ADVENTURES IN LOGIC FOR UNDERGRADUATES

by
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Lecture 2. Logical Quantifiers

LECTURE 1. LOGICAL CONNECTIVES. Jan. 18, 2011

LECTURE 2. LOGICAL QUANTIFIERS. Jan. 25, 2011

LECTURE 3. TURING MACHINES. Feb. 1, 2011

LECTURE 4. GÖDEL'S BLESSING AND GÖDEL'S CURSE. Feb. 8, 2011

LECTURE 5. FOUNDATIONS OF MATHEMATICS Feb. 15, 2011

SAME TIME - 10:30AM

SAME ROOM - Room 355 Jennings Hall

WARNING: CHALLENGES RANGE FROM EASY, TO MAJOR PARTS OF COURSES

PROPOSITIONAL CALCULUS - PREDICATE CALCULUS

In the first lecture, we worked within the framework of propositional calculus, based on

i. Statement letters $A_1, A_2, \ldots$.

ii. Connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

In this talk, we present the far richer framework called *predicate calculus*.

We will approach the full predicate calculus in small bite sized stages, which are important in their own right.

Propositional Calculus is NOT powerful enough to support the logic behind mathematics. Predicate Calculus IS.

We will have some SPECIAL FUN at the end of the talk.

VARIABLES AND EQUALITY

In this talk, we will scrap the sentential letters $A_1, A_2, \ldots$. But we will retain the five connectives

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow.$$

from Lecture 1. We now introduce the variables $v_1, v_2, \ldots$, and the equality symbol =. That's it. A very gentle first step.

The *variable equations* take the form $v_i = v_j$.

The *variable formulas* are built up from the variable equations using the give logical connectives. Examples of variable formulas: $v_3 = v_3$ $\qquad$ $\neg(v_3 = v_4) \vee v_3 = v_4$

$v_3 = v_4 \rightarrow v_4 = v_3$ $\qquad$ $(v_2 = v_7 \wedge v_9 = v_1) \leftrightarrow \neg(v_8 = v_2)$

Everybody likes to write $v_i \neq v_j$ instead of $\neg(v_i = v_j)$.

VARIABLE FORMULA LOGIC

In Variable Formula Logic, we use only variable formulas. To *interpret* the variable formulas, we must designate a nonempty domain D. The variables are to range over the elements of D.

Variable formulas will be true or false in (D,=) depending on which objects from D are assigned to which variables. We use the terminology: D assignments.

We say that a variable formula is *true* in (D,=) if and only if it is true in (D,=) under ALL D assignments.

Note that among our example variable formulas,
$v_3 = v_3$, $v_3 \neq v_4 \vee v_3 = v_4$, $v_3 = v_4 \rightarrow v_4 = v_3$ **are each true in (D,=) for all D assignments in (D,=).**

Hence each of these three formulas is true in (D,=).

VARIABLE FORMULA LOGIC

More holds of these three examples of variable formulas:

$v_3 = v_3$, $v_3 \neq v_4 \lor v_3 = v_4$, $v_3 = v_4 \rightarrow v_4 = v_3$

are all true in ALL (D,=).

Such a variable formula is said to be *valid*.

THEOREM. If a variable formula is true in (Z,=) then it is valid. This claim is false for any (D,=), D finite.

CHALLENGE: Prove the above Theorem.

Can all valid variable formulas be "proved"?

To address this, we first apply the notion of tautology from Lecture 1 to this context.

VALID VARIABLE FORMULAS - AXIOMATIZATION

Consider $v_5 = v_2 \vee v_5 \neq v_2$, $v_3 = v_4 \rightarrow v_3 = v_4$. These become propositional formulas, if we view $v_5 = v_2$ and $v_3 = v_4$ as propositional letters. (Recall $v_5 \neq v_2$ abbreviates $\neg(v_5 = v_2)$).

These two examples are *tautologies*. They are true for any truth assignment to the letters; i.e., to $v_5 = v_2$ and $v_3 = v_4$.

AXIOMS AND RULES

$x = x$. $x = y \rightarrow y = x$. $(x = y \wedge y = z) \rightarrow x = z$. (Any variables).
All tautologies. (All variable formulas that are tautologies).
Modus Ponens. (From α, $\alpha \rightarrow \beta$, derive β).

CHALLENGE: Prove that these axioms and rules generate only valid variable formulas.

CHALLENGE: Prove that these axioms and rules generate all of the valid variable formulas.

EQUATIONAL LOGIC

So far, our equations only involve variables. Mathematics needs functions on a domain. E.g., +,• on Z (the integers).

The general setup for *Equational Logic* is to use these SYMBOLS:

i. Variables $v_1, v_2, \ldots$.

ii. Constant symbols $c_1, c_2, \ldots$.

iii. Function symbols $F^n_1, F^n_2, \ldots$, where $n \geq 1$.

Here F^n_m is an n-place (or n-ary) function symbol. Typically, we avoid writing subscripts and superscripts.

iv. Equality. =.

v. NO CONNECTIVES! (a little step backwards for us!!).

The terms are formed by combining the above. E.g.,

$$v \quad c \quad F(v) \quad F(c) \quad F(F(w)) \quad F(F(d))$$

$$H(x,c) \quad H(J(y,e),G(K(z)))$$

Indicate that v,x,y,z,w are variables, and c,d,e are constants.

EQUATIONAL LOGIC

Equational Logic (EL) is an important branch of logic involving term equations only, and WITHOUT our connectives ¬,∧,∨,→,↔. The power of EL comes from using *proper axioms*.

In EL, typically we use only some constant and function symbols. Infix notation for 2-ary function symbols is common.

An important example of proper axioms in EL are the abelian group axioms. These are based on the symbols 0,+,-. Here 0 is a constant symbol, + is a 2-ary function symbol, and - is a 1-ary function symbol. The abelian group axioms are

$$x+y = y+x \quad x+(y+z) = (x+y)+z \quad x+0 = x \quad x+(-x) = 0$$

In fact, an *abelian group* is just a MODEL of the above proper axioms.

EQUATIONAL LOGIC

$x+y = y+x \quad x+(y+z) = (x+y)+z \quad x+0 = x \quad x+(-x) = 0$

In EL, a model of these proper axioms is a quadruple (D, 0,+,-), where D is a nonempty set, $0 \in D$, $+:D^2 \to D$, $-:D \to D$, such that these proper axioms are true in (D,0,+,-). I.e., they are true for ALL assignments of elements of D to the variables.

In this way, the *models* of these proper axioms are the abelian groups (D,0,+,-).

Note that --x = x is a CONSEQUENCE of these proper axioms, by the following "proof" (where various tiny steps are omitted).

$-x+(--x) = 0$
$x+(-x+(--x)) = x+0 = x$
$x+(-x+(--x)) = (x+(-x))+(--x)) = 0+(--x) = --x.$
$\therefore --x = x.$

AXIOMATIZATION OF EQUATIONAL LOGIC

$x+y = y+x \quad x+(y+z) = (x+y)+z \quad x+0 = x \quad x+(-x) = 0$

$-x+(--x) = 0$
$x+(-x+(--x)) = x+0 = x$
$x+(-x+(--x)) = (x+(-x))+(--x)) = 0+(--x) = --x.$
$\therefore\ --x = x.$

AXIOMS AND RULES FOR EL USING THE PROPER AXIOM SET K

AXIOMS. Any element of K.
AXIOMS. x = x, where x is any variable.
RULE. From s = t, p = q, derive p′ = q′, where p′ = q′ results from p = q by replacing one or more occurrences of s by t.
RULE. From s = t, derive s* = t*, where variables are replaced by terms that use only the chosen symbols. It is crucial that the same variables be replaced by the same terms.

CHALLENGE: Show that these axioms and rules support the above proof.

COMPLETENESS OF EQUATIONAL LOGIC

AXIOMS. Any element of K.
AXIOMS. x = x, where x is any variable.
RULE. From s = t, p = q, derive p′ = q′, where p′ = q′ results from p = q by replacing one or more occurrences of s by t.
RULE. From s = t, derive s* = t*, where variables are replaced by terms that use only the chosen symbols. It is crucial that the same variables be replaced by the same terms.

We say that K *proves* s = t iff s = t can be derived from K according to the above axioms and rules of inference.

We say that K *logically implies* s = t iff s = t is true in every model in which all equations in K are true.

CHALLENGE: Show: if K proves s = t then K logically implies s = t.

CHALLENGE: Show: if K logically implies s = t then K proves s = t.

QUANTIFIER FREE LOGIC

We have discussed
i. Variable Formula Logic. Only variables, =, and connectives. No constant or function symbols.
ii. Equational Logic. Only variables, =, constant symbols, and function symbols. No connectives.

Quantifier Free Logic (QFL) combines both, and also adds relation symbols. Thus QFL has

i. Variables $v_1, v_2, \ldots$.
ii. Constant symbols $c_1, c_2, \ldots$.
iii. Equality. =.
iv. Function symbols F^n_m, $n,m \geq 1$. These are n-ary.
v. Relation symbols. R^n_m, $n,m \geq 1$. These are n-ary.

vi. Connectives. $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

We have terms as in equational logic (EL), using just i,ii,iv.

QUANTIFIER FREE LOGIC

i. Variables $v_1, v_2, \ldots$.
ii. Constant symbols $c_1, c_2, \ldots$.
iii. Equality. =.
iv. Function symbols F^n_m, $n,m \geq 1$. These are n-ary.
v. Relation symbols. R^n_m, $n,m \geq 1$. These are n-ary.
vi. Connectives.
We have terms as in EL, using just i,ii,iv.

The atomic formulas of QFL are of two forms:

a. s = t, where s,t are terms.
b. $R^n_m(t_1,\ldots,t_n)$, where $t_1,\ldots,t_n$ are terms.

The QFL formulas result from the atomic formulas by using the connectives. We avoid using subscripts and superscripts. E.g.,

$$F(x,y) = G(x,y,H(x,c)) \rightarrow (\neg R(F(z,c),d,G(w,x,w)) \wedge S(e,F(c,c))).$$

QUANTIFIER FREE LOGIC

The QFL models take the form $(D,=,d_n,f^n_m,r^n_m)$, where D is a nonempty set (the domain), c_n is interpreted by the element d_n of D, F^n_m is interpreted by the function $f^n_m:D^n \to D$, and R^n_m is interpreted by the relation $r^n_m \subseteq D^n$. In particular applications, only certain constant, function, and relation symbols are actually used.

As in EL, we have the *tautologies*, by treating the atomic formulas as letters. QFL has a lot more atomic formulas than EL.

We retain the crucial feature of EL - treating proper axioms. As in EL these are interpreted universally.

EXAMPLE OF PROPER AXIOMS IN QFL
THE ORDERED ABELIAN GROUP AXIOMS

The ordered abelian groups play an important role in mathematics. They are MODELS of these proper axioms (ordered abelian group axioms), using the symbols 0,+,-,< (where < is a 2-ary relation symbol):

$$x+y = y+x \quad x+(y+z) = (x+y)+z \quad x+0 = x \quad x+(-x) = 0$$

$$\neg x < x \quad (x < y \land y < z) \rightarrow x < z \quad x < y \lor y < x \lor x = y$$

$$x < y \rightarrow x+z < y+z$$

The models of this set of proper axioms - the ordered abelian group axioms - are just the ordered abelian groups (D,=, 0,+,-,<).

SAMPLE "PROOF" FROM THE ORDERED ABELIAN GROUP AXIOMS IN QFL

In every ordered abelian group, $x < y \rightarrow -y < -x$. Here is one of many ways to argue this (where various tiny steps are omitted).

$$x < y \rightarrow x+(-y) < y+(-y)$$
$$x+(-y) < y+(-y) \rightarrow x+(-y) < 0$$
$$x+(-y) < 0 \rightarrow (x+(-y))+(-x) < 0+(-x)$$
$$(x+(-y))+(-x) < 0+(-x) \rightarrow -y < -x$$
$$x < y \rightarrow -y < -x$$

What is a proof in quantifier free logic (QFL)?

AXIOMATIZATION OF QUANTIFIER FREE LOGIC WITH PROPER AXIOM SET K

$x < y \rightarrow x+(-y) < y+(-y)$
$x+(-y) < y+(-y) \rightarrow x+(-y) < 0$
$x+(-y) < 0 \rightarrow (x+(-y))+(-x) < 0+(-x)$
$(x+(-y))+(-x) < 0+(-x) \rightarrow -y < -x$
$x < y \rightarrow -y < -x$

AXIOMS. Any element of K.

AXIOMS. x = x, where x is a variable.

AXIOMS. $(x = y \wedge \alpha) \rightarrow \alpha'$, where α' is the result of replacing one or more occurrences of the variable x by the variable y in the QFL atomic formula α.

AXIOMS. All tautologies.

RULE. Substitution. From α, derive α^*, where α^* results from replacing variables by terms in the QFL formula α. It is crucial that the same variables be replaced by the same terms.

RULE. Modus Ponens. From α, $\alpha \rightarrow \beta$, derive β.

CHALLENGE: Show that these axioms and rules support the above proof.

COMPLETENESS OF QUANTIFIER FREE LOGIC

AXIOMS. Any element of K.
AXIOMS. x = x, where x is a variable.
AXIOMS. $(x = y \wedge \alpha) \rightarrow \alpha'$, where α' is the result of replacing one or more occurrences of the variable x by the variable y in the QFL atomic formula α.
AXIOMS. All tautologies.
RULE. Substitution. From α, derive α*, where α* results from replacing variables by terms in the QFL formula α. It is crucial that the same variables be replaced by the same terms.
RULE. Modus Ponens. From α, $\alpha \rightarrow \beta$, derive β.

We say that K *proves* α iff α can be derived from K according to the above axioms and rules of inference.

We say that K *logically implies* α iff α is true every model in which all formulas in K are true.

CHALLENGE: Show: if K proves α then K logically implies α.

CHALLENGE: Show: if K logically implies α then K proves α.

FULL PREDICATE CALCULUS

i. Variable Formula Logic. Only variables, =, and connectives. No constant or function symbols.
ii. Equational Logic. Only variables, =, constant symbols, and function symbols. No connectives.
iii. Quantifier Free Logic. Only variables, =, constant symbols, function symbols, relations symbols, connectives.

Predicate Calculus (PC) extends Quantifier Free Logic by adding QUANTIFIERS. Thus QFL has

i. Variables $v_1,v_2,...$.
ii. Constant symbols $c_1,c_2,...$.
iii. Equality. =.
iv. Function symbols F^n_m, n,m ≥ 1. These are n-ary.
v. Relation symbols. R^n_m, n,m ≥ 1. These are n-ary.
vi. Connectives.
vii. Quantifiers. ∀,∃. For all, and there exists.

FULL PREDICATE CALCULUS

i. Variables $v_1, v_2, \ldots$.
ii. Constant symbols $c_1, c_2, \ldots$.
iii. Equality. =.
iv. Function symbols F^n_m, $n,m \geq 1$. These are n-ary.
v. Relation symbols. R^n_m, $n,m \geq 1$. These are n-ary.
vi. Connectives.
vii. Quantifiers. $\forall, \exists$.

PC formulas are obtained by combining i-vii. E.g.,

$$(\forall x)(\neg R(x,y) \rightarrow (\exists z)(F(z,H(y),c) = d \vee \neg S(z,c)))$$

is a PC formula. PC borrows a lot from the more primitive QFL.

Just as in QFL, we will use sets of axioms K, and also the same models $(D,=,d_n,f^n_m,r^n_m)$, where D is a nonempty set (the domain), c_n is interpreted by the element d_n of D, F^n_m is interpreted by the function $f^n_m:D^n \rightarrow D$, and R^n_m is interpreted by the relation $r^n_m \subseteq D^n$.

FULL PREDICATE CALCULUS

i. Variables $v_1, v_2, \ldots$.
ii. Constant symbols $c_1, c_2, \ldots$.
iii. Equality. =.
iv. Function symbols F^n_m, $n,m \geq 1$. These are n-ary.
v. Relation symbols. R^n_m, $n,m \geq 1$. These are n-ary.
vi. Connectives.
vii. Quantifiers. $\forall, \exists$.
Models $(D,=,d_n,f^n_m,r^n_m)$, D nonempty.

Even if we have no proper axioms (e.g., the ordered abelian group axioms), PC is rather subtle and powerful. We say that a PC formula is valid iff it holds in every model under every variable assignment (mapping from variables into the domain D).

Let K be a set of proper axioms in PC, and α be a PC formula. We say that K logically implies α iff α is true in every PC model in which all elements of K are true.

There are nice complete axiomatizations of PC in the spirit of the ones we saw for QFL. A bit too much for one lecture!

HAVING FUN WITH PREDICATE CALCULUS

Everybody loves everybody.
Somebody loves somebody.
Everybody is loved by everybody.
Somebody is loved by somebody.

Everybody loves somebody.
Somebody loves everybody.
Everybody is loved by somebody.
Somebody is loved by everybody.

Let's transcribe these eight into PC, and work out their logical relationships.

We use L(x,y) for "x loves y". Note that "x is loved by y" is expressed by L(y,x).

HAVING FUN WITH PREDICATE CALCULUS

1. Everybody loves everybody. $(\forall x)(\forall y)(L(x,y))$.
2. Somebody loves somebody. $(\exists x)(\exists y)(L(x,y))$.
3. Everybody is loved by everybody. $(\forall x)(\forall y)(L(y,x))$.
4. Somebody is loved by somebody. $(\exists x)(\exists y)(L(y,x))$.

5. Everybody loves somebody. $(\forall x)(\exists y)(L(x,y))$.
6. Somebody loves everybody. $(\exists x)(\forall y)(L(x,y))$.
7. Everybody is loved by somebody. $(\forall x)(\exists y)(L(y,x))$.
8. Somebody is loved by everybody. $(\exists x)(\forall y)(L(y,x))$.

The models are (D,R), where D is a nonempty set, and R is a binary relation on D. Quantifiers range over D, and L is interpreted as R.

CHALLENGE: 1,3 have the same models. 2,4 have the same models.
CHALLENGE: Every model of 1 (or of 3) is a model of 1-8.
CHALLENGE. Every model of any of 1-8 is a model of 2,4.

HAVING FUN WITH PREDICATE CALCULUS

1. Everybody loves everybody. $(\forall x)(\forall y)(L(x,y))$.
2. Somebody loves somebody. $(\exists x)(\exists y)(L(x,y))$.
3. Everybody is loved by everybody. $(\forall x)(\forall y)(L(y,x))$.
4. Somebody is loved by somebody. $(\exists x)(\exists y)(L(y,x))$.
5. Everybody loves somebody. $(\forall x)(\exists y)(L(x,y))$.
6. Somebody loves everybody. $(\exists x)(\forall y)(L(x,y))$.
7. Everybody is loved by somebody. $(\forall x)(\exists y)(L(y,x))$.
8. Somebody is loved by everybody. $(\exists x)(\forall y)(L(y,x))$.

How do we show that one of these does not logically imply a second? Give a model of the first that is not a model of the second.

EXAMPLE: 2) does not logically imply 1). A model of 2 is ({b,c}, {(b,b)}). This is not a model of 1) since, here, b does not love c.

CHALLENGE: {2,4,5,6,7,8} does not logically imply 1).

CHALLENGE: 2) does not logically imply $1 \vee 3 \vee 5 \vee 6 \vee 7 \vee 8$.

HAVING FUN WITH PREDICATE CALCULUS

1. **Everybody loves everybody.** $(\forall x)(\forall y)(L(x,y))$.
2. **Somebody loves somebody.** $(\exists x)(\exists y)(L(x,y))$.
3. **Everybody is loved by everybody.** $(\forall x)(\forall y)(L(y,x))$.
4. **Somebody is loved by somebody.** $(\exists x)(\exists y)(L(y,x))$.
5. **Everybody loves somebody.** $(\forall x)(\exists y)(L(x,y))$.
6. **Somebody loves everybody.** $(\exists x)(\forall y)(L(x,y))$.
7. **Everybody is loved by somebody.** $(\forall x)(\exists y)(L(y,x))$.
8. **Somebody is loved by everybody.** $(\exists x)(\forall y)(L(y,x))$.

CHALLENGE: Determine which subsets of 1-8 logically imply which entries in 1-8.

CHALLENGE: Determine which subsets of 1-8 logically imply the disjunction of which subsets of 1-8.

MORE FUN WITH PREDICATE CALCULUS

1. Everybody loves everybody who loves everybody.
2. Everybody loves everybody who loves somebody.
3. Everybody loves somebody who loves everybody.
4. Everybody loves somebody who loves somebody.
5. Somebody loves everybody who loves everybody.
6. Somebody loves everybody who loves somebody.
7. Somebody loves somebody who loves everybody.
8. Somebody loves somebody who loves somebody.

1. $(\forall x)(\forall y)((\forall z)(L(y,z)) \rightarrow L(x,y))$.
2. $(\forall x)(\forall y)((\exists z)(L(y,z)) \rightarrow L(x,y))$.
3. $(\forall x)(\exists y)((\forall z)(L(y,z) \land L(x,y)))$.
4. $(\forall x)(\exists y)((\exists z)(L(y,z) \land L(x,y)))$.
5. $(\exists x)(\forall y)((\forall z)(L(y,z)) \rightarrow L(x,y))$.
6. $(\exists x)(\forall y)((\exists z)(L(y,z)) \rightarrow L(x,y))$.
7. $(\exists x)(\exists y)((\forall z)(L(y,z)) \land L(x,y))$.
8. $(\exists x)(\exists y)((\exists z)(L(y,z)) \land L(x,y))$.

MORE FUN WITH PREDICATE CALCULUS

1. $(\forall x)(\forall y)((\forall z)(L(y,z)) \rightarrow L(x,y))$.
2. $(\forall x)(\forall y)((\exists z)(L(y,z)) \rightarrow L(x,y))$.
3. $(\forall x)(\exists y)((\forall z)(L(y,z) \land L(x,y)))$.
4. $(\forall x)(\exists y)((\exists z)(L(y,z) \land L(x,y)))$.
5. $(\exists x)(\forall y)((\forall z)(L(y,z)) \rightarrow L(x,y))$.
6. $(\exists x)(\forall y)((\exists z)(L(y,z)) \rightarrow L(x,y))$.
7. $(\exists x)(\exists y)((\forall z)(L(y,z)) \land L(x,y))$.
8. $(\exists x)(\exists y)((\exists z)(L(y,z)) \land L(x,y))$.

CHALLENGE: Determine which of these logically imply which of these.

CHALLENGE: Determine which sets of these logically imply which of these.

CHALLENGE. Determine which sets of these logically imply which disjunctions of sets of these.