THE UPPER SHIFT KERNEL THEOREMS

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October 9, 2010
DRAFT

1. INFINITE UPPER SHIFT KERNEL THEOREM.

Here we sketch a proof of the Infinite Upper Shift Kernel Theorem from a suitable large cardinal assumption. The reversal will be available later.

A digraph is a pair G = (V,E), where V is a nonempty set of vertices and $E \subseteq V^2$ is a set of edges. We say that G is on V.

A kernel in (V,E) is commonly defined in graph theory as a set $S \subseteq V$ such that

i. No element of S connects to any element of S.
ii. Every element of V\S connects to some element of S.

We now fix $A \subseteq Q$. We study a fundamental class of digraphs associated with A, which we call the A-digraphs. An A,k-digraph is a digraph (A^k,E), where E is an order invariant subset of A^{2k} in the following sense. For all $x,y \in A^{2k}$, if x,y have the same order type then $x \in E \leftrightarrow y \in E$.

An A-digraph is an A,k-digraph for some $k \geq 1$.

Note that for each $A \subseteq Q$, there are finitely many A,k-digraphs.

A downward A-digraph (A,k-digraph) is an A-digraph (A,k-digraph) where for every edge (x,y), max(x) > max(y).

The upper shift ush:Q → Q is defined by ush(q) = q+1 if $q \geq 0$; q otherwise. This lifts to $ush:Q^k \to Q^k$ by acting coordinatewise. We can now use ush to give forward images of subsets of Q^k.

SRP stands for "stationary Ramsey property". We say that λ has the k-SRP if and only if λ is an infinite cardinal, and every $f:[\lambda]^k \to 2$ is constant on some $[S]^k$, where S is a stationary subset of λ. Here $[S]^k$ is the set of all unordered k tuples from S.

INFINITE UPPER SHIFT KERNEL THEOREM. There exists $0 \in A \subseteq Q$ such that every downward A-digraph has a kernel containing its upper shift.

Note that we are talking about Con(SRP), rather than 1-Con(SRP).

Also, $A \subseteq Q$ can be taken to be recursive in the jump, as well as the sequence of kernels for the A-digraphs.

INFINITE UPPER SHIFT KERNEL THEOREM(k). There exists $0 \in A \subseteq Q$ such that every downward A,k-digraph has a kernel containing its upper shift.

There is a small k such that the Infinite Upper Shift Kernel Theorem(k) also has the same metamathematical properties. I.e., is also provably equivalent, over ACA_0, to Con(SRP). Thus we do NOT get a hierarchy on the dimension k.

It remains to give a small k. I have been postponing this kind of investigation for some time, waiting for the independent statements to stabilize.

We will now prove the Infinite Upper Shift Kernel Theorem in ACA_0 + Con(SRP). Here SRP = ZFC + $\{$there exists λ with the k-SRP$\}_k$.

We expect to use the ideas and techniques of the BRT book to show that the Infinite Upper Shift Kernel Theorem, and the Infinite Upper Shift Kernel Theorem(k) for small k, are provably equivalent to Con(SRP) over ACA_0.

LEMMA 1. Let λ be least with the 2k+1-SRP. Let $f:[\lambda]^k \to \lambda$ obey $\min(A) > 0 \to f(A) < \min(A)$. There exists stationary $S \subseteq \lambda$ such that f is constant on $[S]^k$. (This can be improved with k+1 instead of 2k+1).

Proof: Let λ,f be as given. Then λ is an uncountable regular cardinal. Let $g:[\lambda]^{k+1} \to 2$ be defined as follows. Let $\alpha_1 < \ldots < \alpha_{2k+1}$. Set $g(\alpha_1,\ldots,\alpha_{k+1}) = 0$ if $f(\alpha_1,\ldots,\alpha_k) = f(\alpha_1,\alpha_{k+1},\ldots,\alpha_{2k})$; 1 otherwise. Let $E \subseteq \lambda\backslash\{0\}$ be stationary, where g is constant on $[E]^{2k+1}$.

case 1. g is constantly 0. Then the values of f(A), $A \in [S]^k$, depend only on min(A). Let $h:S \to \lambda$ be given accordingly. Then h is regressive, and so h is constant on some stationary subset of E. Then f is constant on $[E]^k$.

case 2. f is constantly 1. Let $\alpha \in E$. By a cardinality argument, there exists $\alpha < \beta_1 < \ldots < \beta_{2k-2}$ such that $f(\alpha,\beta_1,\ldots,\beta_{k-1}) = f(\alpha,\beta_k,\ldots,\beta_s)$. This is a contradiction.

QED

LEMMA 2. Let $k \geq 1$. SRP proves the existence of an uncountable regular cardinal λ and a stationary subset S of λ with the following indiscernibility condition. Let $\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_k < \lambda$, where $(\alpha_1,\ldots,\alpha_k)$ and $(\beta_1,\ldots,\beta_k)$ are elements of S^k with the same order type, and $\gamma_1,\ldots,\gamma_k < \min(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$. Let φ be a formula in $\in,=$ with only the free variables $v_1,\ldots,v_{3k}$. Then in $L(\lambda)$, $\varphi(\alpha_1,\ldots,\alpha_k,\gamma_1,\ldots,\gamma_k) \leftrightarrow \varphi(\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_k)$.

LEMMA 3. ACA_0 + Con(SRP) proves the existence of a countable model M of ZFC + V = L (perhaps with nonstandard integers), and an unbounded set S of ordinals of M, with the following indiscernibility condition. Let $k,n \geq 1$ and $\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_n$ be ordinals of M, where $(\alpha_1,\ldots,\alpha_k)$ and $(\beta_1,\ldots,\beta_k)$ are elements of S^k with the same order type, and $\gamma_1,\ldots,\gamma_n < \min(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$. Let φ be a formula in $\in,=$ with only the free variables $v_1,\ldots,v_{k+n}$. Then $\varphi(\alpha_1,\ldots,\alpha_k,\gamma_1,\ldots,\gamma_n) \leftrightarrow \varphi(\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_n)$ holds in M. Furthermore, for all $\alpha < \beta$ from S, $L(\alpha)$ is an elementary submodel of $L(\beta)$.

We now work within ACA_0 + Con(SRP), and let M,S be as given by Lemma 3.

LEMMA 4. Let $\alpha_0,\alpha_1,\ldots$ be the first ω elements of S, with limit δ. Then the $L(\delta)$ of M satisfies ZFC, and the same indiscernibility condition as in Lemma 3 holds for $\alpha_0,\alpha_1,\ldots$ in $L(\delta)$. Also, each $L(\alpha_i)$ is an elementary submodel of $L(\alpha_{i+1})$.

LEMMA 5. There is a countable model M of ZFC + V = L, with indiscernibles $\alpha_0 < \alpha_1 < \ldots$ as in Lemma 3, which are unbounded in M, and generate M in the sense that every element of M is definable over M from finitely many α's.

Also, each $L(\alpha_i)$ is an elementary submodel of $L(\alpha_{i+1})$.

LEMMA 6. There is a countable model M of ZFC + V = L, and an elementary embedding $j:M \to M$, with a critical point α, where for all $\beta \geq \alpha$, we have $\beta < j(\beta) < j(j(\beta)) < \ldots$ is unbounded in M.

Let $D = \mathrm{dom}(M) \cap \mathrm{On}$.

LEMMA 7. The unique internal M kernel of every downward D-digraph contains its image under j.

LEMMA 8. $(D,<,j,\alpha)$ is isomorphic to $(A,<,\mathrm{ush},0)$ for some $0 \in A \subseteq Q$, where ush is the upper shift.

We will prove Lemma 8 below. But it is now clear A is as required by the Infinite Upper Shift Kernel Theorem. We have thus proved the Infinite Upper Shift Kernel Theorem in ACA_0 + Con(SRP).

A thread is a set $\{\beta,j(\beta),j(j(\beta)),\ldots\}$, where $\beta \notin \mathrm{rng}(j)$. Clearly $\beta \geq \alpha$, and $\beta < j(\beta) < \ldots$ is unbounded in $M \cap \mathrm{On}$. We say that this thread is generated by β.

Obviously, D is made up of $\beta < \alpha$, and infinitely many distinct threads $\sigma_1,\sigma_2,\ldots$. We will assume that σ_1 is the thread generated by α.

By ordinary induction, for each $k \geq 0$, we define a function f_k from $\sigma_1 \cup \ldots \cup \sigma_k$ into Q which is order preserving, maps α to 0, and is the successor function, +1, along each thread.

Start by defining f_1 on thread σ_1 by setting $f_1(j^p(\alpha)) = p$.

Suppose f_k has been defined, $k \geq 1$. Clearly σ_{k+1} is disjoint from $\mathrm{dom}(f_k)$. Let α_{k+1} be generated by β. Clearly $\beta > \alpha$.

Fix $p \geq 0$ such that the threads $\sigma_2,\ldots,\sigma_{k+1}$ each meet $(j^p(\alpha),j^{p+1}(\alpha))$. They each have exactly one element in this open interval. Let these k ordinals of M be the γ's in the inequality chain

1) $j^p(\alpha) < \gamma_1 < \ldots < \gamma_r < \beta^* < \gamma_{r+2} < \ldots < \gamma_k < j^{p+1}(\alpha)$

where β^* is in the thread generated by β. Clearly $p < f(\gamma_1) < \ldots < f(\gamma_k) < p+1$. If we replace p with p+t, $t \geq 0$, then we get the inequality chain

$$2)\ j^{p+t}(\alpha) < j^t(\gamma_1) < \ldots < j^t(\gamma_r) < j^t(\beta^*) < j^t(\gamma_{r+2}) < \ldots < j^t(\gamma_k) < j^{p+t+1}(\alpha).$$

If we lower p to p+t, $-p \leq t \leq -1$, then we also get the same inequality chain, 2), but with perhaps one or more terms missing.

Every element of $\sigma_2 \cup \ldots \cup \sigma_k$ appears exactly once in exactly one of these inequality chains, but not at the endpoints; the elements of σ_1 appear at the endpoints.

Now set f_{k+1} to extend f_k by defining $f_{k+1}(\beta^*)$ so that

$$p < f_k(\gamma_1) < \ldots < f_k(\gamma_r) < f_{k+1}(\beta^*) < f_k(\gamma_{r+2}) < \ldots < f_k(\gamma_k) < p+1.$$

I.e., so that the inequality chain 1) is preserved under f. This also defines f_{k+1} on the entire thread generated by β, using shift, +1. Obviously, all of the inequality chains for p = 0,1,..., must be preserved under f by the induction hypothesis on f_k. Take $g = \cup_k f_k$, and extend g in any way so that the ordinals of M strictly below α are sent to negative rationals in an order preserving way. QED

2. FINITE UPPER SHIFT KERNEL THEOREM.

There are general principles that allow us to look at the form of the
Infinite Upper Shift Kernel Theorem(k) and see that it must be provably
equivalent to a Π^0_1 sentence over ACA_0.This is more awkward for the Infinite Upper Shift Kernel Theorem.

However, we also want to find a simple explicitly Π^0_1 form of the Infinite Upper Shift Kernel Theorem or the Infinite Upper Shift Kernel Theorem(k).

We have settled on a quantitative approach. We are also considering non quantitative approaches, but they haven't yet jelled into something suitably simple.

We use convenient norms on $Q^{\wedge k}$. Let $|q|$, $q \in Q$, be the least integer n such that q can be written as a ratio of two integers of magnitude at most n. Let $|x|$, $x \in Q^k$, be $\max(|x_1|,\ldots,|x_k|)$.

The n-upper shift of $B \subseteq Q^k$ consists of the upper shifts of the elements of B of norm at most n.

Let (B,E), $B \subseteq Q^k$, be a digraph. An n-kernel in (B,E) is a set $S \subseteq B$ such that

i. No element of S connects to any element of S.
ii. Every x in V\S of norm at most n connects to some element of S of norm at most $|x|^k + k$.

FINITE UPPER SHIFT KERNEL THEOREM. For each $n > k > 1$, there exists finite $0 \in A \subseteq Q$ such that every downward A,k-digraph has an n-kernel containing its n-upper shift. Furthermore, we can require that every element of A have norm at most n^k+1.

Note that the Finite Upper Shift Kernel Theorem is explicitly Π^0_1.

The Finite Upper Shift Kernel Theorem is proved from the Infinite Upper Shift Kernel Theorem by starting with $0 \in A \subseteq Q$ from the Infinite Upper Shift Kernel Theorem, choosing the provided kernels, and building a tower of simultaneous approximations for A and the kernels. Since the conditions in question are purely order theoretic, we can move the elements of A around to conform to the norm ||.

For any fixed $k > 1$, we can use a compactness argument to go from the Finite Upper Shift Kernel Theorem, to the Infinite Upper Shift Kernel Theorem(k).

In this way, we see that the Finite Upper Shift Kernel Theorem is also equivalent to Con(SRP) over ACA_0. We expect to be able to replace ACA_0 here by EFA = exponential function arithmetic.

3. TEMPLATES.

The obvious item to Template is the upper shift. Recall that we first defined the upper shift as a function ush:$Q \to Q$, and then lifted it to higher dimensions, and then used it for forward images.

Ush is a special case of a rational pieceise, or even rational partial piecewise linear function from Q into Q.

TEMPLATE. Let $f_1,...,f_n:Q \to Q$ be partial rational piecewise linear functions. There exists $0 \in A \subseteq Q$ such that every downward A-digraph has a kernel containing its images under $f_1,...,f_n$.

CONJECTURE. Every instance of the above Template is provable or refutable in SRP^+ = ZFC + "for all k there exists λ with the k-SRUP".

This should be within our existing technology.

* This research was partially supported by Templeton Grant ID#15557.