardly finite length list of distinct parameters $u_1,\ldots,u_n$, $n \geq 1$, then every x codes an n-tuple, $V(x,u_1),\ldots,V(x,u_n)$.
3. Ternary function symbol C (change). C(x,y,z) has the same values (V) as x except that at y it is z. In the intended models, C is internal to $(D,\in)$.
4. Binary relation ε. This is not intended to be related to the membership relation in set theory, but rather the external membership relation in K1. Thus every x not only codes a function from objects to objects internal to $(D,\in)$, but also codes a set of objects, $\{y\colon y \mathrel{\varepsilon} x\}$, with the coding not internal to $(D,\in)$. In the intended models, these sets are exactly the subsets of D that are definable over $(D,\in)$ without =, but with parameters. So in that sense, each individual $\{y\colon y \mathrel{\varepsilon} x\}$ is internal to $(D,\in)$, but the actual coding as represented by the binary relation ε, is not internal to $(D,\in)$.
5. Constant symbol 0. This is used as a distinguished parameter, and also as the empty set in ZF.

6. Equality symbol =.

In this section, ∈ is merely a binary relation on a nonempty domain D, and may not have anything to do with membership. Consequently, in order to avoid any confusion, we will use the word "in" when talking about actual sets here.

The axioms of K1 are as follows.

K[1,1]. Change. $(\forall x,y,z,w)(V(C(x,y,z),y) = z \wedge (w \neq y \rightarrow V(C(x,y,z),w) = V(x,w)))$.
K[1,2]. Boolean. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow \neg(w\ \varepsilon\ x \wedge w\ \varepsilon\ y))$.
K[1,3]. Epsilon. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow V(w,x) \in V(w,y))$.
K[1,4]. Quantification. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow (\exists u)(C(w,x,u)\ \varepsilon\ y))$.
K[1,5]. Projection. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow C(y,0,w)\ \varepsilon\ x)$.

LEMMA 4.1.1. (BSEP) Every countable model of K1 is infinite. Let $M = (D,\in,V,C,0,\varepsilon)$ be countable model of K1 with complete diagram. Let φ be a formula in ∈ and no =, with free variables among $v_1,\ldots,v_k$, $k \geq 1$. Let $d_1,\ldots,d_k$ in D be distinct. There exists x in D such that for all y in D, $y\ \varepsilon\ x$ if and only if $(D,\in)$ satisfies $\varphi[V(y,d_1),\ldots,V(y,d_s)]$.

Proof: From K[1,2] we obtain

$(\forall x)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow \neg(w\ \varepsilon\ x))$.
$(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow w\ \varepsilon\ x \wedge w\ \varepsilon\ y)$.

by using x = y in K[1,2], and then going back to K[1,2], taking the z, and applying the first. So the various $\{x: x\ \varepsilon\ y\}$ form a Boolean algebra of sets. In particular, it includes the empty set and the universe. Hence there is at least one element other than 0, and so we let 0,1 be two distinct elements. By K[1,1], given any x,y, we can find z such that $V(z,0) = x \wedge V(z,1) = y$. This gives us a one-one binary function from elements to elements, which establishes that there are infinitely many elements.

We prove the following by induction on formulas φ in ∈ (no =). Assume the free variables of φ are among $v_1,\ldots,v_k$, $k \geq 1$. Let $d_1,\ldots,d_k$ in D be distinct. There exists x in D such that for all y in D, y ε x if and only if (D,∈) satisfies $\varphi[V(y,d_1),\ldots,V(y,d_k)]$. Here we quantify over $k \geq 1$.

a. $\varphi = v_i \in v_j$. Let $k \geq i,j$. Let $d_1,\ldots,d_k$ in D be distinct. Want x in D such that for all y in D, y ε x ↔ (D,∈) satisfies $v_i \in v_j[d_1,\ldots,d_k] \leftrightarrow V(y,d_i) \in V(y,d_j)$. Apply K[1,3] with x,y set to d_i,d_j. Here we quantify over $k \geq 1$.

Now assume that the assertion is true for φ,ψ. I.e.,

1) Let the free variables of φ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k$ in D be distinct. There exists x in D such that for all y in D, y ε x if and only if (D,∈) satisfies $\varphi[V(y,d_1),\ldots,V(y,d_k)]$. Here we have quantified over $k \geq 1$.

2) Let the free variables of ψ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k$ in D be distinct. There exists x in D such that for all y in D, y ε x if and only if (D,∈) satisfies $\psi[V(y,d_1),\ldots,V(y,d_k)]$. Here we have quantified over $k \geq 1$.

b. Show 1) holds for ¬φ. Let the free variables of ¬φ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k$ in D be distinct. Then the free variables of φ are among $v_1,\ldots,v_k$. Apply 1) to obtain x in D, and then use the complementation of x (for ε). Here we have quantified over $k \geq 1$.

c. Show 1) holds for φ ∧ ψ. Let the free variables of φ ∧ ψ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k$ in D be distinct. Then the free variables of φ,ψ are among $v_1,\ldots,v_k$. Apply 1),2) to obtain x,x', and then use the intersection of x,x' (for ε). Here we have quantified over $k \geq 1$.

d. Show 1) holds for $(\exists v_i)(\varphi)$. Let the free variables of $(\exists v_i)(\varphi)$ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k$ in D be distinct. We want: there exists x in D such that for all y in D, y ε x if and only if (D,∈) satisfies $\varphi[V(y,d_1),\ldots,V(y,d_k)]$. Instead of showing this under the given assumptions, it is more convenient to show this with the further assumption that $k \geq i$. Fortunately, it is obvious that this suffices.

Since $k \geq i$, the free variables of φ are among $v_1,\ldots,v_k$, and we apply the induction hypothesis to φ. Let x in D be given by 1).

We apply K[1,4] with x set to d_i and y set to x. By K[1,4], let z in D be such that

3) for all w in D, w ε z if and only if (∃u)(C(w,d_i,u) ε x).

By 1),

4) C(w,d_i,u) ε x if and only if (D,∈) satisfies φ[V(C(w,d_i,u),d_1),...,V(C(w,d_i,u),d_k)] if and only if (D,∈) satisfies φ[V(w,d_1),...,V(w,d_{i-1}),u,V(w,d_{i+1}),...,V(w,d_k)].

By 4),

5) (∃u)(C(w,d_i,u) ε x) if and only if (D,∈) satisfies (∃v_i)(φ)[V(w,d_1),...,V(w,d_k)].

By 3),5),

6) for all w in D, w ε z if and only if (D,∈) satisfies (∃v_i)(φ)[V(w,d_1),...,V(w,d_k)].

QED

THEOREM 4.1.2. (BSEP) Let M = (D,∈,V,C,0,ε) be a countable model of K1 with complete diagram. Let A ⊆ D be definable over (D,∈) without =, but with parameters. (∃x in D)(∀y in D)(y ε x ↔ y in A). Let φ be a formula in ∈ and no =, without x. K1 proves (∃x)(∀y)(y ε x ↔ φ).

Proof: Let M,A be as given. Let φ be a formula in ∈ (without =) whose free variables are among v_1,...,v_{k+1}, and u_1,...,u_k ∈ D, where

1) (∀y)(y in A ↔ (D,∈) satisfies φ[y,u_1,...,u_k]}.

Let 0,d_1,...,d_k in D be distinct. By Lemma 4.1.1, let x such that

2) (∀y)(y ε x ↔ (D,∈) satisfies φ[V(y,0),V(y,d_1),...,V(y,d_k)]).

By K[1,1], let y be such that V(y,d_1) = u_1 ∧ ... ∧ V(y,d_k) =

u_k. By K[1,5], let z be such that

3) (∀w)(w ε z ↔ C(y,0,w) ε x).

We claim (∀w)(w ε z ↔ w in A). Let w in D. By 1),3) rewrite this as (D,∈) satisfies C(y,0,w) ε x ↔ (D,∈) satisfies φ[w,u_1,...,u_k]. This is immediate by 2).

For the last claim, it is clear from the above that every countably infinite model M of K1 with complete diagram satisfies each such (∃x)(∀y)(y ε x ↔ φ). By the Gödel Completeness Theorem in ACA_0, we must have provability. QED

THEOREM 4.1.3. (ACA_0) Let (D,∈) be a countable model of BSEP with complete diagram. There is a model (D,∈,V,C,0,ε) of K1 where the various {y ∈ D: y ε x}, x in D, comprise exactly the subsets of D that are definable over (D,∈) without =, but with parameters.

Proof: Let (D,∈) be as given. Recall that BSEP has equality, and because of the presence of Extensionality in BSEP, equality is definable without = and no parameters, over (D,∈). Interpret 0 as the empty set in (D,∈). Interpret V(x,y) internally in (D,∈) by V(x,y) is the value of the function x at y if x is a finite function and x(y) exists; otherwise 0. Interpret C(x,y,z) internally in (D,∈) as the unique function x' = {<a,b>: x(a) = b ∧ a ≠ y} ∪ {<y,z>}. Obviously K[1,1] holds.

Since D is countably infinite, let A_x, x in D, comprise the subsets of D that are definable over (D,∈) without =, but with parameters. This construction goes through in ACA_0 using crucially the complete diagram of (D,∈). Interpret x ε y if and only if y lies in A_x. Obviously K[1,2] holds since the A's are closed under Boolean combinations. Let x,y in D. Since V(w,x) ∈ V(w,y) is definable (on w) without =, but with parameters x,y, K[1,3] holds. Also for K[1,4], (∃u)(C(w,x,u) ε y is definable (on w) without =, but with parameter x. Hence K[1,4] holds. In addition, C(y,0,w) ε x is definable (on w) without =, but with parameters y,0. Hence K[1,5] holds. QED

In Theorem 4.1.3, we are making no claim that the model of K1 so constructed has a complete diagram. However, in the next section, we will apply Theorem 4.1.3 in such a way that we will be ultimately stay on track by getting models with complete diagrams - with the help of the discussion at the beginning of section 3.

4.2. EQUICONSISTENCY WITH ZFC

Recall the discussion and definition at the beginning of section 3.

ZFC denotes the usual ZFC in $\in,=$. Let T1 be the following theory in the language $\in,0,\infty,\{\ \},\cup,\subseteq,\wp$, with $=$, where $\{\ \}$ is binary, $\cup$ is unary, and ∞ is a constant symbol.

We adhere to the following conventions concerning the presentation of relation schemes. The schematic relation letters appear in the form $\varphi(y_1,\ldots,y_n)$ where $y_1,\ldots,y_n$, $n \geq 0$ are distinct variables. When implemented, φ is to be replaced by any formula in the language of the theory (without schematic letters), in which $y_1,\ldots,y_n$ are allowed to appear as well as any variable that does not appear in the entire relation scheme itself. Such a scheme is called an n-ary relation scheme. Thus T[1,7],T[1,8] below are unary relation schemes. The usual Replacement and Collection and schemes for ZF are binary schemes. It is muchmore convenient for our purposes to work entirely with unary schemes.

T[1,1]. Definitional. $(\forall x)(x \notin 0) \wedge (\forall x,y)(x \subseteq y \leftrightarrow (\forall z)(z \in x \rightarrow z \in y))$.
T[1,2]. Extensionality. $(\forall x,y)(x \subseteq y \wedge y \subseteq x \rightarrow x = y)$.
T[1,3]. Pairing. $(\forall x,y)(x,y \in \{x,y\})$.
T[1,4]. Union. $(\forall x,y,z)(y \in z \wedge z \in x \rightarrow y \in \cup x)$.
T[1,5]. Power Set. $(\forall x,y)(y \subseteq x \rightarrow y \in \wp(x))$.
T[1,6]. Infinity. $0 \in \infty \wedge (\forall x)(x \in \infty \rightarrow \{x,x\} \in \infty))$.
T[1,7]. Separation. $(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow w \in y \wedge \varphi(w))$, where φ is a formula in $\in$, without $=$, in which y,z do not appear.
T[1,8]. Unary Collection. $(\forall y)(\exists z)(\forall w \in y)((\exists u)(w \in u \wedge \varphi(u)) \rightarrow (\exists u \in z)(w \in u \wedge \varphi(u)))$, where φ is a formula in $\in$, without $=$, in which y,z,w do not appear.

Thus T1 results from ZFC by adding a few standard symbols, dropping Choice and Foundation, replacing Replacement with

Unary Collection, simplifying Infinity, and disallowing = in Separation and Unary Collection. But = is used in T1 (see T[1,2]).

THEOREM 4.2.1. (EFA) T1 proves ZF without Foundation.

Proof: First note that by T[1,2], we have a definition of = (see T[1,2]), so that T_1 proves T[1,7], T[1,8] with = allowed. We now show that T1 proves these two usual ZF axioms (with =):

INFINITY. $(\exists x)(\emptyset \in x \wedge (\forall y \in x)(y \cup \{y\} \in x))$, expanded out with the usual abbreviations.

REPLACEMENT. $(\forall y \in x)(\exists ! z)(\varphi(x,y) \rightarrow (\exists w)(\forall y \in x)(\exists z \in w)(\varphi(x,y))$, where ! is expanded out in the usual way.

Actually, we prove the well known simpler sharper form, which does not use !.

COLLECTION. $(\forall y \in x)(\exists z)(\varphi(x,y) \rightarrow (\exists w)(\forall y \in x)(\exists z \in w)(\varphi(x,y))$, where ! is expanded out in the usual way.

For Collection, assume $(\forall y \in x)(\exists z)(\varphi)$, where x,w do not appear in φ. We can assume v does not appear in φ. Then for all $y \in x$ there exists w such that $(\exists z \in w)(\varphi)$. By T[1,8], let v be such that for all $y \in x$ there exists $w \in v$ such that $(\exists z \in w)(\varphi)$. Hence for all $y \in x$ there exists $z \in \cup v$ such that φ. Now apply T[1,4],T[1,7].

For Infinity, first note that $0 = \emptyset$ by T[1,1]. Now we say that x is a finite ordinal if and only if
i. x is epsilon connected and transitive.
ii. Every nonempty subset of x has an epsilon minimal element and an epsilon maximal element.

Obviously $\emptyset$ is a finite ordinal, and if x is a finite ordinal then $x \cup \{x\}$ is a finite ordinal. It is well known that all standard facts about finite ordinals can be proved using i-iii in the presence of the axioms of T without T[1,5].

It is clear by induction that for all finite ordinals n there exists a unique function f_n with domain n+1 satisfying $f(0) = 0 \wedge (\forall i \in n)(f(i+1) = \{f(i)\})$. Furthermore, these unique f_n are comparable, and are into ∞. We claim that each f_n is one-one. Assume fi is one-one, and we show that f_{i+1} is one-one. By comparability, we can assume that $f_{i+1}(i+2) = f_{i+1}(j)$, $0 < j < i+2$. Then $f_{i+1}(i+1) = f_{i+1}(j-1)$, $0 \leq j-1 < i+1$, which contradicts that f_i is one-one.

Thus some of the y ∈ ∞ are some $f_n(n+1)$, n unique, and every finite ordinal n gets used in this way, uniquely. So by Collection, we obtain a set that includes all finite ordinals as elements. Hence we obtain the set ω of all finite ordinals. It is easily verified that ω witnesses INFINITY. QED

THEOREM 4.2.2. ZFC,T1 are equiconsistent. T1 proves 0 ≠ ∞.

Proof: We argue in ACA_0. Let M be a countable model of ZFC with complete diagram. Let M' be the expansion of M by interpreting 0 as ∅, ∞ as ω, { } as unordered pairing, ∪ as unary union, ⊆ as inclusion, and ℘ as power set, all in the sense of M. Because of the definability of these symbols, there is no problem creating the complete diagram of M'. Obviously M' satisfies T1 as all of the axioms above easily follow from ZF under this interpretation.

Conversely, let M be a countable model of T1 with complete diagram. By Theorem 4.2.1, we obtain a countable model of ZF without Foundation, with complete diagram, by merely dropping information. It is well known how to construct a countable model of ZFC from M' by relativizing to the cumualative hierarchy (the V's) and then to the inner model of Godel's constructible sets. Since this is an explicitly definable relativization, there is no problem obtaining its complete diagram.

The last claim is immediate from T[1,1] and T[1,6]. QED

We now appropriately combine T1 and K1 into the following system K1T2 with finitely many axioms, whose language is that of K1 and T1 combined. This requires that the schemes T[1,7],T[1,8] be replaced by the axioms T[2,7],T[2,8] below. The other six axioms of T1 are retained but relabeled as T[2,1] - T[2,6].

K[1,1]. Change. (∀x,y,z,w)(V(C(x,y,z),y) = z ∧ (w ≠ y → V(C(x,y,z),w) = V(x,w))).
K[1,2]. Boolean. (∀x,y)(∃z)(∀w)(w ε z ↔ ¬(w ε x ∧ w ε y)).

K[1,3]. Epsilon. (∀x,y)(∃z)(∀w)(w ε z ↔ V(w,x) ∈ V(w,y)).
K[1,4]. Quantification. (∀x,y)(∃z)(∀w)(w ε z ↔ (∃u)(C(w,x,u) ε y)).
K[1,5]. Projection. (∀x,y)(∃z)(∀w)(w ε z ↔ C(y,0,w) ε x).
T[2,1]. Definitional. (∀x)(x ∉ 0) ∧ (∀x,y)(x ⊆ y ↔ (∀z)(z ∈ x → z ∈ y)).
T[2,2]. Extensionality. (∀x,y)(x ⊆ y ∧ y ⊆ x → x = y).
T[2,3]. Pairing. (∀x,y)(x,y ∈ {x,y}).
T[2,4]. Union. (∀x,y,z)(y ∈ z ∧ z ∈ x → y ∈ ∪x).
T[2,5]. Power Set. (∀x,y)(y ⊆ x → y ∈ ℘(x)).
T[2,6]. Infinity. 0 ∈ ∞ ∧ (∀x)(x ∈ ∞ → {x,x} ∈ ∞).
T[2,7]. Separation. (∀x,y)(∃z)(∀w)(w ∈ z ↔ w ∈ y ∧ w ε x).
T[2,8]. Unary Collection. (∀x,y)(∃z)(∀w ∈ y)((∃u)(w ∈ u ∧ u ε x) → (∃u ∈ z)(w ∈ u ∧ u ε x)).

THEOREM 4.2.3. ZFC,T1,K1T2 are equiconsistent. K1T2 proves T1 proves 0 ≠ ∞.

Proof: We argue in ACA_0, and let M be a countable model of T1 with complete diagram. As in the proof of Theorem 4.2.2, we create a countable model M', (D,∈,0,∞,{ },∪,⊆,℘), with =, of T1, using extensionality in T1, and with complete diagram. By Theorem 4.1.3, let M'' be a countable expansion of (D,∈) satisfying K1 and where the various {y ∈ D: y ε x}, x in D, comprise exactly the subsets of D that are definable over (D,∈) without equality, but with parameters.

M'' is a model of K1, which has finitely many axioms, but we are not claiming that ACA_0 is sufficient to construct its complete diagram. It is now clear that T[2,7],T[2,8] hold in M''. Also we can interpret ∈,0,∞,{ },∪,⊆,℘ in the obvious way to provde a further expansion M''' of M that satisfies the finitely axiomatized theory K1T2. We obtain a countable model of K1T2 with complete diagram according to the discussion at the beginning of section 3.

Conversely, it is clear that K1T2 proves T1 by the second claim of Theorem 4.1.2. We obtain a countable model with complete diagram of T1 from a countable model of K1T2 with complete diagram by dropping information. Also T1 proves 0 ≠ ∞ by the last claim of Theorem 4.2.1. QED

We are now going to move into the realm of universal sentences and universal theories, where we eliminate all quantifiers except the outer layer of zero or more universal quantifiers. We will have constant, relation, and function symbols of various arities. We need to be careful about how they are used in universal sentences. Eventually, after a long series of equiconsistent conversions, we will arrive at sentences in Y(4,7).

DEFINITION 4.2.1. A strict sentence is a universal sentence whose universal quantifiers are distinct, and whose atomic subformulas are of the form $x = y$, $F(x_1,...,x_k) = y$, $y = F(x_1,...,x_k)$, $F(x_1,...,x_k) = G(y_1,...,y_n)$, $R(z_1,...,z_m)$, where F is a k-ary function symbol, G is an n-ary function symbol, R is a m-ary relation symbol, $x_1,...,x_k,y_1,...,y_n,z_1,...,z_m$ are variables, and x,y are variables or constants. A strict theory is a finite set of strict sentences.

DEFINITION 4.2.2. A k-sentence is a universal sentence whose universal quantifiers are distinct, and whose atomic subformulas are of the form $x = y$, $G^k(x_1,...,x_k) = y$, $y = G_k(x_1,...,x_k)$, $G^k(x_1,...,x_k) = G^k(y_1,...,y_k)$, where G^k is a k-ary function symbol and $x,y,x_1,...,x_k,y_1,...,y_k$ are either variables or constants. A k-theory is a finite set of k-sentences, where it is understood that all elements use the same G^k.

Note that in strict sentences, constant symbols are not allowed to be in the scope of function symbols, but in k-sentences, they are allowed to be in the scope of the unique function symbol G^k.

Note that k-sentences are a lot closer to Y(k,r) than strict sentences. But still all of the constant symbols need to be removed.

We will ultimately arrive at a 4-theory S in 7 variables x,y,z,w,u,v,e with no constant symbols. Note that $S \subseteq Y(4,7)$, and so we will apply Theorem 3.2.5.

We have already passed from ZFC, a theory in predicate calculus with = with finitely many axiom schemes, to a theory K1T2 with finitely many axioms. We now pass to a strict theory by performing Skolemization/denesting on the finitely many axioms of K1T2. This is a well known procedure that can be applied to any set of sentences in first order predicate calculus with =, and always results in a strict theory that is equiconsistent. The

Skolemization operations applied to φ always yields ψ that logically implies φ. The denesting operations applied to φ always yields ψ logically equivalent to φ.

We strive to keep all of the variables among the five variables x,y,z,w,u, and keep all of the Skloem functions of arity ≤ 3. However, we do run into some trouble accomplishing this, and so we use an additional device. We introduce new auxiliary functions or relations defining - and substituting for - certain subformulas. As in Skolemization, this results in ψ that logically implies φ.

We will not introduce any new constant symbols, and will keep the number of relation and function symbols introduced under control. The equiconsistency will always be apparent.

For K[1,1], we have

K[1,1]. (∀x,y,z,w)(V(C(x,y,z),y) = z ∧ (w ≠ y → V(C(x,y,z),w) = V(x,w))).
K[1,1]. (∀x,y,z,w,u)(u = C(x,y,z) → V(u,y) = z ∧ (w ≠ y → V(u,w) = V(x,w))).

A logical equivalence.

For K[1,2], we have

K[1,2]. (∀x,y)(∃z)(∀w)(w ε z ↔ ¬(w ε x ∧ w ε y)).
K[1,2]. (∀x,y,w)(w ε F1(x,y) ↔ ¬(w ε x ∧ w ε y)).
K[1,2]. (∀x,y,w,u)(u = F1(x,y) → (w ε u ↔ ¬(w ε x ∧ w ε y))).
K[1,2]. (∀x,y,z,w)(w = F1(x,y) → (z ε w ↔ ¬(z ε x ∧ z ε y))).

A logical strengthening.

For K[1,3], we have

K[1,3]. (∀x,y)(∃z)(∀w)(w ε z ↔ V(w,x) ∈ V(w,y)).
K[1,3]. (∀x,y,w)(w ε F2(x,y) ↔ V(w,x) ∈ V(w,y)).
K[1,3]. (∀x,y,w)(w ε F2(x,y) ↔ R1(x,y,w))) ∧ (∀x,y,z)(R1(x,y,z) ↔ V(z,x) ∈ V(z,y)).
K[1,3]. (∀x,y,z,w)(z = F2(x,y) → (w ε z ↔ R1(x,y,w))) ∧ (∀x,y,z,w,u)(w = V(z,x) ∧ u = V(z,y) → (R1(x,y,z) ↔ w ∈ u)).

A logical strengthening.

For K[1,4], we have

K[1,4]. (∀x,y)(∃z)(∀w)(w ε z ↔ (∃u)(C(w,x,u) ε y)).
K[1,4]. (∀x,y,w)(w ε F3(x,y) ↔ (∃u)(C(w,x,u) ε y)).
K[1,4]. (∀x,y,z)(z ε F3(x,y) ↔ (∃w)(C(z,x,w) ε y)).
K[1,4]. (∀x,y,z)(z ε F3(x,y) ↔ R2(x,y,z)) ∧
(∀x,y,z)(R2(x,y,z) ↔ (∃w)(C(z,x,w) ε y)).
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧
(∀x,y,z)(R2(x,y,z) → (∃w)(C(z,x,w) ε y)) ∧
(∀x,y,z)((∃w)(C(z,x,w) ε y) → R2(x,y,z)).
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧
(∀x,y,z)(∃w)(R2(x,y,z) → C(z,x,w) ε y) ∧
(∀x,y,z,w)(C(z,x,w) ε y → R2(x,y,z)).
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧
(∀x,y,z)(R2(x,y,z) → C(z,x,F4(x,y,z)) ε y) ∧ (∀x,y,z,w,u)(u
= C(z,x,w) ∧ u ε y → R2(x,y,z)).
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧
(∀x,y,z,w)(w = F4(x,y,z) ∧ R2(x,y,z) → C(z,x,w) ε y) ∧
(∀x,y,z,w,u)(u = C(z,x,w) ∧ u ε y → R2(x,y,z)).
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧
(∀x,y,z,w,u)(u = C(z,x,w) ∧ w = F4(x,y,z) ∧ R2(x,y,z) → u ε
y) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ u ε y → R2(x,y,z)).

A logical strengthening.

For K[1,5], we have

K[1,5]. (∀x,y)(∃z)(∀w)(w ε z ↔ C(y,0,w) ε x).
K[1,5]. (∀x,y,w)(w ε F5(x,y) ↔ C(y,0,w) ε x).
K[1,5]. (∀x,y,z,w)(z = F5(x,y) → (w ε z ↔ C(y,0,w) ε x)).
K[1,5]. (∀x,y,z,w,u)(u = C(y,0,w) ∧ z = F5(x,y) → (w ε z ↔
u ε x)).
K[1,5]. (∀x,y,z,w,u)(u = F6(y,w) ∧ z = F5(x,y) → (w ε z ↔
u ε x)) ∧ (∀x,y)(F6(x,y) = C(x,0,y)).
K[1,5]. (∀x,y,z,w,u)(u = F6(y,w) ∧ z = F5(x,y) → (w ε z ↔
u ε x))) ∧ (∀x,y,z,w,u)(z = 0 ∧ w = C(x,z,y) → F6(x,y) =
w).

A logical strengthening.

For T[2,1], we have

T[2,1]. $(\forall x)(x \notin 0) \wedge (\forall x,y)(x \subseteq y \leftrightarrow (\forall z)(z \in x \rightarrow z \in y))$.
T[2,1]. $(\forall x,y)(y = 0 \rightarrow x \notin y) \wedge (\forall x,y)(x \subseteq y \rightarrow (\forall z)(z \in x \rightarrow z \in y)) \wedge (\forall x,y)((\forall z)(z \in x \rightarrow z \in y) \rightarrow x \subseteq y)$.
T[2,1]. $(\forall x,y)(y = 0 \rightarrow x \notin y) \wedge (\forall x,y,z)(x \subseteq y \wedge z \in x \rightarrow z \in y)) \wedge (\forall x,y)(\exists z)((z \in x \rightarrow z \in y) \rightarrow x \subseteq y)$.
T[2,1]. $(\forall x,y)(y = 0 \rightarrow x \notin y) \wedge (\forall x,y,z)(x \subseteq y \wedge z \in x \rightarrow z \in y)) \wedge (\forall x,y)((F7(x,y) \in x \rightarrow F7(x,y) \in y) \rightarrow x \subseteq y)$.
T[2,1]. $(\forall x,y)(y = 0 \rightarrow x \notin y) \wedge (\forall x,y,z)(x \subseteq y \wedge z \in x \rightarrow z \in y) \wedge (\forall x,y,z)(z = F7(x,y) \wedge (z \in x \rightarrow z \in y) \rightarrow x \subseteq y)$.

A logical strengthening.

For T[2,2], we have

T[2,2]. $(\forall x,y)(x \subseteq y \wedge y \subseteq x \rightarrow x = y)$.

For T[2,3], we have

T[2,3]. $(\forall x,y)(x,y \in \{x,y\})$.
T[2,3]. $(\forall x,y.z)(z = \{x,y\} \rightarrow x \in z \wedge y \in z)$.

A logical equivalence.

For T[2,4], we have

T[2,4]. $(\forall x,y,z)(y \in z \wedge z \in x \rightarrow y \in \cup x)$.
T[2,4]. $(\forall x,y,z,w)(w = \cup x \wedge y \in z \wedge z \in x \rightarrow y \in w)$.

A logical equivalence.

For T[2,5], we have

T[2,5]. $(\forall x,y)(y \subseteq x \rightarrow y \in \wp(x))$.
T[2,5]. $(\forall x,y,z)(z = \wp(x) \wedge y \subseteq x \rightarrow y \in z)$.

A logical equivalence.

For T[2,6], we have

T[2,6]. $0 \in \infty \wedge (\forall x)(x \in \infty \rightarrow \{x,x\} \in \infty))$.
T[2,6]. $(\forall x,y)(x = 0 \wedge y = \infty \rightarrow x \in y) \wedge (\forall x,y,z)(y = \infty \wedge z$

= {x,x} ∧ x ∈ y → z ∈ y).

A logical equivalence.

For T[2,7], we have

T[2,7]. (∀x,y)(∃z)(∀w)(w ∈ z ↔ w ∈ y ∧ w ε x).
T[2,7]. (∀x,y,w)(w ∈ F8(x,y) ↔ w ∈ y ∧ w ε x).
T[2,7]. (∀x,y,z,w)(z = F8(x,y) → (w ∈ z ↔ (w ∈ y ∧ w ε x))).

A logical strengthening.

For T[2,8], we have

T[2,8]. (∀x,y)(∃z)(∀w ∈ y)((∃u)(w ∈ u ∧ u ε x) → (∃u ∈ z)(w ∈ u ∧ u ε x)).
T[2,8]. (∀x,y)(∃z)(∀w)(w ∈ y ∧ (∃u)(w ∈ u ∧ u ε x) → (∃u)(u ∈ z ∧ w ∈ u ∧ u ε x)).
T[2,8]. (∀x,y)(∃z)(∀w)(w ∈ y ∧ (∃v)(w ∈ v ∧ v ε x) → (∃u)(u ∈ z ∧ w ∈ u ∧ u ε x)).
T[2,8]. (∀x,y)(∃z)(∀w)((∃v)(w ∈ y ∧ w ∈ v ∧ v ε x) → (∃u)(u ∈ z ∧ w ∈ u ∧ u ε x)).
T[2,8]. (∀x,y)(∃z)(∀w)(∃u)((∃v)(w ∈ y ∧ w ∈ v ∧ v ε x) → u ∈ z ∧ w ∈ u ∧ u ε x).
T[2,8]. (∀x,y)(∃z)(∀w)(∃u)(∀v)(w ∈ y ∧ w ∈ v ∧ v ε x → u ∈ z ∧ w ∈ u ∧ u ε x).
T[2,8]. (∀x,y,w)(∃u)(∀v)(w ∈ y ∧ w ∈ v ∧ v ε x → u ∈ F9(x,y) ∧ w ∈ u ∧ u ε x).
T[2,8]. (∀x,y,w,v)(w ∈ y ∧ w ∈ v ∧ v ε x → F10(x,y,w) ∈ F9(x,y) ∧ w ∈ F10(x,y,w) ∧ F10(x,y,w) ε x).
T[2,8]. (∀x,y,w,v,u)(u = F10(x,y,w) ∧ w ∈ y ∧ w ∈ v ∧ v ε x → u ∈ F9(x,y) ∧ w ∈ u ∧ u ε x).
T[2,8]. (∀x,y,w,v,u)(u = F10(x,y,w) ∧ w ∈ y ∧ w ∈ v ∧ v ε x → R3(u,x,y) ∧ w ∈ u ∧ u ε x) ∧ (∀x,y,z)(R3(x,y,z) ↔ x ∈ F9(y,z)).
T[2,8]. (∀x,y,w,v,u)(u = F10(x,y,z) ∧ w ∈ y ∧ z ∈ w ∧ w ε x → R3(u,x,y) ∧ z ∈ u ∧ u ε x) ∧ (∀x,y,z,w)(w = F9(y,z) → (R3(x,y,z) ↔ x ∈ w)).

A logical strengthening.

We now organize these results the system K2T3. Note that

K2T3 is strict.

K[2,1]. (∀x,y,z,w,u)(u = C(x,y,z) → V(u,y) = z ∧ (w ≠ y → V(u,w) = V(x,w))).
K[2,2]. (∀x,y,z,w)(w = F1(x,y) → (z ε w ↔ ¬(z ε x ∧ z ε y))).
K[2,3]. (∀x,y,z,w)(z = F2(x,y) → (w ε z ↔ R1(x,y,w))) ∧ (∀x,y,z,w,u)(w = V(z,x) ∧ u = V(z,y) → (R1(x,y,z) ↔ w ∈ u)).
K[2,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ w = F4(x,y,z) ∧ R2(x,y,z) → u ε y) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ u ε y → R2(x,y,z)).
K[2,5]. (∀x,y,z,w,u)(u = F6(y,w) ∧ z = F5(x,y) → (w ε z ↔ u ε x))) ∧ (∀x,y,z,w,u)(z = 0 ∧ w = C(x,z,y) → F6(x,y) = w).
T[3,1]. (∀x,y)(y = 0 → x ∉ y) ∧ (∀x,y,z)(x ⊆ y ∧ z ∈ x → z ∈ y) ∧ (∀x,y,z)(z = F7(x,y) ∧ (z ∈ x → z ∈ y) → x ⊆ y).
T[3,2]. (∀x,y)(x ⊆ y ∧ y ⊆ x → x = y).
T[3,3]. (∀x,y.z)(z = {x,y} → x ∈ z ∧ y ∈ z).
T[3,4]. (∀x,y,z,w)(w = ∪x ∧ y ∈ z ∧ z ∈ x → y ∈ w).
T[3,5]. (∀x,y,z)(z = ℘(x) ∧ y ⊆ x → y ∈ z).
T[3,6]. (∀x,y)(x = 0 ∧ y = ∞ → x ∈ y) ∧ (∀x,y,z)(y = ∞ ∧ z = {x,x} ∧ x ∈ y → z ∈ y).
T[3,7]. (∀x,y,z,w)(z = F8(x,y) → (w ∈ z ↔ (w ∈ y ∧ w ε x))).
T[3,8]. (∀x,y,w,v,u)(u = F10(x,y,z) ∧ w ∈ y ∧ z ∈ w ∧ w ε x → R3(u,x,y) ∧ z ∈ u ∧ u ε x) ∧ (∀x,y,z,w)(w = F9(y,z) → (R3(x,y,z) ↔ x ∈ w)).

The language of K2T3 is as follows.

2 constant symbols. 0,∞.
2 unary function symbols. ∪,℘.
10 binary function symbols. V,{ },F1,F2,F3,F5,F6,F7,F8,F9.
3 ternary function symbols. C,F4,F10.
3 binary relation symbols. ∈,ε,⊆.
3 ternary relation symbols. R1,R2,R3.
variables x,y,z,w,u.

THEOREM 4.2.4. ZFC,T1,K1T2,K2T3 are equiconsistent. K2T3 proves K1T2 proves T1 proves 0 ≠ ∞. K2T3 is strict.

Proof: It is clear by inspection that any countable model of K2T3 with complete diagram is a model of K1T2 by merely dropping information. Now let M be a countable model of K1T2 with complete diagram. We can obviously build an expansion which is a countable model in the standard way of K2T3 armed with arithmetic comprehension, but we will lose all semblance of a complete diagram. But we do have a countable model of K2T3, and K2T3 is given by finitely many axioms. So as discussed at the beginning of section 3, we obtain a countable model of K2T3 with complete diagram as is required for equiconsistency. T1 proves $0 \neq \infty$ by T[3,1],T[3,6], setting $x = 0$ and $y = \infty$. QED

We now remove all of the relation symbols from K2T3 in favor of new function symbols of the same arity by the obvious crude general method. We replace each $R(x_1,...,x_k)$ by $F(x_1,...,x_k) = x_1$, where F is associated with R. We have already used F1,...,F10, and so the numbering of the new function symbols starts with 11. This results in the system W whose language consists of

2 constant symbols $0,\infty$.
2 unary function symbols $\cup,\wp$.
13 binary function symbols V,{
},F1,F2,F3,F5,F6,F7,F8,F9,F11,F12,F13.
6 ternary function symbols C,F4,F10,F14,F15,F16.
variables x,y,z,w,u.

THEOREM 4.2.5. ZFC,T1,K1T2,K2T3,W are equiconsistent. W proves $0 \neq \infty$. W is strict.

Proof: The second claim is clear by the conversions of T[3,1],T[3,6] for W, setting $x = 0$ and $y = \infty$. Let M be a countable model of K2T3 with complete diagram. By Theorem 4.2.3, M has at least two elements. So we can interpret the 2,3-ary relation symbols R^2,R^3 in K2T3 by $F_{R^\wedge 2}(x,y) = 0$ if $R^2(x,y)$; ∞ otherwise, $F_{R^\wedge 3}(x,y,z) = 0$ if $R^3(x,y,z)$; ∞ otherwise. There is no problem obtaining the complete diagram. Let M be a countable model of W with complete diagram. We can interpret the function symbols that replace relation symbols by $R^2(x,y) \leftrightarrow F_{R^\wedge 2}(x,y) = x$, $R^3(x,y,z) \leftrightarrow F_{R^\wedge 3}(x,y,z) = x$). Here $F_{R^\wedge 2},F_{R^\wedge 3}$ are the function symbols in W associated with the relation symbols R2,R3 in K2T3. Again there is no problem obtaining the complete diagram. QED

4.3. IMPOSSIBLE COUNTING

We now transform W into an equiconsistent 4-system W[G,1,2] in 5 variables and 2 constants, getting a step closer to Y(4,7) and being able to apply Theorem 3.2.5.

W1*. 1 ≠ 2.
W2*. For all x,y,z ∉ {1,2}, the interpretation of all 23 symbols of W, given below, are ∉ {1,2}.
W3*. Every sentence of W holds for all x,y,z,w,u ∉ {1,2} when interpreted as given below.
0 = G(1,1,2,2)
∞ = G(2,2,1,1)
∪x = G(1,1,1,x)
℘x = G(1,1,x,1)
V(x,y) = G(1,1,x,y)
{x,y} = G(1,x,1,y)
F1(x,y) = G(1,x,y,1)
F2(x,y) = G(x,1,1,y)
F3(x,y) = G(x,1,y,1)
F5(x,y) = G(x,y,1,1)
F6(x,y) = G(2,2,x,y)
F7(x,y) = G(2,x,2,y)
F8(x,y) = G(2,x,y,2)
F9(x,y) = G(1,x,2,y)
F11(x,y) = G(x,2,2,y)
F12(x,y) = G(x,2,y,2)
F13(x,y) = G(x,y,2,2)
C(x,y,z) = G(1,2,x,y)
F4(x,y,z) = G(1,x,y,z)
F10(x,y,z) = G(x,1,y,z)
F14(x,y,z) = G(x,y,1,z)
F15(x,y,z) = G)x,y,z,1)
F16(x,y,z) = G(2,x,y,z)

THEOREM 4.3.1. ZFC,T1,K1T2,K2T3,W,W[G,1,2] are equiconsistent. W[G,1,2] is a 4-system in x,y,z,w,u,1,2.

Proof: Let M be a countable model of W with countable domain D, with complete diagram, which we can assume has 1,2 ∉ D. We define a model M' of W[G,1,2] as follows. The domain of M' is D ∪ {1,2}. The above 23 equations determine the values of G on 23 pairwise disjoint subsets of $(D \cup \{1,2\})^4$, where the letters x,y,z represent elements of D only, and these values lie in D. The disjointness is clear from the pattern of displayed 1's and 2's. We set G to be 1 at all remaining elements of $(D \cup \{1,2\})^4$. Clearly M' is a countable model of W[G,1,2], and we can easily obtain the complete diagram.

Now let M be a model of W[G,1,2] with countable domain D, with complete diagram. By W1*, 1 ≠ 2 are distinct elements of D. We construct a model of W with domain D\{1,2} by interpreting the function symbols of W according to the above 21 equations, and also interpreting 0,∞ as G(1,1,2,2),G(1,2,1,2). By W2*, this interprets all of the function symbols of W by functions from and into D\{1,2}, and also the constants 0,∞ are interpreted as elements of D\{1,2}. In addition, D\{1,2} is nonempty since by W2*, 0 ∉ {1,2}. By W3* all of the axioms of W hold in this structure. There is no problem obtaining the complete diagram. QED

LEMMA 4.3.2. (BSEP) In any countable model of W[G,1,2], we can change values of G(x,x,x,x), and of G(x,y,z,w) for x,y,z,w ∈ {1,2}, other than at 1,1,2,2 and 2,2,1,1, in any way, and still satisfy W[G,1,2].

Proof: Left to the reader. Of course, in so doing, we may not immediately have a complete diagram. QED

Finally, let S be the following 4-system in variables x,y,z,w,u,v,e with no constants.

S1. (∀x)(G(x,x,x,x) ≠ x).
S2. (∀x,y,z)(G(x,x,x,x),G(y,y,y,y),G(z,z,z,z) are not distinct).
S3. (∀x,y,v,e)(G(x,x,x,x) = v ≠ e = G(y,y,y,y) → G(v,v,v,e) = G(e,e,e,v) = v ∨ G(v,v,v,e) = G(e,e,e,v) = e).
S4. Let G(v,v,v,v) ≠ G(e,e,e,e) ∧ G(v,v,v,e) = G(e,e,e,v) = v. Then the axioms of W[G,1,2] hold with 1 replaced by v and 2 replaced by e.

The axioms of W[G,1,2] are explicitly compiled in section 8.

THEOREM 4.3.3. ZFC,T1,K1T2,K2T3,W,W[G,1,2],S are equiconsistent. S is a 4-system in x,y,z,w,u,v,e. S ⊆ Y(4,7).

Proof: Let M be a countable model of W[G,1,2] with complete diagram. Change G so that G(x,x,x,x) = 1 if x ≠ 1; 2 if x = 1. Change G so that G(1,1,1,2) = G(2,2,2,1) = 1. We stil have a model of W[G,1,2] by Lemma 4.3.2. By the first change, 1,2 comprise the values of G(x,x,x,x), and therefore S1,S2 hold. For S3, let G(x,x,x,x) ≠ G(y,y,y,y) ∧ G(x,x,x,x) = v ∧ G(y,y,y,y) = e. Then {v,e} = {1,2}. If v,e = 1,2 then G(v,v,v,e) = G(e,e,e,v) = v. If v,e = 2,1 then G(v,v,v,e) = G(e,e,e,v) = e. For S4, let G(v,v,v,v) ≠ G(e,e,e,e) ∧ G(v,v,v,e) = G(e,e,e,v) = v. Then {v,e} = {1,2}, and so v,e = 1,2. Since W[G,1,2] holds, clearly W[G,1,2] holds with 1 replaced by v and 2 replaced by e. It is clear that the complete diagram of the resulting model of S is easily obtained.

Conversely, let M be a countable model of S with domain D, with complete diagram. By S1, D has at least two elements. By S1, G(x,x,x,x) cannot be constant (if the constant is c, then G(c,c,c,c) = c). Hence there are at least two values of G(x,x,x,x). By S3, there are exactly two values of G(x,x,x,x), v,e. By S3, G(v,v,v,e) = G(e,e,e,v) = v ∨ G(v,v,v,e) = G(e,e,e,v) = e. In either case, we find two distinct values of G(x,x,x,x), v ≠ e, such that G(v,v,v,e) = G(v,v,v,e) = v. By S4, the axioms of W[G,1,2] hold with 1 replaced by v and 2 replaced by e. Thus we have a countable model of W[G,1,2], with the complete diagram readily obtained. QED

THEOREM 4.3.4. (BSEP) Assuming ZFC is consistent, ZFC does not correctly evaluate Θ(4,7,N). Assuming ZFC does not prove its own inconsistency, ZFC does not evaluate Θ(4,7,N).

Proof: Since S ⊆ Y(4,7), this is immediate from Theorem 3.2.5. QED

4.4. GENERALIZATION

Here we generalize the later stages of the proof of Theorem 4.3.4 in preparation for section 5, where we treat ZFC + I1 instead of just ZFC. Here are the later stages.

1. We start with the strict system K2T3, equiconsistent with ZFC, in x,y,z,w,u, with quite a number of constant, relation, and function symbols, each of arity ≤ 3.
2. We pass to the strict system W, equiconsistent with K2T3, in x,y,z,w,u, with quite a number of constant and function symbols, each of arity ≤ 3.
3. We pass to the 4-system W[G,1,2], equiconsistent with W, in x,y,z,w,u,1,2.

4. We pass to the 4-system S, equiconsistent with W[G,1,2], in variables x,y,z,w,u,v,e.
5. We then apply Theorem 3.2.5, since $S \subseteq Y(4,7)$.

THEOREM 4.4.1. (EFA) Let X be a strict system in variables $x_1,...,x_k$. X is equiconsistent with a strict system X' in variables $x_1,...,x_k$, with the constant symbols of X, no relation symbols, function symbols of X, and new function symbols corresponding to the relation symbols of X, the latter preserving the arities of the relation symbosl of X.

Proof: The proof of Theorem 4.2.4 uses that K2T3 proves the existence of at least two elements. Here we avoid that assumption. Firstly, by taking the conjunction, we can assume that X consists of a single sentence $(\forall x1,...,xk)(\varphi)$. We consider truth assignments to the relation symbols in φ, only; not to atomic subformulas R(y1,...,yn). Let f be such a truth assignment that makes φ true. We associate function symbols F_R to the R in φ, and use two alternatives for replacing the $R(y_1,...,y_k)$ in φ. If f(R) = T then replace each $R(y_1,...,y_k)$ in φ by $F_R(y_1,...,y_k) = y_1$. If f(R) = F then replace each $R(y_1,...,y_k)$ in φ by $F_R(y_1,...,y_k) \neq y_1$. If there is no such truth assignement f (to the relation symbols only) that makes φ true, then replace each $R(y_1,...,y_k)$ in φ by $F_R(y_1,...,y_k) = y_1$. Let X' = $(\forall x_1,...,x_k)(\psi)$ result from these substitutions.

Now let M be a model of X with countable domain D, with complete diagram. If D has at least two elements then we form the model M' from M as in the proof of Theorem 4.2.4, and we don't need to know how f was chosen. Truth values of atomic formulas will be preserved. Now suppose D has exacty one element. Obvously there must be a truth assignment f that makes φ true in the sense above. Since |D| = 1, the truth value of any $R(y_1,...,y_k)$ is f(R), which is the same as the truth value of the associated $F_R(y_1,...,y_k) \neq y_1$, the latter being independent of the choice of $y_1,...,y_k$. The complete diagram of the result is easily obtained.

As in the proof of Theorem 4.2.4, passing from a model of X' to a model of X creates no issues. QED

DEFINITION 4.4.1. A k-system X with at least constants 1,2 is said to be free at the G(x,x,x,x), $x \notin \{1,2\}$, if and only if the following holds. Let M be a model of X. Then we can change the values G(x,x,x,x), $x \notin \{1,2\}$, in any way, and still have a model of X.

THEOREM 4.4.2. (EFA) Let X be a strict system in x,y,z,w,u satisfying the quantitative restrictions below. Then X is equiconsistent with a 4-system X[G,1,2] in x,y,z,w,u,1,2 containing the axiom $1 \neq 2$, where none of G(1,1,1,1),G(2,2,2,2),G(1,1,1,2),G(2,2,2,1) appear, and which is free at the G(x,x,x,x), $x \notin \{1,2\}$.
i. There are at most 12 constant symbols. W of section 4.2 has 2.
ii. There are at most 32 unary function symbols. W has 2.
iii. There are at most 24 binary function symbols. W has 13.
iv. There are at most 8 ternary function symbols. W has 6.

Proof: We follow the proof of Theorem 4.3.1. This is a matter of creating a corresponding table of equations that define the various symbols of X, given just before the statement of Theorem 4.3.1. Since G(1,...,1),G(2,...,2),G(1,...,1,2),G(2,...,2,1) are off limits, we can accommodate 16-4 = 12 constant symbols. We use G(i,j,k,x),G(i,j,x,k),G(i,x,j,k),G(x,i,j,k), $i,j,k \in \{1,2\}$, to accommodate 32 unary function symbols. We use G(i,j,x,y),G(i,x,j,y),G(i,x,y,j),G(x,i,j,y),G(x,i,y,j),G(x,y,i,j), $i,j \in \{1,2\}$, to accommodate 24 binary function symbols. We use G(i,x,y,z),G(x,i,y,z),G(x,y,i,z),G(x,y,z,i), i = 1,2, to accommodate 8 ternary function symbols. We do need to prove that there exists $x \neq 1,2$ in X[G,1,2], and for this purpose, we add $G(1,2,1,2) \notin \{1,2\}$. QED

For section 5, we need a variant of Theorem 4.4.2, with one more ternary function symbol.

THEOREM 4.4.3. (EFA) Let X be a strict system in x,y,z,w,u satisfying the quantitative restrictions below. Then X is equiconsistent with a 4-system X[G,1,2] in x,y,z,w,u,1,2 containing the axiom $1 \neq 2$, where G(1,1,1,1),G(2,2,2,2),G(1,1,1,2),G(2,2,2,1) are not subterms. Furthermore, X is free at the G(x,x,x,x), $x \notin \{1,2\}$.
i. There are at most 12 constant symbols. W of section 4.2 has 2.
ii. There are at most 31 unary function symbols. W has 2.
iii. There are at most 24 binary function symbols. W has

13.
iv. There are at most 9 ternary function symbols. W has 6.

Proof: With the proof of Theorem 4.4.2, we wind up short one ternary function symbol. To squeeze out one more ternary function F, we try to use G(x,y,z,z). This almost works. The problem is that we are not going to be free at the G(x,x,x,x), x $\notin$ {1,2}, as we have to accommodate values F(x,x,x) for ternary F. We get around this problem by using one dedicated unary function set aside for accommodating values F(x,x,x). This brings the 32 down to 31. QED

THEOREM 4.4.4. (BSEP) Let X be a 4-system in x,y,z,w,u,1,2 with the axiom 1 ≠ 2. Assume that X is free at the G(x,x,x,x), x $\notin$ {1,2}, and G(1,...,1),G(2,...,2),G(1,...,1,2),G(2,...,2,1) are not subterms of X. Then X is equiconsistent with a 4-system S in variables x,y,z,w,u,v,e. X' $\subseteq$ Y(4,7).

Proof: By the proof of Theorem 4.3.3. No issues arise with regard to complete diagrams. QED

COROLLARY 4.4.5. (BSEP) Let T be a consistent recursively presented system extending BSEP which is equiconsistent with some system X obeying the conditions in Theorem 4.4.2 or Theorem 4.4.3. Then T does not correctly evaluate Θ(4,7,N). Assume T does not prove its own inconsistency, T does not evaluate Θ(4,7,N).

Proof: By Theorems 4.4.2, 4.4.3, 4.4.4, and Theorem 3.2.5. QED

5. In ZFC + I1.

We adapt the Impossible Counting in ZFC of section 4 to the far stronger system ZFC + I1. To accomplish this, we first strengthen section 4.1 by axiomatizing the additional definability needed to control the complexity of the formalization of I1. We then develop a system equiconsistent with ZFC + I1 analogous to the one for ZFC in section 5.2. There will be considerably more constant, relation, and function symbols used. But we are able to strengthen the arguments in section 5.3 to show that the same Θ values are impossible to count even in ZFC + I1.

5.1. AXIOMATIZING MORE DEFINABILITY

We start with a system K1# which is an extension of the system K1 of section 4.1 in a richer language. The language of K1# is as follows.

1. Binary relation $\in$. This is used in the axiomatization of ZFC + I1 in section 5.2.
2. Binary function V (value). As in K1.
3. Ternary function C (change). As in K1.
4. Binary relation ε. As in K1.
5. Constant symbols 0. Used as a special parameter. Will also be used as the empty set in ZFC + I1.
6. Equality symbol =. Not used in the axiomatization of ZF in section 5.2.
7. Constant symbol ∞. Intended to have infinitely many $\in$ predecessors.
8. Constant symbol d. Used as a special constant. Will also be used as a big set in I1.
9. Binary relation symbol ε^*. This is the second external membership relation for our present purposes. In the intended models, the sets of objects so coded by ε^* are exactly the one dimensional sets definable in $(\{x: x \in d\},\in)$ without =, and with only parameters $p \in \infty$.

The axioms of K1# are as follows. The first five axioms are from K1, as indicated by their labels.

K[1,1]. Change. From K1. $(\forall x,y,z,w)(V(C(x,y,z),y) = z \wedge (w \neq y \rightarrow V(C(x,y,z),w) = V(x,w)))$.
K[1,2]. Boolean. From K1. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow \neg(w\ \varepsilon\ x \wedge w\ \varepsilon\ y))$.
K[1,3]. Epsilon. From K1. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow V(w,x) \in V(w,y))$.
K[1,4]. Quantification. From K1. $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon\ z \leftrightarrow (\exists u)(C(w,x,u)\ \varepsilon\ y))$.
K[1,5]. Projection. From K1. $(\forall x,y)(\exists z)(\forall w)(z\ \varepsilon\ y \leftrightarrow C(y,0,w)\ \varepsilon\ x)$.
K[1,6]#. Change(ε^*). $0 \in \infty \wedge (\forall x)(x \in \infty \rightarrow x \in d) \wedge (\forall x,y)(x\ \varepsilon^*\ y \rightarrow x \in d) \wedge (\forall x,y,z)(x,y,z \in d \rightarrow V(x,y),C(x,y,z) \in d))$.
K[1,7]#. Boolean(ε^*). $(\forall x,y)(\exists z)(\forall w)(w\ \varepsilon^*\ z \leftrightarrow (w \in d \wedge \neg(w\ \varepsilon^*\ x \wedge w\ \varepsilon^*\ y)))$.
K[1,8]#. Epsilon(ε^*). $(\forall x,y \in \infty)(\exists z)(\forall w)(w\ \varepsilon^*\ z \leftrightarrow V(w,x) \in V(w,y) \wedge w \in d)$.

K[1,9]#. Quantification(ε*). $(\forall x,y)(x \in \infty \rightarrow (\exists z)(\forall w)(w\ \varepsilon^*\ z \leftrightarrow (\exists u)(w \in d \wedge u \in d \wedge C(w,x,u)\ \varepsilon^*\ y)))$.
K[1,10]#. Projection(ε*). $(\forall x)(\exists y)(\forall z)(z\ \varepsilon^*\ y \leftrightarrow z \in d \wedge C(0,0,z)\ \varepsilon^*\ x)$.

DEFINITION 5.1.1. Let φ be a formula in ∈ with no =. $\varphi^{(d)}$ is the formula in ∈,d which results from relativizing all quantifiers in φ to the α ∈ d.

LEMMA 5.1.1. (BSEP) Let M = (D,∈,V,C,ε,0,∞.ε*,d) be a model of K1# where {x: x ∈ ∞} is infinite, with complete diagram. Let φ be a formula in ∈ and no =, with free variables among v1,...,vk, k ≥ 1. Let $d_1,\ldots,d_k \in \infty$ be distinct. There exists x in D such that for all y in D, y ε* x if and only if y ∈ d and (D,∈) satisfies $\varphi^{(d)}[V(y,d_1),\ldots,V(y,d_k)]$.

Proof: Let M be as given. From K[1,7]# we obtain

$(\forall x)(\exists z)(\forall w)(w\ \varepsilon^*\ z \leftrightarrow w \in d \wedge \neg w\ \varepsilon^*\ x)$. We say that z is a d\x(ε*).
$(\forall x)(\exists u)(\forall w)(w\ \varepsilon^*\ u \leftrightarrow w \in d \wedge \neg w\ \varepsilon^*\ z)$, where z is associated to x via K[1,7]#.
$(\forall x)(\exists u)(\forall w)(w\ \varepsilon^*\ u \leftrightarrow w \in d \wedge \neg(w \in d \wedge \neg(w\ \varepsilon^*\ x \wedge w\ \varepsilon^*\ y)))$.
$(\forall x)(\exists u)(\forall w)(w\ \varepsilon^*\ u \leftrightarrow w \in d \wedge w\ \varepsilon^*\ x \wedge w\ \varepsilon^*\ y)$. We say that u is a d∩x∩y(ε*).

We prove the following by induction on formulas φ in ∈ (no =). Assume the free variables of φ are among $v_1,\ldots,v_k$, k ≥ 1. Let $d_1,\ldots,d_k \in \infty$ be distinct. There exists x in D such that for all y in D, y ε* x if and only if y ∈ d and (D,∈) satisfies $\varphi^{(d)}[V(y,d_1),\ldots,V(y,d_k)]$.

a. $\varphi = v_i \in v_j$. Let k ≥ i,j. Let $d_1,\ldots,d_k \in \infty$ be distinct. Want x in D such that for all y in D, $y\ \varepsilon^*\ x \leftrightarrow y \in d \wedge$ (D,∈) satisfies $v_i \in v_j[d_1,\ldots,d_k] \leftrightarrow y \in d \wedge V(y,d_i) \in V(y,d_j)$. Apply K[1,8]# with x,y set to d_i,d_j. Here we quantify over k ≥ 1.

Now assume that the assertion is true for φ,ψ. I.e.,

1) Let the free variables of φ be among $v_1,\ldots,v_k$, and let

$d_1,\ldots,d_k \in \infty$ be distinct. There exists x in D such that for all y in D, y ε^* x if and only if y $\in$ d and (D,$\in$) satisfies $\varphi^{(d)}[V(y,d_1),\ldots,V(y,d_k)]$. Here we quantify over $k \geq 1$.

2) Let the free variables of ψ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k \in \infty$ be distinct. There exists x in D such that for all y in D, y ε^* x if and only if y $\in$ d $\wedge$ (D,$\in$) satisfies $\psi^{(d)}[V(y,d_1),\ldots,V(y,d_k)]$. Here we quantify over $k \geq 1$.

b. Show 1) holds for $\neg\varphi$. Let the free variables of $\neg\varphi$ be among $v_1,\ldots,v_k$, and let $d_1,\ldots,d_k \in$ d be distinct. Then the free variables of φ are among $v_1,\ldots,v_k$. Apply 1) to obtain x in D, and then use a d\x(ε^*).

c. Show 1) holds for $\varphi \wedge \psi$. Let the free variables of $\varphi \wedge \psi$ be among v1,...,vk, and let d1,...,dk $\in \infty$ be distinct. Then the free variables of φ,ψ are among v1,...,vk. Apply 1),2) to obtain x,x', and then use a d$\cap$x$\cap$y(ε^*).

d. Show 1) holds for ($\exists$vi)(φ). Let the free variables of ($\exists$vi)(φ) be among v1,...,vk, and let $d_1,\ldots,d_k \in \infty$ be distinct. We want: there exists x in D such that for all y in D, y ε x if and only if (D,$\in$) satisfies $\varphi[V(y,d_1),\ldots,V(y,d_k)]$. Instead of showing this under the given assumptions, it is more convenient to show this with the further assumption that $k \geq i$. Fortunately, it is obvious that this suffices.

Since $k \geq i$, the free variables of φ are among $v_1,\ldots,v_k$, and we apply the induction hypothesis to φ. Let x in D be given by 1).

We apply K[1,9]# with x set to d_i and y set to x. By K[1,9]#, let z in D be such that

3) for all w in D, w ε^* z if and only if ($\exists$u)(w,u $\in$ d $\wedge$ $C(w,d_i,u)$ ε^* x).

We want

4) for all w in D, w ε^* z if and only if w $\in$ d and (D,$\in$) satisfies $(\exists v_i)(\varphi)^{(d)}[V(w,d1),\ldots,V(w,dk)]$.

For 4), let w be in D. Assume w ε^* z. By 3), let w,u ∈ d ∧ C(w,di,u) ε^* x. By 1), (D,∈) satisfies $\varphi^{(d)}[V(C(w,d_i,u),d_1),\ldots,V(C(w,d_i,u),d_k)]$, $\varphi^{(d)}[V(w,d_1),\ldots,V(w,d_{i-1}),u,V(w,d_{i+1}),\ldots,V(w,d_k)]$, $(\exists v_i \in d)(\varphi^{(d)})[V(w,d_1),\ldots,V(w,d_{i-1}),V(w,d_i),V(w,d_{i+1}),\ldots,V(w,d_k)]$, $(\exists v_i)(\varphi)^{(d)}[V(w,d_1),\ldots,V(w,d_k)]$.

Conversely, assume w ∈ d and (D,∈) satisfies $(\exists v_i)(\varphi)^{(d)}[V(w,d_1),\ldots,V(w,d_k)]$. Let u ∈ d and (D,∈) satisfies $\varphi^{(d)}[V(w,d_1),\ldots,V(w,d_{i-1}),u,V(w,d_{i+1}),\ldots,V(w,d_k)]$. Then (D,∈) satisfies $\varphi^{(d)}[V(C(w,d_i,u),d_1),\ldots,V(C(w,d_i,u),d_k)]$. Also C(w,i,u) ∈ d by K[1,6]#. Hence by 1), $C(w,d_i,u)$ ε^* x. Since w,u ∈ d, we have w ε^* z using 3).

QED

THEOREM 5.1.2. (ACA_0) Let M = (D,∈,0,∞,d,V,C,ε,ε*) be a countable model of K1# where {x: x ∈ ∞} is infinite, with complete diagram.
i. Let A ⊆ D be definable over (D,∈) by a formula in ∈ without =, and with parameters. Then (∃x in D)(∀y in D)(y ε x ↔ y in A).
ii. Let A be definable over (D,∈) by a formula $\varphi^{(d)}$ in ∈ without =, and without parameters. Then (∃x in D)(∀y in D)(y ε* x ↔ y ∈ d ∧ y in A).
iii. Let φ be a formula in ∈ and no =, without x. K1# proves (∃x)(∀y)(y ε x ↔ φ).
iv. Let φ be a formula in ∈ and no =, without x, with at most the free variable y. K1# proves (∃x)(∀y)(y ε* x ↔ y ∈ d ∧ φ(d)).

Proof: i),iii are from Theorem 4.1.2. For ii, Let A ⊆ D be as given. Let φ be a formula in ∈ (without =) with at most the free variable v_1, where

1) (∀y)(y in A if and only if (D,∈) satisfies $\varphi^{(d)}[y]$.

By Lemma 5.1.1, let x be such that

2) for all y in D, y ε* x if and only if y ∈ d and (D,∈) satisfies $\varphi^{(d)}[V(y,0)]$.

By 2) and K[1,6]#,

3) for all y ∈ D, C(0,0,y) ε* x if and only if (D,∈) satisfies $\varphi^{(d)}$[y].

By 1),3),

4) for all y ∈ D, C(0,0,y) ε* x if and only if y ∈ A.

By K[1,10]#, let y be such that

5) for all z in D, z ε* y if and only if z ∈ d and C(0,0,z) ε* x.

By 4),5),

6) for all z in D, z ε* y if and only if z ∈ d and z ∈ A.

For iv, it is clear from the above that every countable model M of K1 where {x: x ∈ ∞} is infinite, with complete diagram, satisfies each such (∃x)(∀y)(y ε* x ↔ y ∈ d ∧ $\varphi^{(d)}$). By the Gödel Completeness Theorem, we must have provability. QED

THEOREM 5.1.3. (BSEP) Let (D,∈,0,∞,d) be a countable model of BSEP with complete diagram, with the three added constants satisfying the additional first order sentence "0 = ∅ ∈ ∞ ∈ d ∧ ∞ is infinite ∧ d is transitive ∧ every finite subset of d is an element of d" in ∈,=. There is a model (D,∈,0,∞,d,V,C,ε,ε*) of K1# where
i. the various {y in D: y ε x}, x in D, comprise exactly the subsets of D that are definable over (D,∈) without =, but with parameters.
ii. the various {y in D: y ε* x}, comprise exactly the subsets of {x: x ∈ d} that are definable over (D,∈) by a formula $\varphi^{(d)}$ without =, with parameters x ∈ ∞.

Proof: Let (D,∈,0,∞,d) be as given. Since (D,∈) is an equality model of extensionality, definability without = is equivalent to definability with =. Interpret 0 as the empty set in (D,∈) and ∞ as the V(ω) in (D,∈). Interpret V(x,y) internally in (D,∈) by V(x,y) is the value of the function x at y if x is a finite function and x(y) exists; otherwise

0. Interpret C(x,y,z) internally in (D,∈) as the unique function x' = {<a,b>: x(a) = b ∧ a ≠ y} ∪ {<y,z>}.

Since D is countably infinite, let A_x, x in D, comprise the subsets of D that are definable over (D,∈), but with parameters. This construction goes through in ACA_0 using crucially the complete diagram of (D,∈). Interpret x ε y if and only if y lies in A_x. According to the proof of Theorem 4.1.3, (D,∈,0,V,C,ε) satisfies K1.

Now let B_x, x in D, comprise the subsets of {x: x ∈ d} that are definable over (D,∈) by a formula $\varphi^{(d)}$ with parameters from {x: x ∈ ω}. Interpret x ε* y if and only if y lies in B_x and x ∈ d. Again, we crucially use the complete diagram of (D,∈). For K[1,6]#, the first three conjuncts are immediate. Let x,y,z ∈ d. If x is not a finite function then V(x,y) = 0. Suppose x is a finite function. If x(y) does not exist then V(x,y) = 0. Assume x(y) exists. Then <y,x(y)> = {{y},{y,x(y)}} ∈ x, all in the sense of (D,∈). Since d is transitive, x(y) = V(x,y) ∈ d. C(x,y,z) = {<a,b>: x(a) = b ∧ a ≠ y} ∪ {<y,z>}. To see that C(x,y,z) ∈ d, it suffices to prove that C(x,y,z) ⊆ d. as C(x,y,z) is clearly finite. The first set is a subset of x ∈ d. Hence the first set is a finite subset of x ⊆ d. Also {y},{y,z} are finite subsets of d, and therefore elements of d. Hence <y,z> is a finite subset of d, and therefore an element of d. Hence {<y,z>} is a finite subset of d, and therefore an element of d.

For K[1,7]#, let x,y in D. Let B_x,B_y ⊆ d be definable over (D,∈) by φ(d),ψ(d) with parameters from {x: x ∈ ∞}, respectively. Then d\(B_x ∩ B_y) ⊆ d is definable over (D,∈) by $((\exists v_2 \in d)(v_1 \in v_2) \wedge \neg(\varphi \wedge \psi))^{(d)}$ with parameters from {x: x ∈ ∞}. Write d\(B_x ∩ B_y) ⊆ d as B_z, and use z for K[1,7]#.

For K[1,8]#, let x,y ∈ ∞. V(w,x) ∈ V(w,y) ∧ w ∈ d is definable (in the variable w) over (D,∈) by a formula $\varphi^{(d)}$ with paramters x,y ∈ ∞.

For K[1,9]#, let x,y in D, x ∈ ∞. We claim that (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y) is definable over (D,∈) by a formula $\varphi^{(d)}$ with parameters from {x: x ∈ ∞}, in variable w

∈ d. To see this, rewrite as (∃u,v ∈ d)(w ∈ d ∧ v = C(w,x,u) ∧ v ε* y) and use the appropriate definition of v ε* y in the variable v.

For K[1,10]#, let x in D and consider the statement z ∈ d ∧ C(0,0,z) ε* x in variable z. Rewrite this as (∃x ∈ d)(z ∈ d ∧ z = C(0,0,z) ∧ z ε* x), and use the appropriate definition of z ε* x in the variable z. QED

In Theorem 5.1.3, we are making no claim that the model of K1# so constructed has a complete diagram. However, in the next section, we will apply Theorem 4.1.3 in such a way that we will be ultimately stay on track by getting models with complete diagrams - with the help of the discussion at the beginning of section 3.

5.2. EQUICONSISTENCY, Θ(4,7,N)

Let T1# be the following theory in the following language with =.

i. ∈,=,0,{ },⊆,∪,℘,∞ as in T1 in section 5.2. Also binary ∩,UN,< >.
ii. The unary relation symbols sing for "being a singleton".
iii. The unary relation symbol sdns for "being a set of disjoint nonempty sets".
iv. The unary function symbol rk, intended to return V(α)'s.
v. The constant symbols b,c,d. The intention is that b ∈ c and d is the finite closure of rk(c).
vi. The unary relation symbol P for "being an element of d".
vii. The unary function symbol h, intended to be an elementary embedding from d into d which moves b.

We are using a language that is deliberately richer than what we naturally need (especially ii,iii,iv,vi), in order to facilitate the Skolemization/denesting process.

T[1,1]#. Definitional. From T1. (∀x)(x ∉ 0) ∧ (∀x,y)(x ⊆ y ↔ (∀z)(z ∈ x → z ∈ y)).
T[1,2]#. Singleton. (∀x)(sing(x) ↔ (∃y)(x = {y,y}).
T[1,3]#. Binary Intersection/Union. (∀x,y,z)(z ∈ x ∩ y ↔ z

∈ x ∧ z ∈ y) ∧ (∀x,y,z)(z ∈ UN(x,y) ↔ z ∈ x ∨ z ∈ y).

T[1,4]#. Extensionality. From T1. (∀x,y)(x ⊆ y ∧ y ⊆ x → x = y).

T[1,5]#. Pairing. (∀x,y,z)(z ∈ {x,y} ↔ z ∈ x ∨ z ∈ y).

T[1,6]#. Union. From T1. (∀x,y,z)(y ∈ z ∧ z ∈ x → y ∈ ∪x).

T[1,7]#. Power Set. (∀x,y)(y ∈ ℘(x) ↔ y ⊆ x).

T[1,8]#. Infinity. From T1. 0 ∈ ∞ ∧ (∀x)(x ∈ ∞ → {x,x} ∈ ∞)).

T[1,9]#. Foundation. (∀x)(x ≠ 0 → (∃y)(y ∈ x ∧ x ∩ y = 0)).

T[1,10]#. Separation. From T1. (∀y)(∃z)(∀w)(w ∈ z ↔ (w ∈ y ∧ φ(w))), where φ is a formula in ∈, without =, in which y,z do not appear.

T[1,11]#. Unary Collection. From T1. (∀y)(∃z)(∀w ∈ y)((∃u)(w ∈ u ∧ φ(u)) → (∃u ∈ z)(w ∈ u ∧ φ(u))), where φ is a formula in ∈, without =, in which y,z,w do not appear.

T[1,12]#. Preparation. (∀x)(sdns(x) ↔ 0 ∉ x ∧ (∀y,z)(y,z ∈ x ∧ y ≠ z → y ∩ z = 0)) ∧ (∀x)(P(x) ↔ x ∈ d).

T[1,13]#. Axiom of Choice. (∀x)(sdns(x) → (∃y)(∀z)(z ∈ x → sing(y ∩ z))).

T[1,14]#. Ordered Pairs. (∀x,y,z)(<x,y> = {{x,x},{x,y}}).

T[1,15]#. Rank. (∀x,y)(y ∈ rk(x) ↔ (∀z)(z ∈ y → (∃w)(w ∈ x ∧ z ∈ rk(w)))).

T[1,16]#. Rank. ∞,rk({c,c}) ∈ d ∧ (∀x,y)(x ∈ y ∧ y ∈ d → x ∈ d) ∧ (∀x,y)(x,y ∈ d → {x,y},UN(x,y) ∈ d).

T[1,17]#. Internal. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z)).

T[1,18]#. Internal. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = h(z)).

T[1,19]#. Moving. h(b) ≠ b ∧ h(c) = c ∧ b ∈ c ∧ (∀x)(x ∈ d → h(x) ∈ d).

T[1,20]#. Embedding. (∀x,y)(x,y ∈ d → (x ∈ y ↔ h(x) ∈ h(y))).

T[1,21]#. Elementary. (∀y)(y ∈ d → (φ(d) ↔ φ(d)[y/h(y)]), where φ is a formula in ∈,= with only the free variable y, and with all quantifiers bounded to d.

LEMMA 5.2.1. (EFA) T1# proves ZFC + I1.

Proof: Since T1# extends T1 and includes Foundation, we see by Theorem 4.2.1 that T1# proves ZFC. We now argue in T1# and derive I1.

By T[1,17]#, the graph of rk restricted to arguments in any given transitive set A must exist as a set; i.e., a set theoretic function f. We claim that for transitive A, (∀x ∈ A)(f(x) = V(ork(x))), where ork is the ordinal rank of x. Specifically, rk(0) = V(0) = 0 = ork(0). Let x ∈ A be a counterexample with least ork(x) = α. By T[1,15]#, f(0) = 0. Hence α > 0. Suppose α = β+1. We claim rk(x) = ℘(V(β)) = V(α). Let y ∈ rk(x). By T[1,15]#, every z ∈ y is an element of some rk(w), w ∈ x. Hence every z ∈ y is an element of some rk(w), ord(w) ≤ β. Hence every z ∈ y is an element of V(β). Thus we have shown that every y ∈ rk(x) is a subset of V(β). Conversely, suppose y ⊆ V(β). Then y is a subset of some rk(u), u ∈ x, because there exists u ∈ x with ork(u) = β, and for that u we have rk(u) = V(β). We have now verified the claim, rk(x) = V(α), which is a contradiction. So α is a limit ordinal λ. Now we claim that rk(x) = $\cup_{\beta<\lambda}$V(β) = V(λ), proved in the same way, which is also a contradiction. Since every set is an element of a transitive set, clearly every rk(x) = V(ork(x)). By transfinite induction, every x ⊆ rk(x), also proved in the same way.

Let rk({c,c}) = V(α). Clearly α is a successor ordinal. By T[1,16]#, V(α+1) ∈ d and d is transitive and d contains its singletons, doubletons, and pairwise unions as elements. Hence every finite subset of d is an element of d.

By T[1,18]#, the graph of h restricted to arguments in any given transitive set A must exist as a set. In particular, the restriction h:d → d exists as a set, using also T[1,19]#. By T[1,20]#, h is ∈ preserving. By T[1,21]#, h is what we call 1-elementary. Namely, h preserves first order properties (over (d,∈)) of single elements of d, rather than of finite sequences of elements of d. We show below that h is an elementary embedding from d into d.

We first claim that x,y ∈ d → h({x,y}) = {h(x),h(y)}. To see this, fix x,y ∈ d. By ∈ preservation, h(x) ∈ h({x}) and h(x),h(y) ∈ h({x,y}). By 1-elementary, h({x}) is a singleton and if x ≠ y then h({x,y}) is a doubleton. The claim now follows.

We next claim that x,y ∈ d → h(<x,y>) = <h(x),h(y)>. Fix

$x,y \in d$. $h(\langle x,y\rangle) = h(\{\{x\},\{x,y\}\}) = \{h(\{x\}),h(\{x,y\})\} = \{\{h(x)\},\{h(x),h(y)\}\} = \langle h(x),h(y)\rangle$.

For $n \in \omega$, let $n^* = \{\ldots\{\emptyset\}\ldots\}$, where there are n pairs of braces. From the above, clearly $h(n^*) = n^*$.

We are now prepared to verify that $h:d \to d$ is elementary (not just 1-elementary). Let x1,...,xk $\in$ d and φ(v1,...,vk) be given. Note that $\{\langle 1^*,x_1\rangle,\ldots,\langle k^*,x_k\rangle\} \in d$. By $\in$ preservation, $h(\langle 1^*,x_1\rangle),\ldots,h(\langle k^*,x_k\rangle) \in h(\{\langle 1^*,x_1\rangle,\ldots,\langle k^*,x_k\rangle\})$, $\langle 1^*,h(x_1)\rangle,\ldots,\langle k^*,h(x_k)\rangle \in h(\{\langle 1^*,x_1\rangle,\ldots,\langle k^*,x_k\rangle\})$. By 1-elementary, $|h(\{\langle 1^*,x_s\rangle,\ldots,\langle k^*,x_k\rangle\})| = k$. Hence $h(\{\langle 1^*,x_1\rangle,\ldots,\langle k^*,x_k\rangle\}) = \langle 1^*,h(x_1)\rangle,\ldots,\langle k^*,h(x_k)\rangle$. Let $\psi(v) = (\exists v_1,\ldots,v_k)(v = \{(\langle 1^*,v_1\rangle,\ldots,\langle k^*,v_k\rangle\} \wedge \varphi(v_1,\ldots,v_k))$. Since h is 1-elementary, $\psi(v) \leftrightarrow \psi(h(v))$. Hence for $v \in d$, $(\exists v_1,\ldots,v_k \in d)(v = \{(\langle 1^*,v_1\rangle,\ldots,\langle k^*,v_k\rangle\} \wedge \varphi(v_1,\ldots,v_k)) \leftrightarrow (\exists v_1,\ldots,v_k \in d)(h(v) = \{\langle 1^*,v_1\rangle,\ldots,\langle k^*,v_k\rangle\} \wedge \varphi(v_1,\ldots,v_k))$. Apply this biconditional with $v = \{\langle 1^*,x_1\rangle,\ldots,\langle k^*,x_k\rangle\} \in d$. We obtain $\varphi(x_1,\ldots,x_k) \leftrightarrow \varphi(h(x_1),\ldots,h(x_k))$.

Now we have the elementary embedding $h:d \to d$ with $h(c) = c \wedge b \in c \wedge h(b) \neq b \wedge rk(\{c,c\}) \in d \wedge d$ transitive $\wedge$ every finite subset of d lies in d by T[1,16]#,T[1,19]#. Also $rk(\{c,c\}) = V(\alpha+1)$. Now the rank of c, ork(c), is α. Hence $h(\alpha) = \alpha$, and since $V(\alpha+1) \in d$, we have $h(V(\alpha)) = V(\alpha)$, $h(V(\alpha+1)) = V(\alpha+1)$. Therefore $h:V(\alpha+1) \to V(\alpha+1)$ is an elementary embedding with $h(b) \neq b$. This is I1. QED

THEOREM 5.2.2. ZFC + I1 and T1# are equiconsistent.

Proof: We first claim that ZFC + I1 proves the following. There exists a nontrivial elementary embedding $j:V(\alpha+1) \to V(\alpha+1)$ which extends to an elementary embedding $j:V(\alpha+1)^* \to V(\alpha+1)^*$, where $V(\alpha+1)^*$ is the least transitive set with $V(\alpha+1) \in V(\alpha+1)^*$ such that every finite subset of $V(\alpha+1)^*$ is an element of $V(\alpha+1)^*$. To see this, first note that $(V(\alpha+1)^*,\in)$ is isomorphic to some (A,R) where $A \subseteq V(\alpha+1)$, $R \subseteq A^2$, and A,R are definable over $V(\alpha+1)$ with no parameters. So if (A,R) has a nontrivial elementary embedding into itself then so does $(V(\alpha+1)^*,\in)$. By the parameter free definability of A,R, it is clear that $j|A$ is an elementary embedding of (A,R) into itself. It remains to show that j|A is not the identity. Now (A,R) is an extensional well founded relation, defined over V(α+1) with no parameters, and has a point x whose set of R-predecessors has type κ, where κ is the critical point of j. Then j(u) must be the point in (A,R) whose set of R-predecessors has type j(κ) in (A,R). Hence j(u) ≠ u.

We now argue in ACA_0. Let M be a countable model of ZFC + I1 with a complete diagram. In M, let j:V(α+1)* → V(α+1)* be an elementary embedding with critical point κ. Let M' be the expansion of M by interpreting 0 as ∅, [} as unordered pair, ⊆ as inclusion, ∪ as unary union, ℘ as power set, ∞ as V(ω), ∩ as binary intersection, UN as binary union, < > as ordered pair, sing as "being a singleton", sdns as "being a set of disjoint nonempty sets", rk(x) as V(α), where α is the ordinal rank of x, c as α, b as κ, d as V(α+1)*, P as "being an element of d", and h as the extension of j with the default value ∅. It is clear that the model resulting from this interpretation satisfies T1#, and has a complete diagram because of the definability of these symbols with parameters α,κ,j. For the converse, use Theorem 5.2.1, so that given any countable model of T1# with complete diagram, we merely drop information to get a countable model of ZFC + I1 with complete diagram. QED

We now appropriately combine K1# and T1# into the following system K1T2# with finitely many axioms, whose language is that of K1# and T1# combined. This requires that the schemes T[1,10]#,T[1,11]#,T[1,21]# be replaced by the axioms T[2,10]#,T[2,11]#,T[2,21]# below. The other 18 axioms from T1# are merely relabeled.

K[1,1]. Change. From K1. (∀x,y,z,w,u)(u = C(x,y,z) → V(u,y) = z ∧ (u = C(x,y,z) → (w ≠ y → V(u,w) = V(x,w)))).
K[1,2]. Boolean. From K1. (∀x,y)(∃z)(∀w)(w ε z ↔ ¬(w ε x ∧ w ε y)).
K[1,3]. Epsilon. From K1. (∀x,y)(∃z)(∀w)(w ε z ↔ V(w,x) ∈ V(w,y)).
K[1,4]. Quantification. From K1. (∀x,y)(∃z)(∀w)(w ε z ↔ (∃u)(C(w,x,u) ε y)).
K[1,5]. Projection. From K1. (∀x,y)(∃z)(∀w)(w ε z ↔ C(y,0,w) ε x).
K[1,6]#. Change(ε*). 0 ∈ ∞ ∧ (∀x)(x ∈ ∞ → x ∈ d) ∧

(∀x,y)(x ε* y → x ∈ d) ∧ (∀x,y,z)(x,y,z ∈ d → V(x,y),C(x,y,z) ∈ d)).

K[1,7]#. Boolean(ε*). (∀x,y)(∃z)(∀w)(w ε* z ↔ (w ∈ d ∧ ¬(w ε* x ∧ w ε* y))).

K[1,8]#. Epsilon(ε*). (∀x,y ∈ ∞)(∃z)(∀w)(w ε* z ↔ V(w,x) ∈ V(w,y) ∧ w ∈ d).

K[1,9]#. Quantification(ε*). (∀x,y)(x ∈ ∞ → (∃z)(∀w)(w ε* z ↔ (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y))).

K[1,10]#. Projection(ε*). (∀x)(∃y)(∀z)(z ε* y ↔ z ∈ d ∧ C(0,0,z) ε* x).

T[2,1]#. Definitional. From T1. (∀x)(x ∉ 0) ∧ (∀x,y)(x ⊆ y ↔ (∀z)(z ∈ x → z ∈ y)).

T[2,2]#. Singleton. (∀x)(sing(x) ↔ (∃y)(x = {y,y}).

T[2,3]#. Binary Intersection/Union. (∀x,y,z)(z ∈ x ∩ y ↔ z ∈ x ∧ z ∈ y) ∧ (∀x,y,z)(z ∈ UN(x,y) ↔ z ∈ x ∨ z ∈ y).

T[2,4]#. Extensionality. From T1. (∀x,y)(x ⊆ y ∧ y ⊆ x → x = y).

T[2,5]#. Pairing. (∀x,y,z)(z ∈ {x,y} ↔ z ∈ x ∨ z ∈ y).

T[2,6]#. Union. From T1. (∀x,y,z)(y ∈ z ∧ z ∈ x → y ∈ ∪x).

T[2,7]#. Power Set. (∀x,y)(y ∈ ℘(x) ↔ y ⊆ x).

T[2,8]#. Infinity. From T1. 0 ∈ ∞ ∧ (∀x)(x ∈ ∞ → {x,x} ∈ ∞)).

T[2,9]#. Foundation. (∀x)(x ≠ 0 → (∃y)(y ∈ x ∧ x ∩ y = 0)).

T[2,10]#. Separation. From T1. (∀x,y)(∃z)(∀w)(w ∈ z ↔ w ∈ y ∧ w ε x).

T[2,11]#. Unary Collection. From T1. (∀x,y)(∃z)(∀w ∈ y)((∃u)(w ∈ u ∧ u ε x) → (∃u ∈ z)(w ∈ u ∧ u ε x)).

T[2,12]#. Preparation. (∀x)(sdns(x) ↔ 0 ∉ x ∧ (∀y,z)(y,z ∈ x ∧ y ≠ z → y ∩ z = 0)) ∧ (∀x)(P(x) ↔ x ∈ d).

T[2,13]#. Axiom of Choice. (∀x)(sdns(x) → (∃y)(∀z)(z ∈ x → sing(y ∩ z))).

T[2,14]#. Ordered Pairs. (∀x,y,z)(<x,y> = {{x,x},{x,y}}).

T[2,15]#. Rank. (∀x,y)(y ∈ rk(x) ↔ (∀z)(z ∈ y → (∃w)(w ∈ x ∧ z ∈ rk(w)))).

T[2,16]#. Rank. ∞,rk({c,c}) ∈ d ∧ (∀x,y)(x ∈ y ∧ y ∈ d → x ∈ d) ∧ (∀x,y)(x,y ∈ d → {x,y},UN(x,y) ∈ d).

T[2,17]#. Internal. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z)).

T[2,18]#. Internal. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = h(z)).

T[2,19]#. Moving. h(b) ≠ b ∧ h(c) = c ∧ b ∈ c ∧ (∀x)(x ∈ d → h(x) ∈ d).
T[2,20]#. Embedding. (∀x,y)(x,y ∈ d → (x ∈ y ↔ h(x) ∈ h(y))).
T[2,21]#. Elementary. (∀x,y)(C(0,0,y) ε* x ↔ C(0,0,h(y)) ε* x).

THEOREM 5.2.3. ZFC + I1,T1#,K1T2# are equiconsistent.

Proof: We argue in ACA_0. Let (D,∈,0,{ },⊆,∪,℘,∞,∩,UN,<>,sing,sdns,rk,b,c,d,P,h) be a countable model of T1# with complete diagram. By Theorem 5.2.1, (D,∈) satisfies ZFC + I1. We now apply Theorem 5.1.3 to (D,∈,0,∞,d). We can do this since T13 proves BSEP (T[1,10]#), 0 = ∅ (T[1,1]#), 0 ∈ ∞ (T[1,8]#, ∞ ∈ d (T[2,16]#), ∞ is infinite (T[1,8]#), d is transitive (T[1,16]#), every finite subset of d is an element of d (T[1,16]#). Let M be a countable model (D,∈,0,∞,d,V,C,ε,ε*) of K1# where
i. the various {y ∈ D: y ε x}, x in D, comprise exactly the subsets of D that are definable over (D,∈) without =, but with parameters.
ii. the various {C(0,0,y): y ε* x} comprise exactly the subsets of {x: x ∈ d} that are definable over (D,∈) by a formula φ(d) without =, and with parameters from {x: x ∈ ∞}.

It is now clear that M' = (D,∈,0,{ },⊆,∪,℘,∞,∩,UN,<>,sing,sdns,rk,b,c,d,P,h,V,C,ε,ε*) satisfies K1T2# except for T[2,21]#. The problem is that in T[1,21]# of T1#, we do not allow any parameters in φ, yet we have paramters in ii above. So we now go back to T1# and show that M satisfies T[1,21]# with parameters allowed from {x: x < ∞}. Note that in M, ∞ = V(ω). In fact, we will do more. By Theorem 5.2.1, T1# proves ZFC + I1, and we now show in ZFC + I1 that if j:V(α+1) → V(α+1) is an elementary embedding, then we have elementarity with parameters allowed from V(ω). But this is immediate from the obvious fact that x ∈ V(ω) → j(x) = x.

Thus M' is a model of K1T2#, which consists of finitely many axioms. We are not claiming that ACA_0 is sufficient to construct the complete diagram of M'. So as discussed at the beginning of section 3, we can still obtain a countable model of K1T2# with complete diagram in ACA_0, as is required for equiconsistency.

Conversely, let (D,∈,0,{ },⊆,∪,℘,∞,∩,UN,< >,sing,sdns,rk,b,c,d,P,h,V,C,ε,ε*) be a countable model of K1T2# with complete diagram. This obviously satisfies T1# except possibly for T[1,10]#,T[1,11]#,T[1,21]#. Now apply Theorem 5.1.2. Hence M satisfies T1#, and we obtain a model of T1# with complete diagram by dropping information. In fact, we (also) see that K1T2# proves T1#. QED

We now perform Skolemization/denesting on the K1T2# as we did for K1T2. Note that eleven of the axioms of K1T2# are axioms of K1T2, and so we copy those Skolemizations/denestings from section 4. These earlier Skoleminzations/denestings use F1,...,F10,R1,R2,R3, and so the new function and relation symbols that we use here will start with F11 and R4. We shall see also that we can reuse some of F1,...,F10,R1,R2,R3.

There are some opportunities for the reuse of Skolem functions and auxiliary functions and relations, which are duly noted. Of course, this reuse is merely a manifestation of using finite sets of universal sentences, rather than their conjunctions.

We Skolemized/denested K[1,1]-K[1,5] right after Theorem 5.2.2. Here are the results from section 5.

K[1,1]. (∀x,y,z,w,u)(u = C(x,y,z) → V(u,y) = z ∧ (w ≠ y → V(u,w) = V(x,w)))). A logical equivalence.
K[1,2]. (∀x,y,z,w)(w = F1(x,y) → (z ε w ↔ ¬(z ε x ∧ z ε y))). A logical strengthening.
K[1,3]. (∀x,y,z,w)(z = F2(x,y) → (w ε z → R1(x,y,w))) ∧ (∀x,y,z,w,u)(w = V(z,x) ∧ u = V(z,y) → (R1(x,y,z) ↔ w ∈ u)). A logical strengthening.
K[1,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ w = F4(x,y,z) ∧ R2(x,y,z) → u ε y) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ u ε y → R2(x,y,z)). A logical strengthening.
K[1,5]. (∀x,y,z,w,u)(u = F6(y,w) ∧ z = F5(x,y) → (w ε z ↔ u ε x))) ∧ (∀x,y,z,w,u)(z = 0 ∧ w = C(x,z,y) → F6(x,y) = w). A logical strengthening.

Note that (∀x,y,z)(R1(x,y,z) ↔ V(z,x) ∈ V(z,y) ↔ z ε F2(x,y), from K[1,3].

For K[1,6]# we have

K[1,6]#. Change(ε*). 0 ∈ ∞ ∧ (∀x)(x ∈ ∞ → x ∈ d) ∧ (∀x,y)(x ε* y → x ∈ d) ∧ (∀x,y,z)(x,y,z ∈ d → V(x,y),C(x,y,z) ∈ d)).
K[1,6]#. (∀x,y)(x = 0 ∧ y = ∞ → x ∈ y) ∧ (∀x,y,z)(y = ∞ ∧ z = d ∧ x ∈ y → x ∈ z) ∧ (∀x,y,z)(z = d ∧ x ε* y → x ∈ z) ∧ (∀x,y,z)(P(x) ∧ P(y) ∧ P(z) → P(V(x,y)) ∧ P(C(x,y,z))).
K[1,6]#. (∀x,y)(x = 0 ∧ y = ∞ → x ∈ y) ∧ (∀x,y,z)(y = ∞ ∧ z = d ∧ x ∈ y → x ∈ z) ∧ (∀x,y,z)(z = d ∧ x ε* y → x ∈ z) ∧ (∀x,y,z,w,u)(w = V(x,y)) ∧ u = C(x,y,z) ∧ P(x) ∧ P(y) ∧ P(z) → P(w) ∧ P(u)).

A logical equivalence in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#.

For K[1,7]# we have

K[1,7]#. (∀x,y)(∃z)(∀w)(w ε* z ↔ (w ε* d ∧ ¬(w ε* x ∧ w ε* y))).
K[1,7]#. (∀x,y,w)(w ε* F11(x,y) ↔ (w ε* d ∧ ¬(w ε* x ∧ w ε* y))).
K[1,7]#. (∀x,y,w,u)(u = F11(x,y) → (w ε* u ↔ (w ε* d ∧ ¬(w ε* x ∧ w ε* y)))).
K[1,7]#. (∀x,y,z,w,u)(z = d ∧ u = F11(x,y) → (w ε* u ↔ (w ε* z ∧ ¬(w ε* x ∧ w ε* y)))).

A logical strengthening.

For K[1,8]# we have

K[1,8]#. (∀x,y)(∃z)(∀w)(w ε* z ↔ V(w,x) ∈ V(w,y) ∧ w ∈ d).
K[1,8]#. (∀x,y,w)(w ε* F12(x,y) ↔ V(w,x) ∈ V(w,y) ∧ w ∈ d).
K[1,8]#. (∀x,y,w)(w ε* F12(x,y) ↔ R1(x,y,w) ∧ w ∈ d) ∧ (∀x,y,z)(R1(x,y,z) ↔ V(z,x) ∈ V(z,y)).
K[1,8]#. (∀x,y,z,w)(z = F12(x,y) → (w ε* z ↔ R1(x,y,w) ∧ P(w))) ∧ (∀x,y,z,w,u)(w = V(z,x) ∧ u = V(z,y) → (R1(x,y,z) ↔ w ∈ u)).

A logical strengthening in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#. . Note that we have appropriately reused R1 from K[1,3]. In fact, we can just use

K[1,8]#. (∀x,y,z,w)(z = F12(x,y) → (w ε* z ↔ R1(x,y,w) ∧ P(w))).

since the dropped conjunct is from K[1,3] above.

For K[1,9]#, we have

K[1,9]#. (∀x,y)(x ∈ ∞ → (∃z)(∀w)(w ε* z ↔ (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y))).
K[1,9]#. (∀x,y)(∃z)(x ∈ ∞ → (∀w)(w ε* z ↔ (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y))).
K[1,9]#. (∀x,y)(∃z)(∀w)(x ∈ ∞ → (w ε* z ↔ (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y))).
K[1,9]#. (∀x,y,w)(x ∈ ∞ → (w ε* F13(x,y) ↔ (∃u)(w ∈ d ∧ u ∈ d ∧ C(w,x,u) ε* y))).
K[1,9]#. (∀x,y,w)(x ∈ ∞ → (w ε* F13(x,y) ↔ (∃u)(P(w) ∧ P(u) ∧ C(w,x,u) ε* y)))
K[1,9]#. (∀x,y,w)(x ∈ ∞ → (w ε* F13(x,y) ↔ R4(w,x,y))) ∧ (∀w,x,y)(R4(w,x,y) ↔ (∃u)(P(w) ∧ P(u) ∧ C(w,x,u) ε* y)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀w,x,y)(R4(w,x,y) → (∃u)(P(w) ∧ P(u) ∧ C(w,x,u) ε* y)) ∧ (∀w,x,y)((∃u)(P(w) ∧ P(u) ∧ C(w,x,u) ε* y) → R4(w,x,y)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀w,x,y)(∃u)(R4(w,x,y) → P(w) ∧ P(u) ∧ C(w,x,u) ε* y) ∧ (∀w,x,y,u)(P(w) ∧ P(u) ∧ C(w,x,u) ε* y → R4(w,x,y)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀w,x,y)(R4(w,x,y) → P(w) ∧ P(F14(w,x,y)) ∧ C(w,x,F14(w,x,y)) ε* y) ∧ (∀w,x,y,u,v)(v = C(w,x,u) ∧ P(w) ∧ P(u) ∧ v ε* y → R4(w,x,y)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀w,x,y,z)(z = F14(w,x,y) ∧ R4(w,x,y) → P(w) ∧ P(z) ∧ C(w,x,z) ε* y) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ P(x) ∧ P(w) ∧ u ε* z → R4(x,y,z)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀w,x,y,z,u)(u = C(w,x,z) ∧ z = F14(w,x,y) ∧ R4(w,x,y) → P(w) ∧ P(z) ∧ u ε* y) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ P(x) ∧ P(w) ∧ u ε* z → R4(x,y,z)).
K[1,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε*

u ↔ R4(w,x,y))) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ w = F14(x,y,z) ∧ R4(x,y,z) → P(x) ∧ P(w) ∧ u ε* z) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ P(x) ∧ P(w) ∧ u ε* z → R4(x,y,z)).

A logical strengthening in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#.

For K[1,10]#, we have

K[1,10]#. (∀x)(∃y)(∀z)(z ε* y ↔ z ∈ d ∧ C(0,0,z) ε* x).
K[1,10]#. (∀x,z)(z ε* F15(x) ↔ P(z) ∧ C(0,0,z) ε* x).
K[1,10]#. (∀x,y,z)(y = F15(x) → (z ε* y ↔ P(z) ∧ C(0,0,z) ε* x)).
K[1,10]#. (∀x,y,z,w,u)(w = 0 ∧ u = C(w,w,z) ∧ y = F15(x) → (z ε* y ↔ P(z) ∧ u ε* x)).

A logical strengthening in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#.

From T[2,1]#, we have from section 4,

T[2,1]. (∀x,y)(y = 0 → x ∉ y) ∧ (∀x,y,z)(x ⊆ y ∧ z ∈ x → z ∈ y)) ∧ (∀x,y,z)(z = F7(x,y) ∧ (z ∈ x → z ∈ y) → x ⊆ y). A logical strengthening.

From T[2,2]#, we have

T[2,2]#. (∀x)(sing(x) ↔ (∃y)(x = {y,y}).
T[2,2]#. (∀x)(sing(x) → (∃y)(x = {y,y}) ∧ (∀x)((∃y)(x = {y,y}) → sing(x)).
T[2,2]#. (∀x)(∃y)(sing(x) → x = {y,y}) ∧ (∀x,y)(x = {y,y} → sing(x)).
T[2,2]#. (∀x)(sing(x) → x = {F16(x),F16(x)}) ∧ (∀x,y)(x = {y,y} → sing(x)).
T[2,2]#. (∀x,y)(y = F16(x) ∧ sing(x) → x = {y,y}) ∧ (∀x,y)(x = {y,y} → sing(x)).

A logical strengthening.

From T[2,3]#, we have

T[2,3]#. (∀x,y,z)(z ∈ x ∩ y ↔ z ∈ x ∧ z ∈ y) ∧ (∀x,y,z)(z ∈ UN(x,y) ↔ z ∈ x ∨ z ∈ y).

T[2,3]#. $(\forall x,y,z,w)(w = x \cap y \rightarrow (z \in w \leftrightarrow z \in x \wedge z \in y)) \wedge (\forall x,y,z,w)(w = UN(x,y) \rightarrow (z \in w \leftrightarrow z \in x \vee z \in y))$.

A logical equivalence.

From T[2,4]#, we have from section 4,

T[2,2]. $(\forall x,y)(x \subseteq y \wedge y \subseteq x \rightarrow x = y)$. Unchanged.

From T[2,5]#, we have

T[2,5]#. $(\forall x,y,z)(z \in \{x,y\} \leftrightarrow z \in x \vee z \in y)$.
T[2,5]#. $(\forall x,y,z,w)(w = \{x,y\} \rightarrow (z \in w \leftrightarrow z \in x \vee z \in y))$.

A logical strengthening.

From T[2,6]#, we have from section 4,

T[2,4]. $(\forall x,y,z,w)(w = \cup x \wedge y \in z \wedge z \in x \rightarrow y \in w)$. A logical equivalence.

From T[2,7]#, we have

T[2,7]#. $(\forall x,y)(y \in \wp(x) \leftrightarrow y \subseteq x)$.
T[2,7]#. $(\forall x,y,z)(z = \wp(x) \rightarrow (y \in z \leftrightarrow y \subseteq x))$.

A logical equivalence.

From T[2,8]#, we have from section 4,

T[2,6]. $(\forall x,y)(x = 0 \wedge y = \infty \rightarrow x \in y) \wedge (\forall x,y,z)(y = \infty \wedge z = \{x,x\} \wedge x \in y \rightarrow z \in y)$. A logical equivalence.

From T[2,9]#, we have

T[2,9]#. $(\forall x)(x \neq 0 \rightarrow (\exists y)(y \in x \wedge x \cap y = 0))$.
T[2,9]#. $(\forall x)(\exists y)(x \neq 0 \rightarrow y \in x \wedge x \cap y = 0))$.
T[2,9]#. $(\forall x)(x \neq 0 \rightarrow F17(x) \in x \wedge x \cap F17(x) = 0)$.
T[2,9]#. $(\forall x,y)(y = F17(x) \wedge x \neq 0 \rightarrow y \in x \wedge x \cap y = 0)$.

A logical strengthening.

From T[2,10]#, we have from section 4,

T[2,7]. $(\forall x,y,z,w)(z = F8(x,y) \rightarrow (w \in z \leftrightarrow (w \in y \wedge w$ ε x))). A logical strengthening.

From T[2,11]#, we have from section 4,

T[2,8]. (∀x,y,w,v,u)(u = F10(x,y,w) ∧ w ∈ y ∧ w ∈ v ∧ v ε x → R3(u,x,y) ∧ w ∈ u ∧ u ε x) ∧ (∀x,y,z,w)(w = F9(y,z) → (R3(x,y,z) ↔ x ∈ w)). A logical strengthening.

From T[2,12]#, we have

T[2,12]#. (∀x)(sdns(x) ↔ 0 ∉ x ∧ (∀y,z)(y,z ∈ x ∧ y ≠ z → y ∩ z = 0)) ∧ (∀x)(P(x) ↔ x ∈ d).
T[2,12]#. (∀x)(sdns(x) → 0 ∉ x ∧ (∀y,z)(y,z ∈ x ∧ y ≠ z → y ∩ z = 0)) ∧ (∀x)(0 ∉ x ∧ (∀y,z)(y,z ∈ x ∧ y ≠ z → y ∩ z = 0) → sdns(x)) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)) .
T[2,12]#. (∀x)(sdns(x) → (∀y,z)(0 ∉ x ∧ (y,z ∈ x ∧ y ≠ z → y ∩ z = 0))) ∧ (∀x)((∀y,z)(0 ∉ x ∧ y,z ∈ x ∧ y ≠ z → y ∩ z = 0) → sdns(x)) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)).
T[2,12]#. (∀x,y,z)(sdns(x) → 0 ∉ x ∧ (y,z ∈ x ∧ y ≠ z → y ∩ z = 0)) ∧ (∀x)(∃y,z)((0 ∉ x ∧ y,z ∈ x ∧ y ≠ z → y ∩ z = 0) → sdns(x)) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)).
T[2,12]#. (∀x,y,z,w)(w = 0 ∧ sdns(x) → w ∉ x ∧ (y,z ∈ x ∧ y ≠ z → y ∩ z = w)) ∧ (∀x)((0 ∉ x ∧ F18(x),F19(x) ∈ x ∧ F18(x) ≠ F19(x) → F18(x) ∩ F19(x) = 0) → sdns(x)) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)).
T[2,12]#. (∀x,y,z,w)(w = 0 ∧ sdns(x) → w ∉ x ∧ (y,z ∈ x ∧ y ≠ z → y ∩ z = w)) ∧ (∀x,y,z)(y = F18(x) ∧ z = F19(x) ∧ w = 0 → ((w ∉ x ∧ y,z ∈ x ∧ y ≠ z → y ∩ z = w) → sdns(x))) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)).

A logical strengthening since (∀x,y)(y = d → (P(x) ↔ x ∈ y)) logically implies (logically equivalent to) (∀x)(P(x) ↔ x ∈ d).

For T[2,13]#, we have

T[2,13]#. (∀x)(sdns(x) → (∃y)(∀z)(z ∈ x → sing(y ∩ z))).
T[2,13]#. (∀x)(∃y)(sdns(x) → (∀z)(z ∈ x → sing(y ∩ z))).
T[2,13]#. (∀x)(sdns(x) → (∀z)(z ∈ x → sing(F20(x) ∩ z))).
T[2,13]#. (∀x,z)(sdns(x) ∧ z ∈ x → sing(F20(x) ∩ z)).
T[2,13]#. (∀x,y,z,w)(y = F20(x) ∧ w = y ∩ z ∧ sdns(x) ∧ z ∈ x → sing(w)).

A logical strengthening.

For T[2,14]#, we have

T[2,14]#. (∀x,y,z)(<x,y> = {{x,x},{x,y}}).
T[2,14]#. (∀x,y,z,w,u)(w = {x,x} ∧ u = {x,y} → <x,y> = {w,u}).

A logical equivalence.

For T[2,15]#, we have

T[2,15]#. (∀x,y)(y ∈ rk(x) ↔ (∀z)(z ∈ y → (∃w)(w ∈ x ∧ z ∈ rk(w)))).
T[2,15]#. (∀x,y)(y ∈ rk(x) ↔ (∀z)(∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w))).
T[2,15]#. (∀x,y)(y ∈ rk(x) → (∀z)(∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w))) ∧ (∀x,y)((∀z)(∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w)) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z)(y ∈ rk(x) → (∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w))) ∧ (∀x,y)((∀z)(∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w)) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z)(∃w)(y ∈ rk(x) ∧ z ∈ y → w ∈ x ∧ z ∈ rk(w)) ∧ (∀x,y)(∃z)((∃w)(z ∈ y → w ∈ x ∧ z ∈ rk(w))) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z)(y ∈ rk(x) ∧ z ∈ y → F21(x,y,z) ∈ x ∧ z ∈ rk(F21(x,y,z))) ∧ (∀x,y)(∃z)(∀w)((z ∈ y → w ∈ x ∧ z ∈ rk(w)) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z,w,u)(w = F21(x,y,z) ∧ u = rk(w) ∧ y ∈ rk(x) ∧ z ∈ y → w ∈ x ∧ z ∈ u) ∧ (∀x,y,w)((F22(x,y) ∈ y → w ∈ x ∧ F22(x,y) ∈ rk(w)) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z,w,u)(w = F21(x,y,z) ∧ u = rk(w) ∧ y ∈ rk(x) ∧ z ∈ y → w ∈ x ∧ z ∈ u) ∧ (∀x,y,z,w,u)(z = F22(x,y) ∧ u = rk(w) ∧ (z ∈ y → w ∈ x ∧ z ∈ u) → y ∈ rk(x)).
T[2,15]#. (∀x,y,z,w,u)(w = F21(x,y,z) ∧ u = rk(w) ∧ R5(y,x) ∧ z ∈ y → w ∈ x ∧ z ∈ u) ∧ (∀x,y)(R5(x,y) ↔ x ∈ rk(y)) ∧ (∀x,y,z,w,u)(z = F22(x,y) ∧ u = rk(w) ∧ (z ∈ y → w ∈ x ∧ z ∈ u) → R5(y,x)).
T[2,15]#. (∀x,y,z,w,u)(w = F21(x,y,z) ∧ u = rk(w) ∧ R5(y,x) ∧ z ∈ y → w ∈ x ∧ z ∈ u) ∧ (∀x,y,z)(z = rk(y) → (R5(x,y) ↔ x ∈ z)) ∧ (∀x,y,z,w,u)(z = F22(x,y) ∧ u = rk(w) ∧ (z ∈ y

→ w ∈ x ∧ z ∈ u) → z ∈ y).

A logical strengthening.

For T[2,16]#, we have

T[2,16]#. ∞,rk({c,c}) ∈ d ∧ (∀x,y)(x ∈ y ∧ y ∈ d → x ∈ d) ∧ (∀x,y)(x,y ∈ d → {x,y},UN(x,y) ∈ d).
T[2,16]#. P(∞) ∧ P(rk({c,c})) ∧ (∀x,y)(x ∈ y ∧ P(y) → P(x)) ∧ (∀x,y)(P(x) ∧ P(y) → P({x,y}) ∧ P(UN(x,y))).
T[2,16]#. (∀x,y,z,w)(x = ∞ ∧ y = c ∧ z = {y,y} ∧ w = rk(z) → P(x) ∧ P(w)) ∧ (∀x,y)(x ∈ y ∧ P(y) → P(x)) ∧ (∀x,y,z,w)(z = {x,y} ∧ w = UN(x,y) ∧ P(x) ∧ P(y) → P(z) ∧ P(w)).

A logical equivalence in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#.

T[2,17]#. Internal. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z)).
T[2,17]#. Internal. (∀x,z,w)(<z,w> ∈ F23(x) ↔ z ∈ x ∧ w = rk(z)).
T[2,17]#. Internal. (∀x,y,z,w)(y = F23(x) → (<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z))).

A logical strengthening.

For T[2,18]#, we have

T[2,18]#. (∀x)(∃y)(∀z,w)(<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z)).
T[2,18]#. (∀x,z,w)(<z,w> ∈ F24(x) ↔ z ∈ x ∧ w = rk(z)).
T[2,18]#. (∀x,y,z,w,u)(y = <z,w> ∧ u = F24(x) → (y ∈ u ↔ z ∈ x ∧ w = rk(z))).

A logical strengthening.

For T[2,19]#, we have

T[2,19]#. h(b) ≠ b ∧ h(c) = c ∧ b ∈ c ∧ (∀x)(x ∈ d → h(x) ∈ d).
T[2,19]#. (∀x,y)(x = b ∧ y = c → h(x) ≠ x ∧ h(y) = y ∧ x ∈ y) ∧ (∀x,y,z)(y = d ∧ z = h(x) ∧ x ∈ y → z ∈ y).

A logical equivalence.

For T[2,20]#, we have

T[2,20]#. (∀x,y)(x,y ∈ d → (x ∈ y ↔ h(x) ∈ h(y)).
T[2,20]#. (∀x,y)(P(x) ∧ P(y) → (x ∈ y ↔ h(x) ∈ h(y)).
T[2,20]#. (∀x,y,z,w)(z = h(x) ∧ w = h(y) → (x ∈ y ↔ z ∈ w)).

A logical equivalence in the presence of (∀x)(P(x) ↔ x ∈ d). See T[2,12]#.

For T[2,21]#, we have

T[2,21]#. Elementary. (∀x,y)(C(0,0,y) ε* x ↔ C(0,0,h(y)) ε* x).
T[2,21]#. Elementary. (∀x,y)(F25(y) ε* x ↔ F25(h(y)) ε* x) ∧ (∀x)(F25(x) = C(0,0,x)).
T[2,21]#. Elementary. (∀x,y,z,w,u)(z = F25(y) ∧ w = h(y) ∧ u = F25(w) → (z ε* x ↔ u ε* x)) ∧ (∀x,y,z)(y = 0 ∧ z = C(y,y,x) → F25(x) = z).

A logical eequivalence.

We organize these results into the following system K2T3#.

K[2,1]#. From K[2,1]. (∀x,y,z,w,u)(u = C(x,y,z) → V(u,y) = z ∧ (w ≠ y → V(u,w) = V(x,w)))).
K[2,2]#. From K[2.2]. (∀x,y,z,w)(w = F1(x,y) → (z ε w ↔ ¬(z ε x ∧ z ε y))).
K[2,3]#. From K[2,3]. (∀x,y,z,w)(z = F2(x,y) → (w ε z → R1(x,y,w))) ∧ (∀x,y,z,w,u)(w = V(z,x) ∧ u = V(z,y) → (R1(x,y,z) ↔ w ∈ u)).
K[2,4]#. From K[2,4]. (∀x,y,z,w)(w = F3(x,y) → (z ε w ↔ R2(x,y,z))) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ w = F4(x,y,z) ∧ R2(x,y,z) → u ε y) ∧ (∀x,y,z,w,u)(u = C(z,x,w) ∧ u ε y → R2(x,y,z)).
K[2,5]#. From K[2,5]. (∀x,y,z,w,u)(u = F6(y,w) ∧ z = F5(x,y) → (w ε z ↔ u ε x))) ∧ (∀x,y,z,w,u)(z = 0 ∧ w = C(x,z,y) → F6(x,y) = w).
K[2,6]#. (∀x,y)(x = 0 ∧ y = ∞ → x ∈ y) ∧ (∀x,y,z)(y = ∞ ∧ z = d ∧ x ∈ y → x ∈ z) ∧ (∀x,y,z)(z = d ∧ x ε* y → x ∈ z) ∧ (∀x,y,z,w,u)(w = V(x,y)) ∧ u = C(x,y,z) ∧ P(x) ∧ P(y) ∧ P(z) → P(w) ∧ P(u)).

K[2,7]#. (∀x,y,z,w,u)(z = d ∧ u = F11(x,y) → (w ε* u ↔ (w ε* z ∧ ¬(w ε* x ∧ w ε* y)))).

K[2,8]#. (∀x,y,z,w)(z = F12(x,y) → (w ε* z ↔ R1(x,y,w) ∧ P(w))).

K[2,9]#. (∀x,y,z,w,u)(z = ∞ ∧ u = F13(x,y) ∧ x ∈ z → (w ε* u ↔ R4(w,x,y))) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ w = F14(x,y,z) ∧ R4(x,y,z) → P(x) ∧ P(w) ∧ u ε* z) ∧ (∀x,y,z,w,u)(u = C(x,y,w) ∧ P(x) ∧ P(w) ∧ u ε* z → R4(x,y,z)).

K[2,10]#. (∀x,y,z,w,u)(w = 0 ∧ u = C(w,w,z) ∧ y = F15(x) → (z ε* y ↔ P(z) ∧ u ε* x)).

T[3,1]#. From T[3,1]. (∀x,y)(y = 0 → x ∉ y) ∧ (∀x,y,z)(x ⊆ y ∧ z ∈ x → z ∈ y)) ∧ (∀x,y,z)(z = F7(x,y) ∧ (z ∈ x → z ∈ y) → x ⊆ y).

T[3,2]#. (∀x,y)(y = F16(x) ∧ sing(x) → x = {y,y}) ∧ (∀x,y)(x = {y,y} → sing(x)).

T[3,3]#. (∀x,y,z,w)(w = x ∩ y → (z ∈ w ↔ z ∈ x ∧ z ∈ y)) ∧ (∀x,y,z,w)(w = UN(x,y) → (z ∈ w ↔ z ∈ x ∨ z ∈ y)).

T[3,4]#. From T[3,2]. (∀x,y)(x ⊆ y ∧ y ⊆ x → x = y).

T[3,5]#. (∀x,y,z,w)(w = {x,y} → (z ∈ w ↔ z ∈ x ∨ z ∈ y)).

T[3,6]#. From T[3,4]. (∀x,y,z,w)(w = ∪x ∧ y ∈ z ∧ z ∈ x → y ∈ w).

T[3,7]#. (∀x,y,z)(z = ℘(x) → (y ∈ z ↔ y ⊆ x)).

T[3,8]#. From T[3,6]. (∀x,y)(x = 0 ∧ y = ∞ → x ∈ y) ∧ (∀x,y,z)(y = ∞ ∧ z = {x,x} ∧ x ∈ y → z ∈ y).

T[3,9]#. (∀x,y)(y = F17(x) ∧ x ≠ 0 → y ∈ x ∧ x ∩ y = 0).

T[3,10]#. From T[3,7]. (∀x,y,z,w)(z = F8(x,y) → (w ∈ z ↔ (w ∈ y ∧ w ε x))).

T[3,11]#. From T[3,8]. (∀x,y,w,v,u)(u = F10(x,y,w) ∧ w ∈ y ∧ w ∈ v ∧ v ε x → R3(u,x,y) ∧ w ∈ u ∧ u ε x) ∧ (∀x,y,z,w)(w = F9(y,z) → (R3(x,y,z) ↔ x ∈ w)).

T[3,12]#. (∀x,y,z,w)(w = 0 ∧ sdns(x) → w ∉ x ∧ (y,z ∈ x ∧ y ≠ z → y ∩ z = w)) ∧ (∀x,y,z)(y = F18(x) ∧ z = F19(x) ∧ w = 0 → ((w ∉ x ∧ y,z ∈ x ∧ y ≠ z → y ∩ z = w) → sdns(x))) ∧ (∀x,y)(y = d → (P(x) ↔ x ∈ y)).

T[3,13]#. (∀x,y,z,w)(y = F20(x) ∧ w = y ∩ z ∧ sdns(x) ∧ z ∈ x → sing(w)).

T[3,14]#. (∀x,y,z,w,u)(w = {x,x} ∧ u = {x,y} → <x,y> = {w,u}).

T[3,15]#. (∀x,y,z,w,u)(w = F21(x,y,z) ∧ u = rk(w) ∧ R5(y,x) ∧ z ∈ y → w ∈ x ∧ z ∈ u) ∧ (∀x,y,z)(z = rk(y) → (R5(x,y)

↔ x ∈ z)) ∧ (∀x,y,z,w,u)(z = F22(x,y) ∧ u = rk(w) ∧ (z ∈ y → w ∈ x ∧ z ∈ u) → z ∈ y).
T[3,16]#. (∀x,y,z,w)(x = ∞ ∧ y = c ∧ z = {y,y} ∧ w = rk(z) → P(x) ∧ P(w)) ∧ (∀x,y)(x ∈ y ∧ P(y) → P(x)) ∧ (∀x,y,z,w)(z = {x,y} ∧ w = UN(x,y) ∧ P(x) ∧ P(y) → P(z) ∧ P(w)).
T[3,17]#. (∀x,y,z,w)(y = F23(x) → (<z,w> ∈ y ↔ z ∈ x ∧ w = rk(z))).
T[3,18]#. (∀x,y,z,w,u)(y = <z,w> ∧ u = F24(x) → (y ∈ u ↔ z ∈ x ∧ w = rk(z))).
T[3,19]#. (∀x,y)(x = b ∧ y = c → h(x) ≠ x ∧ h(y) = y ∧ x ∈ y) ∧ (∀x,y,z)(y = d ∧ z = h(x) ∧ x ∈ y → z ∈ y).
T[2,20]#. (∀x,y,z,w)(z = h(x) ∧ w = h(y) → (x ∈ y ↔ z ∈ w)).
T[3,21]#. (∀x,y,z,w,u)(z = F25(y) ∧ w = h(y) ∧ u = F25(w) → (z ε* x ↔ u ε* x)) ∧ (∀x,y,z)(y = 0 ∧ z = C(y,y,x) → F25(x) = z).

The language of K2T3# is as follows.

5 constant symbols. 0,∞,b,c,d.
12 unary function symbols. ℘,rk,h,F15,F16,F17,F18,F19,F20,F23,F24,F25.
18 binary function symbols. V,{ },< >,∩,UN,∪,F1,F2,F3,F5,F6,F7,F8,F9,F11,F12,F13,F22.
5 ternary function symbols. C,F4,F10,F14,F21.
3 unary relation symbols. P,sing,sdns.
5 binary relation symbols. ∈,ε,ε*,⊆,R5.
4 ternary relation symbols. R1,R2,R3,R4.

THEOREM 5.2.4. ZFC + I1,T1#,K1T2#,K2T3# are equiconsistent. K2T3# proves K1T2# proves T1# proves 0 ≠ ∞. K2T3# is strict.

Proof: It is clear by inspection that any countable model of K2T3# with complete diagram is a model of K1T2# with complete diagram by merely dropping information. Now let M be a countable model of K1T2# with complete diagram. We can obviously build an expansion which is a countable model in the standard way of K2T3 armed with arithmetic comprehension, but we will lose all semblance of a complete diagram. But we do have a countable model of K2T3#, and K2T3# is given by finitely many axioms. So as discussed at the beginning of section 3, we obtain a countable model of

K2T3# with complete diagram as is required for equiconsistency. QED

We now remove all the relation symbols from K2T3# in favor of new function symbols of the same arity. This corresponds to the construction of W from K1T2 in section 4. This results in the system W# with the following language.

5 constant symbols.
15 unary function symbols.
23 binary function symbols.
9 ternary function symbols.

THEOREM 5.2.5. ZFC,T1#,K1T2#,K2T3#,W# are equiconsistent. W# proves $0 \neq \infty$. W# is strict.

Proof: See the proof of Theorem 4.2.5. QED

THEOREM 5.2.6. (BSEP) Assuming ZFC + I1 is consistent, ZFC + I1 does not correctly evaluate $\Theta(4,7,N)$. Assuming ZFC + I1 does not prove its own inconsistency, ZFC + I1 does not evaluate $\Theta(4,7,N)$.

Proof: Clearly W# is a strict system in x,y,z,w,u satisfying the quantitative restrictions i-iv listed in Theorem 4.4.3. Also ZFC + I1 is a recursively presented system extending BSEP which is equionsistent with W#. Now apply Corollary 4.4.5. QED

6. $\Theta(k,r)$, $\Theta(k,\leq r)$, $\Theta(k,r,\infty)$, $\Theta(k,\leq r,\infty)$, $\Theta(k,r,N)$, $\Theta(k,\leq r,N)$, $\Theta(k,r,\subseteq N)$, $\Theta(k,\leq r,\subseteq N)$

We have already defined the last four of the above quantities in Definition 1.5. We proved that they are finite integers (Theorem 1.3). The idea there is to associate a set of functions $f:\{1,...,r\}^k \to \{1,...,r^k+r\}$ to any given k-ary operation on a subset of N so that two operations are ≤r-isomorphic if and only if their associates are identical.

Throughout the paper, we have simplified the development by focusing on $\Theta(k,r,N)$, and in detail on $\Theta(4,7,N)$. We now define the first four of the above quantities.

DEFINITION 6.1. $\Theta(k,r)$ is the number of k-ary operations up to r-isomorphism, $\Theta(k,\leq r)$ is the number of k-ary operations up to ≤r-isomorphism, $\Theta(k,r,\infty)$ is the number of infinite k-ary operations up to r-isomoprhism, and $\Theta(k,\leq r,\infty)$ is the number of infinite k-ary operations up to ≤r-isomorphism.

Certainly the simplest looking of these eight quantities is $\Theta(k,r)$. However, $\Theta(k,r)$ is considerably more abstract than the ones with N and ⊆N, in terms of what it is counting. It is not counting the elements of a finite set, but instead is counting the elements of a collection of proper classes (of sets). Each of these proper classes is a proper class of finite functions. It is not counting the elements of a finite set, but instead is counting the elements of a finite collection of proper classes (of sets). Each of these proper classes is a proper class of finite functions.

In order to avoid these issues, we stay well within set theory and define $\Theta(k,r),\Theta(k,\leq r),\Theta(k,r,\infty),\Theta(k,\leq r,\infty)$ as follows. $\Theta(k,r) = t$ if and only if there exists operations $f_1,\ldots,f_t$ such that every operation is r-isomorphic to exactly one f_i, and the other three analogously. With these natural definitions, the natural base theory is BSEP instead of ACA_0 (we view BSEP as an extension of ACA_0).

THEOREM 6.1. (BSEP) Two infinite k-ary operation is r-isomorphic if and only if they are ≤r-isomorphic. $\Theta(k,r,\infty) = \Theta(k,\leq r,\infty)$, $\Theta(k,r,N) = \Theta(k,\leq r,N)$.

Proof: Suppose $f:A^k \to A$ and $g:B^k \to B$ are r-isomorphic, where A,B are infinite. Let h be an i element restriction of f, $0 \leq i \leq r$. Extend h to an r element restriction h' of f. Let h' be isomorphic to an r-element restriction H' of g via α. Then α is also an isomorphism from h to a restriction H of H'. So every ≤r-element restriction of f is isomorphic to a ≤r-element restriction of g. The other direction is established the same way. Now let $f_1,\ldots,f_n$ be infinite k-ary operations such that every infinite k-ary operation is r-isomorphic to a unique f_i. Then $n = \Theta(k,r,\infty)$. By the first claim, every infinite k-ary operation is ≤r-isomorphic to a unique fi, and so $n = \Theta(k,\leq r,\infty)$. The second equation is established in the same way, working only with operations $f:N^k \to N$. QED

THEOREM 6.2. (BSEP) Every infinite k-ary operation is r-isomorphic to some k-ary operation on N. $\Theta(k,r,\infty) = \Theta(k,\leq r,\infty) = \Theta(k,r,N) = \Theta(k,\leq r,N)$.

Proof: Let $f:A^k \to A$, A infinite, and $r \geq 1$. By Theorem 1.5, let φ be a sentence in PC(=) such that a nonempty $g:B^k \to B$ is r-isomoprhic to f if and only if (B,g) satisfies φ. By the upward and downward Skolem Lowenheim theorems, φ has models (B,g) with B of any infinite cardinality. Now let $f_1,\ldots,f_n$ be infinite k-ary operations such that every infinite k-ary operation is r-isomoprhic to exactly one of the f's. Then $n = \Theta(k,r,\infty)$. By the first claim, we can arrange that $f_1,\ldots,f_n$ are countably infinite. Then every countably infinite k-ary operation is $\leq$r-isomorphic to exactly one f_i. Hence $n = \Theta(k,r,N)$. The equations now follow with the help of those in Theorem 6.1. QED

THEOREM 6.3. (BSEP) Every k-ary operation is r-isomorphic to some k-ary operation on a subset of N. $\Theta(k,r) = \Theta(k,r,\subseteq N)$, $\Theta(k,\leq r) = \Theta(k,\leq r,\subseteq N)$.

Proof: Let $f_1,\ldots,f_n$ be k-ary operations such that every k-ary operation is r-isomorphic to exactly one f_i. Then $n = \Theta(k,r)$. By Theorem 6.2, we can arrange that $f_1,\ldots,f_n$ are countable. Hence $n = \Theta(k,r,\subseteq N)$. This establishes the first equation. Now let $f_1,\ldots,f_n$ be k-ary operations such that every k-ary operation is $\leq$r-isomorphic to exactly one f_i. Then $n = \Theta(k,\leq r)$. By Theorem 6.2, we can replace the infinite f_i by countable f_i. Hence $n = \Theta(k,\leq r,\subseteq N)$. This establishes the second equation. QED

From Theorems 6.1 - 6.3, we now need only consider $\Theta(k,r,N)$, $\Theta(k,r,\subseteq N)$, $\Theta(k,\leq r,\subseteq N)$. This now brings our investigation back down to earth.

THEOREM 6.4. (BSEP) $\Theta(k,\leq r,\subseteq N) = \Theta(k,r,\subseteq N) + \Theta(k,\leq r,\subseteq\{1,\ldots,r-1\}) - 1$.

Proof: We partition the k-ary operations on subsets of N into the $<$r-element k-ary operations and the $\geq$r-element k-ary operations. For the $\geq$r-element k-ary operations, $\leq$r-isomorphism and r-isomorphism are the same as can be seen from the proof of Theorem 6.1 where the extension of h to h' needs only that f is a $\geq$r-element k-ary operation. For the <r-element k-ary operations, there is exactly one up to r-isomorphism. Also the k-ary <r-element and ≥r-element operations have no common elements up to r-isomorphism. So in comparing $\Theta(k,\leq r,\subseteq N)$ versus $\Theta(k,r,\subseteq N)$, the former has an excess of $\Theta(k,\leq r,\subseteq\{1,...,r-1\})$ - 1 over the latter. QED

So now we only need to consider $\Theta(k,r,N)$ and $\Theta(k,r,\subseteq N)$.

THEOREM 6.5. (BSEP) For fixed $k \geq 2$, the unary function $\Theta(k,r,\subseteq N)$ has Turing degree 0'. The function $\Theta(1,r,\subseteq N)$ is primitive recursive.

Proof: This is Theorem 2.1.7 for $\Theta(k,r,\subseteq N)$ instead of the former $\Theta(k,r,N)$. That was proved using the development from Lemma 2.1.1 through Lemma 2.1.6. Basically, the same proof works for $\Theta(k,r,\subseteq N)$ by replacing N throughout with ⊆N. This begins with Definition 2.1.1, where we define (r,S) to be $(k,\subseteq N)$-special if and only if there exists $G:E^k$ E, $E \subseteq N$, such that $S = \alpha r(G)$. Lemma 2.1.1 no w asserts that the number of $(k,\subseteq N)$-speical (r,S) is $\Theta(k,r,\subseteq N)$. In Lemma 2.1.4, we replace N satisfiable by ⊆N satisfiable and (k,N)-speical by $(k,\subseteq N)$-special. The same proof goes through. For Lemma 2.1.5, we use the ⊆N satisfiable universal sentences in a single binary function symbol. The upper bound is still immediate. The lower bound was obtained by reducing nonhalting to N satisfiability. The same proof reduces nonhalting to ⊆N satisfiability by the same argument since the models involved there are all infinite. Then we have Lemma 2.1.6 for ⊆N satisfiability as in the proof of Lemma 2.1.6. We arrive at the present claim with the same proof. QED

THEOREM 6.6. (BSEP) Let T be a consistent extension of BSEP and $k,r \geq 1$. The following are equivalent.
i. Every satisfiable sentence in Y(k,r) is provably N satisfiable in T.
ii. T correctly evaluates $\Theta(k,r,\subseteq N)$.

Proof: This is Theorem 3.2.4 for $\Theta(k,r,\subseteq N)$ instead of $\Theta(k,r,N)$. That was proved using the development from Theorem 3.2.1 through Theorem 3.2.3. Essentially the same proof works for $\Theta(k,r,\subseteq N)$ by replacing N throughout with ⊆N. Theorem 3.2.1 uses the same well known decision procedure for satisfiability in one unary function symbol, [BGG01], p. 315. In Theorem 3.2.2, we replace $\Theta(k,r,N)$,(k,N)-special by $\Theta(k,r,\subseteq N)$,(k,⊆N)-special, respectively. The same proof goes through. In Theorem 3.2.3, we replace (k,N)-special, N satisfiable by (k,⊆N)-special, satisfiable, respectively. Again the same proof goes through. We arrive at the present claim with the same proof. QED

We now use Theorem 6.6 to compare the evaluation of $\Theta(k,r,N)$ with the evaluation of $\Theta(k,r,\subseteq N)$. One direction is very easy, and the other is not.

LEMMA 6.7. (BSEP) Let T be an extension of BSEP. Let $k,r \geq 1$. Suppose every N satisfiable $\varphi \in Y(k,r)$ is provably N satisfiable in T. Then every ⊆N satisfiable $\varphi \in Y(k,r)$ is provably ⊆N satisfiable in T.

Proof: Let T,k,r be as given and suppose every N satisfiable $\varphi \in Y(k,r)$ is provably N satisfiable in T. Let $\psi \in Y(k,r)$ be satisfiable. If ψ is N satisfiable then we are done. Otherwise, ψ has a finite model, and ψ is provably satisfiable. QED

LEMMA 6.8. (BSEP) Let $1 \leq k < r$. Suppose every N satisfiable $\varphi \in Y(k,r)$ is provably N satisfiable in T. Then every ⊆N satisfiable $\varphi \in Y(k,r)$ is provably ⊆N satisfiable in T.

Proof: Let T,k,r be as given. By the well known decision procedure for PC(=) in one unary function symbol, both statements hold for k = 1 ([BGG01], p. 315). So we assme $2 \leq k < r$. Assume every ⊆N satisfiable $\varphi \in Y(k,r)$ is provably ⊆N satisfiable in T. Let $\varphi \in Y(k,r)$ be N satisfiable. Then φ is provably ⊆N satisfiable in T. ψ be the conjunction of the following.

i. $(\exists x)(F(x,\ldots,x) = x) \wedge (\exists x,y)(x \neq y)$.
ii. $F(x,\ldots,x) = x \wedge F(y,\ldots,y) = y \rightarrow x = y$.
iii. If we restrict F to arguments x with $F(x,\ldots,x) \neq x$, then we get a value y with $F(y,\ldots,y) \neq y$.
iv. If we restrict F to arguments x with $F(x,\ldots,x) \neq x$, then φ holds.
v. Let $F(x,\ldots,x) = x$ and $y,z \neq x$. $(F(x,y,\ldots,y) = F(x,z,\ldots,z) \rightarrow y = z) \wedge (F(y,\ldots,y,x) = F(z,\ldots,z,x) \rightarrow y = z) \wedge F(x,y,\ldots,y) \neq F(z,\ldots,z,x)$.

Note that we need r ≥ 2 for i, r ≥ 2 for ii, r ≥ k for iv, and r ≥ 3 for v. But we need r ≥ k+1 for iii. Since 2 ≤ k < r, we have ψ ∈ Y(k,r).

We claim that ψ is provably ⊆N satisfiable in T. We argue in T. Let (N,F) be a model of φ. We construct a model (N ∪ {-1},F') of ψ as follows. Let F' extend F. Define F'(-1,...,-1) = -1. Define F'(-1,n,...,n) = 2n, F'(n,...,n,-1) = 2n+1, for n ≥ 0. Define F' elsewhere to be -1.

Now i,ii hold since F(n,...,n) = -1 for all n ≥ -1. iii,iv hold because F' extends F. For v, let F(x,...,x) = x. Then x = -1, and the functions 2n, 2n+1 are one-one with disjoint ranges.

We now claim T proves that every model of ψ is infinite. To see this, let (D,F) be a model of ψ. According to i,ii, let 0 be unique such that F(0,...,0) = 0. According to v, the unary functions F(0,x,...,x),F(x,...,x,0) are one-one for x ≠ 0, and have disjoint ranges on the x ≠ 0. This implies that there are infinitely many x ≠ 0 or no x ≠ 0. The latter is impossible by the second conjunct of i.

T must prove that φ is N satisfiable because of iv, which provides a model of φ with domain D\{0}. QED

THEOREM 6.9. (BSEP) Let T be a consistent extension of BSEP and k,r ≥ 1. If T correctly evaluates Θ(k,r,N) then T correctly evaluates Θ(k,r,⊆N). If 1 ≤ k < r then T correctly evaluates (k,r,N) if and only if T correctly evaluates (k,r,⊆N).

Proof: By Theorem 6.6 and Lemmas 6.7, 6.8. QED

COROLLARY 6.10. (BSEP) Let T be a consistent extension of BSEP and 1 ≤ k < r. T correctly evaluates all or none of Θ(k,r),Θ(k,≤r),Θ(k,r,∞),Θ(k,≤r,∞),Θ(k,r,N),Θ(k,≤r,N),Θ(k,r,⊆N),Θ(k,≤r,⊆N).

Proof: By Theorem 6.2,6.3,6.4,6.9. QED

COROLLARY 6.11. (BSEP) Assuming ZFC is consistent, ZFC does not correctly evaluate any of

$\Theta(4,7),\Theta(4,\leq 7),\Theta(4,7,\infty),\Theta(4,\leq 7,\infty),\Theta(4,7,N),\Theta(4,\leq 7,N),\Theta(4,7,\subseteq N),\Theta(4,\leq 7,\subseteq N)$. Assuming ZFC + I1 is consistent, ZFC + I1 does not correctly evaluate any of $\Theta(4,7),\Theta(4,\leq 7),\Theta(4,7,\infty),\Theta(4,\leq 7,\infty),\Theta(4,7,N),\Theta(4,\leq 7,N),\Theta(4,7,\subseteq N),\Theta(4,\leq 7,\subseteq N)$.

Proof: By Theorem 4.3.4, 5.2.6, and Corollary 6.10. QED

7. FORMAL SYSTEMS USED

EFA. Our Exponential Function Arithmetic, corresponding to $I\Sigma_0(\exp)$. [HP98], p. 37, 405.

RCA_0. Our base theory for Reverse Mathematics. Recursive Comprehension Axiom. [Fr75], [Fr76], [Frxx], [Si09].

ACA_0. One of our five major Reverse Mathematics theories. Arithemtic Comprehension Axiom. [Fr75], [Fr76], [Frxx], [Si09].

BSEP. In $\in,=$, with Extensionality, Pairing, Union, Foundation, Bounded Separation, $(\forall x)(\exists y)(y$ is transitive $\wedge$ $x \in y \wedge (\forall z,w \in y)(\{z,w\},z \cup w \in y))$. Note that the last axiom here implies a strong form of Infinity.

ZC. Zermelo set theory with the Axiom of Choice, in $\in,=$. Extensionality, Pairing, Union, Infinity, Foundation, Separation, Power Set, Choice. [Ka94]. [Le79].

ZFC. Zermelo Frankel set theory with the Axiom of Choice, in $\in,=$. Extensionality, Pairing, Union, Infinity, Foundation, Separation, Power Set, Replacement, Choice. [Le79].

ZFC + I1. ZFC together with I1, which asserts that there exists an elementary embedding $j:V(\alpha+1) \to V(\alpha+1)$ which is not the identity. [Ka94], p. 325.

By the well known Kunen Inconsistency, the α in ZFC + I1 must be a limit ordinal λ. [Ka94], p. 322.

There are some interesting points in the development of BSEP.

LEMMA 7.1. BSEP proves the usual Infinity, which asserts

$(\exists x)(\emptyset \in x \wedge (\forall y \in x)(y \cup \{y\} \in x))$.

Proof: By Bounded Separation, $\emptyset$ exists. Apply the last axiom of BSEP to obtain y with $\emptyset \in y$ and y closed under unordered pair and pairwise union. Thus let $z \in y$. Then $\{z\}, z \in y$, and therefore $z \cup \{z\} \in y$. QED

We will now develop the nonnegative integers and the usual inductively ordered semiring structure using only the usual Infinity, which we have secured by Lemma 7.1.

An ordinal is an $\in$ connected transitive set. We first prove some facts about ordinals.

LEMMA 7.2. Let x,y be ordinals. Let z be an $\in$ minimal element of y/x. Then z = x.

Proof: Let x,y,z be as given. Every element of z must lie in y (y is transitive), and so must lie in x. I.e., $z \subseteq x$. We claim $x \subseteq z$. Let $w \in x$. Then $w,z \in y$, and so $w \in z \vee z \in w \vee w = z$. If $z \in w$, then $z \in x$ (x is transitive), contradicting the choice of z. If z = w then $z \in x$, contradicting the choice of x. Hence $w \in z$. This establishes the claim, and hence $x \subseteq z$. QED

LEMMA 7.3. Let x,y be ordinals.
i. $x \subseteq y \vee y \subseteq x$.
ii. $x \subseteq y \leftrightarrow x \in y \vee x = y$.
iii. $x \in y \vee y \in x \vee x = y$ exclusively.
iv. Every nonempty set of ordinals has an $\in$ least element.
v. Every $z \in x$ is an ordinal.
vi. $x \cup \{x\}$ is an ordinal.

Proof: For i, assume $\neg y \subseteq x$. Let z be an $\in$ minimal element of y/x. By Lemma 7.2, z = x, and so $x \in y$. Since y is transitive, $x \subseteq y$.

For ii, assume $x \subseteq y$. If $x \neq y$ then $\neg y \subseteq x$, so by Lemma 7.2, x is an $\in$ minimal element of y/x. In particular, if $x \neq y$ then $x \in y$. The converse is immediate from the transitivity of y.

For iii, we use $x \subseteq y \vee y \subseteq x$. If $x \subseteq y$ then by ii, $x \in y \vee$

$x = y$. If $y \subseteq x$ then by ii, $y \in x \vee x = y$. Exclusivity is by Foundation.

For iv, let A be a nonempty set of ordinals. By Foundation, A has an $\in$ minimal element, which must be the $\in$ least element by iii.

For v, let $z \in x$. Suppose $a \in b \in z$. Then $a,b,z \in x$, and so $a \in z \vee z \in a \vee a = z$. By Foundation, $a \in z$. Now suppose $a,b \in z$. Then $a,b \in x$, and so $a \in b \vee b \in a \vee a = b$.

For vi, $x \cup \{x\}$ is transitive by inspection. Now let $y,z \in x \cup \{x\}$. If $y,z \in x$ then they are $\in$ comparable. Otherwise, y,z are $\in$ comparable by inspection. QED

We now define $0 = \emptyset$, and $Sx = x \cup \{x\}$ for ordinals x. Define $x < y \leftrightarrow x \in y$, $x \leq y \leftrightarrow x < y \vee x = y$, for ordinals x,y.

LEMMA 7.4. $<$ on the ordinals forms a strict linear ordering on all ordinals, with least element $\emptyset$, where every ordinal has the immediate successor Sx. $<$ forms a strict linear ordering on every ordinal. Every set of ordinals has a $<$ least element.

Proof: Strict linear order is from Lemma 7.3 iii. Obviously $\emptyset$ is an ordinal, and $x < \emptyset$ is impossible. We now claim that Sx is the immediate successor of x in $<$. By Lemma 7.2 v, Sx is an ordinal. Obviously $y < x \cup \{x\} \leftrightarrow y < x \vee y = x \leftrightarrow y \leq x$. For the second claim, let x be an ordinal. Then x consists of ordinals by Lemma 7.3 v. Hence $<$ is a strict linear ordering on x. The last claim is from Lemma 7.3 iv. QED

We are now ready to define ω. Let ω^* witness Infinity. I.e, $\emptyset \in \omega^* \wedge (\forall x \in \omega^*)(x \cup \{x\} \in \omega^*)$. We define $\omega = \{x \in \omega^*$: x is an ordinal $\wedge$ $(\forall y)(0 < y \leq x \rightarrow y$ has an immediate predecessor in $<)\}$.

LEMMA 7.5. The following holds.
i. ω is an ordinal.
ii. $(\omega,<)$ is a strict well ordering with least element $\emptyset$.
iii. Every $x \in \omega$ has the immediate successor $Sx \in \omega$.
iv. Every nonempty $x \in \omega$ has an immediate predecessor $y \in \omega$

under <.
v. Every $A \subseteq \omega$ with $\emptyset \in A \wedge (\forall x \in A)(Sx \in A)$ is ω.

Proof: For i, we first claim ω is transitive. let $x \in y \in \omega$. Then y is an ordinal, and so by Lemma 7.3 v, x is an ordinal. Suppose $0 < z \leq x$. Then $0 < z \leq y$ and so z has an immediate predecessor in <. Hence $x \in \omega$ and establishes the claim. Now let $x,y \in \omega$. Then x,y are ∈ comparable because x,y are ordinals and Lemma 7.3 iii.

For ii, use i and Lemma 7.4. For iii, use Lemma 7.4 and that for all $x \in \omega$, $x \cup \{x\} \in \omega$. To see this, first note that $x \cup \{x\} \in \omega^*$, and $Sx = x \cup \{x\}$ is an ordinal. Now let $0 < y \leq Sx$. If $y < Sx$ then $0 < y \leq x$, and so y has an immediate predecessor in <. If $y = Sx$ then y has the immediate predecessor x in <.

For iv, note that the immediate predecessor y of x is an element of x, and therefore by i, lies in ω.

For v, let $A \subseteq \omega$, $\emptyset \in A$, $(\forall x \in A)(Sx \in A)$. Supppose $\omega \backslash A$ is nonempty, and let y be an ∈ minimal element of $\omega \backslash A$. Then $y \neq \emptyset$, and so y has the immediate predecessor z under < and $z \in \omega$. Hence $Sz = z \cup \{z\} = y$. By the hypothesis on A, we have $y \in A$, which is a contradiction. QED

We can now go on and develop the ordered semiring structure on ω. We sketch some main points. How do we develop +? The idea is to develop functions $f:n \times n \rightarrow \omega$, where $n \in \omega$. This requires that we work with ordered triples. This is why we have added the last axiom of BSEP. In particular, we have a transitive set A with $\omega \in A$, and where A is closed under pairwise unions and unordered pairs. The rather rich set A is easily enough to support what we need here using Bounded Separation.

From here we have the obvious natural interpretation of ACA_0 in BSEP.

To interpret BSEP in ACA_0, we first construct the standard model of BSEP. For any set x, let $\Gamma(x)$ be the least transitive set y such that x and all of the finite subsets of y lie in y. We start with ∅, and build $\emptyset, \Gamma\emptyset, \Gamma\Gamma\emptyset, \ldots$. Of course, $\Gamma\emptyset = V(\omega)$. The standard model consists of all subsets of the various terms in this infinite series. There is no difficulty in making this construction within ACA_0 as follows. Each element of the setsin this series has a canonical finitary name, with an obvious membership relation, and where for any two different names, we can effectively provide a name of an element of one that is not an element of the other. So the sets here, as far as ACA0 is concerned, are named simply by a set of names. The essential point here is that no inductive extensionalization process need be performed.

REFERENCES

[BGG01] E. Borger, E. Gradel, Y. Gurevich, The Classical Decision Problem, Universitext, Springer, 2001 (second printing).

[BT10] D. Berend, T. Tassa, "Improved bounds on Bell number and on moments of sums of random variables". Probability and Mathematical Statistics. 30 (2): 185 - 205.

[Fr75] H. Friedman, Some Systems of Second Order Arithmeitc and Their Use, Proceedings of the 1974 International Congress of Mathematicians, Vol. 1, (1975), pp. 235-242.

[Fr76] H. Friedman, Subsystems of Second Order Arithmetic with Restricted Induction I,II, abstracts, J. of Symbolic Logic, Vol. 41, No. 2 (1976), pp. 557-559

[Frxx] H. Friedman, Downloadable Manuscripts, Information on the Reverse Mathematics program. https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ u.osu.edu.

[HP98] P. Hajek, P. Pudlak, Metamathematics of First-Order Arithmetic, Perspectives in Mathematical Logic, 1998.

[Ka94] A. Kanamori, The Higher Infinite, Perspectives in Mathematical Logic, Springer-Verlag, 1994.

[Le79] A. Levy, Basic Set Theory, Perspectives in Mathematical Logic, Berlin, Springer-verlag 1979.

[Si09] S. Simpson, Subsystems of Second Order Arithmetic, Perspectives in Logic, ASL, 2009.