THE INTERPRETATION OF SET THEORY IN MATHEMATICAL PREDICATION THEORY

Preliminary Report

by

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here we interpret set theory formal systems using no extensionality. Parameters are allowed.

NOTE: We have only the one important result about Subworld Separation + Reducibility.

INTRODUCTION

This paper was referred to in the Introduction in our paper [Fr97a], “The Axiomatization of Set Theory by Separation, Reducibility, and Comprehension.” In [Fr97a], all systems considered used the axiom of Extensionality. This is appropriate in a set theoretic context.

Here we view the underlying concept as that of “mathematical predication.” instead of set. This is the concept of predication that underlies the comprehension and separation axiom schemes, where first order definability (with parameters) approximates mathematical predication.

Obviously there is nothing extensional about this concept. This is why none of the systems for mathematical predication considered here have the axiom of Extensionality.

In [Fr97a] we were preoccupied with outright derivations of set theoretic axioms. In this context we will not be concerned with outright derivations. Instead we will be exclusively concerned with interpretability. In each case, we opt for the simplest and cleanest set of axioms. The interpretations will at least preserve the arithmetic of natural numbers.

There is the legitimate concern of the strength of the set theories based on ZF with the higher of the large cardinal axioms that are interpreted here. Results of Hugh Woodin about forcing over universes with very large cardinals suggest that they are very strong - probably as strong as when the axiom of Choice is added. However at the time of this writing, this has not been resolved. In any case, it is knwon that these systems based on ZF are at least as strong as all large cardinals for which there exists a current inner model theory. One can of course merely add the axiom of Choice in its usual formulation to the axiomatizations considered here.

We close the paper with some conceptual interpretations of the axiomatizations proposed here. Although we regard this discussion as perhaps premature at this stage, it is intended to provide motivation for the axiomatizations, and the choice of terminology: reducibility and comprehension.

SECTION 1. ZF.

We consider the following system S in $L(\in,W)$.

1. SS. $x \in W \rightarrow (\exists y \in W)(\forall z)(z \in y \leftrightarrow (z \in x \ \& \ \varphi))$, where φ is a formula in $L(\in,W)$ in which x is not free.
2. RED. $x_1,\ldots,x_k \in W \rightarrow ((\exists y)(\varphi) \rightarrow (\exists y \in W)(\varphi))$, where $k \geq 0$ and φ is a formula in $L(\in)$ with at most the free variables $x_1,\ldots,x_k,y$.

This system is exactly the system K(W) with Extensionality dropped, in “Axiomatization of set theory based on extensionality, separation, and reducibility.” K(W) was shown to interpret ZF\F. We wish to interpret K(W) in S. This will establish an interpretation of ZF\F, or even ZFC within S.

We make the following definitions in S. $x = y$ iff $(\forall z)(z \in x \leftrightarrow z \in y))$. UP(x,y,z) if and only if

i) $x,y \in z$;
ii) $(\forall w)(w \in z \rightarrow (w = x \text{ or } w = y))$.

P(x,y,z) if and only if there exists u,v such that
i) UP(x,x,u);
ii) UP(x,y,v);
iii) UP(u,v,z).

LEMMA 1. Let φ is a formula in $L(\in)$ with at most the free variables $x_1,...,x_k$. The following is provable in S. $x_1,...,x_k \in W \rightarrow (\varphi \leftrightarrow \varphi^W)$.

Proof: By induction on the complexity of φ. □

LEMMA 2. Let φ is a formula in $L(\in,W)$ in which x is not free. The following is provable in S. $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \& \varphi))$.

Proof: It suffices to establish this where φ is a formula in $L(\in)$. Note that by SS, $(\forall u_1,...,u_k,z \in W)(\exists x \in W)(\forall y \in W)(y \in x \leftrightarrow (y \in z \& \varphi^W))$, where the free variables in φ are among $u_1,...,u_k,y,z$. The result follows by Lemma 1. □

LEMMA 3. The following is provable in S.

i) $(\forall x,y)(\exists z)(x,y \in z)$;

ii) $(\forall x,y)(\exists z)(UP(x,y,z))$;

iii) $(UP(x,y,z) \& UP(x',y',z') \& z = z') \rightarrow ((x = x' \& y = y') \text{ or } (x = y' \& y = x'))$;

Proof: Let $x,y \in W$. Then by RED, let $x,y \in z \in W$. Thus we have shown $(\forall x,y \in W)(\exists z \in W)(x,y \in z)$. By Lemma 1, i) is established. ii) follows from i) by Separation (Lemma 2). For iii), let x,y,z,x',y',z' be as given. Every element of z is equaled to x or y, and every element of z′ is equaled to x′ or y′. Since x′,y′ are equaled to elements of z′, they are also equaled to elements of z, and so $(x' = x \text{ or } x' = y) \& (y' = x \text{ or } y' = y)$. There are four cases to consider. Two of them are what we want. By symmetry, we need only consider the remaining case $x' = x = y'$. So any two elements of z′ are equal. Hence any two elements of z are equal. Therefore $x = y$. □

LEMMA 4. The following is provable in S. $(\forall x,y)(\exists z)(P(x,y,z))$. $(P(x,y,z) \& P(x',y',z') \& z = z') \rightarrow (x = x' \& y = y')$.

Proof: From Lemma 3. □

LEMMA 5. The following is provable in S. W is transitive. Union holds; i.e., $(\forall x)(\exists y)(\forall z)(z \in x \leftrightarrow (\exists w \in x)(z \in w))$. In particular, every $a \cup b$ exists.

Proof: Let $x \in W$. By SS, let $x' \in W$ be such that $(\forall b)(b \in x' \leftrightarrow (b \in x \;\&\; b \in W))$. Then x' and x have the same elements from W. By RED, $x = x'$. Let $y \in x$. Then $y \in x'$, and so $y \in W$. For Union, by RED it suffices to construct the union of $x \in W$. This follows from Separation since the elements of the elements of x lie in W. □

We say that x is a transitive closure of y if and only if

i) x is transitive;
ii) $y \subseteq x$;
iii) for all transitive z, if $x \subseteq z$ then $y \subseteq z$.

LEMMA 6. The following is provable in S. Every set is a subset of a transitive set. Every set has a transitive closure. Any two transitive closures of a set are equal.

Proof: We claim that $(\forall x \in W)(\exists y)(y \text{ is transitive} \;\&\; x \subseteq y)$, by taking $y = W$ according to Lemma 5. Hence by RED, $(\forall x \in W)(\exists y \in W)(y \text{ is transitive} \;\&\; x \subseteq y)$. Hence $(\forall x \in W)(\exists y)(y \text{ is transitive} \;\&\; x \in y)$. By RED, $(\forall x)(\exists y)(y \text{ is transitive} \;\&\; x \subseteq y)$.

Now let y be transitive, and $x \subseteq y$. Let $z = \{b \in y: (\forall w)((w \text{ is transitive} \;\&\; x \subseteq w) \rightarrow b \in w\}$. Then $x \subseteq z$ and z is transitive. Also, let w be transitive, $x \subseteq w$. Then $z \subseteq w$. Therefore z is a transitive closure of x.

Finally, it is obvious that any two transitive closures of a set are equal, since they must be contained in each other. □

We write $c[a,b] \leftrightarrow (\exists x \in c)(P(a,b,x))$. Thus we are treating all sets as binary relations. We write $c \sim d$ if and only if $(\forall a,b)(c[a,b] \leftrightarrow d[a,b])$.

We write $x \in TC(y)$ iff $(\exists z)(z \text{ is a transitive closure of } y \text{ and } x \in z)$.

A Cartesian product of x,y is a set z such that

i) $(\forall b \in x)(\forall c \in y)(z[b,c])$.
ii) $(\forall u \in z)(\exists b \in x)(\exists c \in y)(P(b,c,u))$.

LEMMA 7. The following is provable in S.

i) $(\forall x,y \in W)(W[x,y])$;
ii) every x,y has a Cartesian product.

Proof: i) is by RED. For ii), let $x,y \in W$. Now W is transitive. Hence for all $b \in x$ and $c \in y$, $W[b,c]$. Then W can be cut down by Separation to form a Cartesian product of x,y. Therefore by RED, every x,y has a Cartesian product. □

We say that z is a comparison relation for x,y if and only if

i) x,y are transitive;
ii) $(\forall b \in x)(\forall c \in y)(z[b,c] \leftrightarrow ((\forall b' \in b)(\exists c' \in c)(z[b',c']) \ \& \ (\forall c' \in c)(\exists b' \in b)(z[b',c'])))$;
iii) $(\forall u \in z)(\exists b \in x)(\exists c \in y)(P(b,c,u))$.

We say that a transitive set x is well founded if and only if every nonempty subset of x has an $\in$-minimal element. These are called WFT's. We are only interested in comparison relations for WFT's.

LEMMA 8. The following is provable in S. Let z be a comparison relation for the WFT's x,y. Let $b,b^* \in x$, $c,c^* \in y$, where $b = b^*$ and $c = c^*$. Then $z[b,c] \leftrightarrow z[b^*,c^*]$.

Proof: Let b,b^*,c,c^*,x,y,z be as given. Then $z[b,c] \leftrightarrow ((\forall b' \in b)(\exists c' \in c)(z[b',c']) \ \& \ (\forall c' \in c)(\exists b' \in b)(z[b',c']))$, and also $z[b^*,c^*] \leftrightarrow ((\forall b' \in b^*)(\exists c' \in c^*)(z[b',c']) \ \& \ (\forall c' \in c^*)(\exists b' \in b^*)(z[b',c']))$. □

LEMMA 9. The following is provable in S. Let u be a comparison relation for the WFT's x,y, and let v be a comparison relation for the WFT's z,w. Then $(\forall b \in x \cap z)(\forall c \in y \cap w)(u[b,c] \leftrightarrow v[b,c])$.

Proof: Let x,y,z,w,u,v be as given. Suppose $\neg(\forall b \in x \cap z)(\forall c \in y \cap w)(u[b,c] \leftrightarrow v[b,c])$. We will use that the intersection of two WFT's is a WFT. Fix b to be an $\in$-minimal element of $x \cap z$ such that for some $c \in y \cap w$, $u[b,c] \leftrightarrow \neg v[b,c]$. Now fix c to be any element of $y \cap w$ such that $u[b,c] \leftrightarrow \neg v[b,c]$.

Now $u[b,c] \leftrightarrow ((\forall b' \in b)(\exists c' \in c)(u[b',c']) \ \& \ (\forall c' \in c)(\exists b' \in b)(u[b',c']))$, and also $v[b,c] \leftrightarrow ((\forall b' \in b)(\exists c' \in c)(v[b',c']) \ \& \ (\forall c' \in c)(\exists b' \in b)(v[b',c']))$. At the relevant b',c', we see that $u[b',c'] \leftrightarrow v[b',c']$. Hence $u[b,c] \leftrightarrow v[b,c]$. □

LEMMA 10. The following is provable in S. Suppose every element of b has a transitive closure that is a WFT. Then b has a transitive closure that is a WFT. Furthermore, b is an element of a WFT.

Proof: Let b be as given. Let b^* be a transitive closure of b. Let $x \subseteq b^*$ be nonempty. Let $u \in x$. If u is $\in$-minimal then we are done. So assume $u \cap x \neq \emptyset$. Let u^* be a transitive closure of u that is a WFT. Let c be an $\in$-minimal element of $u^* \cap x$. Then c is an $\in$-minimal element of b^*.

For the final claim, let $\{b\}$ be any unordered pair of b and b, and apply the first claim to $b^* \cup \{b\}$. Clearly $b^* \cup \{b\}$ is a WFT. □

LEMMA 11. The following is provable in S. The elements of any transitive closure of b are just the elements of transitive closures of elements of b together with the elements of b.

Proof: Let b be given and x be a transitive closure of b. Let $u \in b$. Then any transitive closure of u must be included in x, since its intersection with x is a transitive set including u. Hence x contains as elements all elements of transitive closures of elements of b together with all elements of b. On the other hand, the set of all elements of transitive closures of elements of b together with all elements of b forms a set by Separation, and is clearly transitive. □

LEMMA 12. The following is provable in S. Let x,y be WFT's. There is a comparison relation for x,y.

Proof: Let x,y be WFT's. First we will prove that for all $b \in x$, and transitive closures b^* of b, there is a comparison relation for b^*,y. Assume this is false, and let b be an $\in$-minimal counterexample. Thus $b \in x$, b^* is a transitive closure of b. We also have that for all $b' \in b$, there is a transitive closure b'^* of b' and a comparison relation for b'^*,y.

We now construct a comparison relation for b^*,y. We now define a relation R between elements of b^* and elements of y as follows. Fix $u \in b^*$.

case 1. u ∈ b′*, b′ ∈ b. Let v be a comparison relation for b′*,y. For z ∈ y, define R(u,z) ↔ v[u,z].

case 2. u ∈ b and case 1 fails. For z ∈ y, define R(u,z) ↔ (∀u′ ∈ u)(∃z′ ∈ z)(R(u′,z′)), where R is given by case 1.

By Lemma 9, we see that in case 1, the choice of b′ does not affect the outcome.

We now verify that R has the properties of a comparison relation for b*,y.

Suppose R(u,z). First assume that case 1 applies to u, and so let u ∈ b′*, b′ ∈ b, and v is a comparison relation for b′*,y. Then v[u,z], and so (∀u′ ∈ u)(∃z′ ∈ z)(v[u′,z′]) & (∀z′ ∈ z)(∃u′ ∈ u)(v[u′,z′]). Hence (∀u′ ∈ u)(∃z′ ∈ z)(R(u′,z′)) & (∀z′ ∈ z)(∃u′ ∈ u)(R(u′,z′)). Now assume that case 2 applies to u. Then (∀u′ ∈ u)(∃z′ ∈ z)(R(u′,z′)) & (∀z′ ∈ z)(∃u′ ∈ u)(R(u′,z′)). Hence R(u,z).

On the other hand, suppose (∀u′ ∈ u)(∃z′ ∈ z)(R(u′,z′)) & (∀z′ ∈ z)(∃u′ ∈ u)(R(u′,z′)), where u ∈ b* and z ∈ y. Suppose first that case 1 applies to u. Then obviously v[u,z] for the appropriate v, and so R(u,z). Now suppose that case 2 applies to u. Also clearly R(u,z).

Now using Separation and Cartesian product, we can realize R as a comparison relation for b*,y.

To complete the proof of the Lemma, let {x} be any unordered pair of x and x and let x′ be x ∪ {x}. Then x′ is a WFT. Now apply what we have just proved to x. We can take x* to be x. So we get a comparison relation for x,y. ∎

Let E(b,c) ↔ (∃x,y,z)(x,y are WFT's, b ∈ x, c ∈ y, z is a comparison relation for x,y, and z[b,c]).

LEMMA 13. The following is provable in S. For all b,c, E(b,c) ↔ ((∀b′ ∈ b)(∃c′ ∈ c)(E(b′,c′)) & (∀c′ ∈ c)(∃b′ ∈ b)(E(b′,c′)).

Proof: First suppose that E(b,c). Let x,y be WFT's, b ∈ x, c ∈ y, z be a comparison relation for x,y, and z[b,c]. Then ((∀b′ ∈ b)(∃c′ ∈ c)(z[b′,c′]) & (∀c′ ∈ c)(∃b′ ∈

b)(z[b′,c′]). Hence ((∀b′ ∈ b)(∃c′ ∈ c)(E[b′,c′]) & (∀c′ ∈ c)(∃b′ ∈ b)(E[b′,c′]).

Now suppose that ((∀b′ ∈ b)(∃c′ ∈ c)(E(b′,c′)) & (∀c′ ∈ c)(∃b′ ∈ b)(E(b′,c′)). By Lemma 10, let x,y be WFT′s where b ∈ x and c ∈ y. By Lemma 12, let z be a comparison relation for x,y. By Lemma 9, ((∀b′ ∈ b)(∃c′ ∈ c)(z[b′,c′]) & (∀c′ ∈ c)(∃b′ ∈ b)(z[b′,c′]). Hence z[b,c]. Therefore E(b,c). □

We now define x ∈* y if and only if (∃z ∈ y)(E(x,z)).

LEMMA 14. The following is provable in S. E is reflexive, symmetric, and transitive. For all b,c, (∀x)(x ∈* b ↔ x ∈* c) ↔ E(b,c). For all b,c, (∀x)(x ∈* b ↔ x ∈* c) → (∀x)(b ∈* x ↔ c ∈* x).

Proof: E is reflexive and symmetric immediately by Lemma 13. For transitivity, let E(b,c) and E(c,d). Then ((∀b′ ∈ b)(∃c′ ∈ c)(E(b′,c′)) & (∀c′ ∈ c)(∃b′ ∈ b)(E(b′,c′)) & ((∀c′ ∈ c)(∃d′ ∈ d)(E(c′,d′)) & (∀d′ ∈ d)(∃c′ ∈ c)(E(c′,d′)). Hence ((∀b′ ∈ b)(∃d′ ∈ d)(E(b′,d′)) & (∀d′ ∈ d)(∃b′ ∈ b)(E(b′,d′)).

For the second claim, first suppose that (∀x)(x ∈* b ↔ x ∈* c). Let b′ ∈ b. Then b′ ∈* b, and so b′ ∈* c. Hence there exists c′ ∈ c with E(b′,c′). Thus (∀b′ ∈ b)(∃c′ ∈ c)(E(b′,c′)). By symmetry, (∀c′ ∈ c)(∃b′ ∈ b)(E(b′,c′)). Hence by Lemma 13, E(b,c).

Now suppose E(b,c). Let x ∈* b. Let E(x,b′), b′ ∈ b. Let c′ ∈ c, E(b′,c′) by Lemma 13. Then E(x,c′) by the first claim. Therefore x ∈* c. Thus we have established x ∈* b → x ∈* c. By symmetry, also x ∈* c → x ∈* b.

For the third claim, we need only show E(b,c) → (∀x)(b ∈* x ↔ c ∈* x). This is obvious using the first claim. □

We now show that ∈* provides an interpretation of K(W) in S.

LEMMA 15. The result of replacing every occurrence of ∈ by ∈* in every axiom of K(W) is provable in S.

Proof: By Lemma 14, this is clear for Extensionality. For SS, let $x \in^* W$ and φ be given. Let $x' \in W$, $E(x,x')$. Let $y \in W$ be such that $(\forall z)(z \in y \leftrightarrow (z \in x' \ \& \ \varphi^*(u_1,\ldots,u_n,z))$. Let z be given. Suppose first that $z \in^* y$. Let $E(z,z')$, $z' \in y$. Then $z' \in x' \ \& \ \varphi^*(u_1,\ldots,u_n,z)$. By Lemma 14, $z' \in^* x \ \& \ \varphi^*(u_1,\ldots,u_n,z)$. Hence $z \in^* x \ \& \ \varphi^*(u_1,\ldots,u_n,z)$. Now suppose that $z \in^* x' \ \&$ $\varphi^*(u_1,\ldots,u_n,z)$. Let $E(z,z')$, $z' \in x'$. Hence $\varphi^*(u_1,\ldots,u_n,z')$. Therefore $z' \in y$. Hence $z \in^* y$.

For RED, let $x_1,\ldots,x_k \in^* W$ and $\varphi^*(x_1,\ldots,x_k,y)$. Let $x_1',\ldots,x_k' \in W$ be such that each $E(x_i,x_i')$. Then $\varphi^*(x_1',\ldots,x_k',y)$. Let $y \in W$ be such that $\varphi^*(x_1',\ldots,x_k',y)$. Then $\varphi(x_1,\ldots,x_k,y)$. $\square$

Note that $\in^*$ provides us with a special kind of interpretation, described as follows. We have an equivalence relation, E, on all sets. We can then define $x \in^* y$ if and only if $(\exists x',y')(E(x,x') \ \& \ E(y,y') \ \& \ x' \in y')$. By Lemma 14, this is equivalent to the official definition of $\in^*$. This is called an interpretation by equivalence relation.

THEOREM. There is an interpretation by equivalence relation of K(W) in S.