INVARIANT MAXIMAL CLIQUES AND INCOMPLETENESS

by

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Abstract. The Invariant Maximal Clique Theorem asserts that every graph on $Q[0,n]^k$ with a specific invariance condition has a maximal clique with another specific invariance condition. Here Q[0,n] consists of the rationals in the interval [0,n]. The invariance conditions are all given by equivalence relations involving only < on [0,n] and the distinguished elements 1,...,n. We prove the Invariant Maximal Clique Theorem using a certain well studied set theoretic hypothesis that goes well beyond the usual axioms for mathematics. The proof is modified so as to rely only on the assumption that this hypothesis is free of contradiction. We show that the Invariant Maximal Clique Theorem is, in fact, equivalent to a slight weakening of this consistency assumption.

1. INTRODUCTION.

This paper represents a major advance in what we call Concrete Mathematical Incompleteness. An extensive discussion of Concrete Mathematical Incompleteness can be found in the Introduction of the forthcoming book [Fr12]. Consequently, the discussion here will be limited, and necessarily omit many important pieces of background information that can be found in [Fr12].

Kurt Gödel was the founder of Incompleteness in the modern sense. In modern terms, Gödel (essentially) established that

A. [Go31]. There exist sentences in mathematical language that are neither provable nor refutable from the usual axioms and rules of mathematics.

The usual axioms and rules of mathematics are represented by the formal system ZFC = Zermelo Frankel set theory with the Axiom of Choice. A asserts the so called incompleteness of ZFC.

The long range relevance of this startling discovery of Gödel to the ongoing development of mathematics depends on the nature of the examples of A.

Gödel (along with Cohen) is credited with two examples of Incompleteness from ZFC, which are very different from each other. The first of the two examples is "ZFC is consistent (free of contradiction)", commonly written as Con(ZFC).

The second example is "the continuum hypothesis", or CH, which asserts that "every infinite set of real numbers is in one-one correspondence with the set of all integers or the set of all real numbers".

The first example is formally mathematical in the sense of being phrased in mathematical language. However, it is not conceptually mathematical, in the sense that it is not about such standard mathematical objects as integers, rationals, vectors, triangles, real numbers, equations, inequalities, vector spaces, trigonometric functions, infinite series, graphs, groups, topological spaces, curves, manifolds, rings, fields, etc. Of course, this example - Con(ZFC) - is of great importance for the foundations of mathematics, particularly in connection with Hilbert's program, which is normally viewed as a failed program to establish beyond doubt that mathematics is free of contradiction. The first example is, however, concrete (down to earth) in that it concerns finite patterns of letters from a finite alphabet.

The second example, CH, is conceptually mathematical, in the sense that it is about sets of real numbers. However, it is not concrete (down to earth) in a way that is compatible with the ongoing development of contemporary mathematics. Such abstract mathematical sentences are commonly referred to as "set theoretic", and are flagged by mathematicians as causing unwanted logical difficulties because of their essential incorporation of pathological objects, incompatible with normal mathematical activity.

We can illustrate the role of abstract, pathological objects in connection with the second example, CH, as follows. If we restrict CH to reasonably concrete sets of real numbers (specifically Borel sets of real numbers), then CH ceases to be an example of Incompleteness: CH becomes provable from ZFC. This is an essential feature of set theoretic problems - their difficulties are almost always removed when recast in terms of the Borel measurable universe. For exceptions to this principle, see [Fr12], section 0.13.

Thus we can summarize Gödel's two main examples of Incompleteness from ZFC in the following way:

B. [Go31]. There exists a formally mathematical sentence which is concrete but not conceptually mathematical, and neither provable nor refutable from the usual axioms and rules of mathematics.
C. [Go40], [Co63,64]. There exists a formally mathematical sentence which is conceptually mathematical but not concrete, and neither provable nor refutable from the usual axioms and rules of mathematics.

In terms of long range impact/relevance of Incompleteness, we arrive at the following critical question:

IS EVERY SENTENCE THAT IS BOTH CONCRETE AND CONCEPTUALLY MATHEMATICAL, PROVABLE OR REFUTABLE FROM THE USUAL AXIOMS AND RULES OF MATHEMATICS?

Hence the phrase "Concrete Mathematical Incompleteness", which is the title of the Introduction to [Fr12]. This is the most recent publication providing examples of Concrete Mathematical Incompleteness from ZFC. These are from Boolean Relation Theory (BRT), the focus of [Fr12].

This paper represents a substantial advance over [Fr12] in two directions. Firstly, the new examples of Concrete Mathematical Incompleteness of ZFC are of a compelling thematic character, with no trace of ad hoc features. They involve only the rationals and the positive integers as a linearly ordered system - not even the group or ring structure. They can be readily grasped by essentially the whole mathematical community, including students from all areas of mathematics.

Of course, we cannot argue at this point that these examples are of ready use for ongoing research projects. However, we do suspect that the Invariant Maximal Clique Theorem will spawn new research programs pursued by a variety of researchers not focused on issues in the foundations of mathematics.

Secondly, the new examples are equivalent to the consistency (freedom from contradiction) of a certain well studied extension of ZFC. The previous examples of Concrete Mathematical Incompleteness from ZFC in [Fr12] are instead equivalent to the 1-consistency of the same or closely related well studied extensions of ZFC. In this sense, the examples here represent a new level of concreteness for Concrete Mathematical Incompleteness of ZFC.

This paper does not obsolete [Fr12] in that BRT represents an entirely new general mathematical framework with vast problem generating power, promising to touch virtually all areas of mathematics - even in contexts where it does not push the boundaries of ZFC.

Here is our motivating General Problem, which we abbreviate by IMCP.

INVARIANT MAXIMAL CLIQUE PROBLEM. IMCP. LET R,R' BE EQUIVALENCE RELATIONS. FOR WHICH R INVARIANT GRAPHS IS IT THE CASE THAT THERE IS AN R' INVARIANT MAXIMAL CLIQUE? IN OTHER WORDS, AN R' INVARIANT SET OF VERTICES WHICH IS ALSO A MAXIMAL CLIQUE?

In section 3, we provide detailed definitions used in the paper. However, we give some definitions in this section, for the reader's convenience.

We say that a graph G is R invariant if and only if for all vertices x,y,z,w, if the ordered pairs (x,y),(z,w) are R equivalent, and (x,y) is an edge, then (z,w) is an edge.

In this paper, we do not attempt to consider IMCP in anything like full generality. Instead, we consider IMCP only in the presence of a linear ordering with a distinguished subset, (A,<,B). In this context, there are natural equivalence relations R,S - called the order theoretic equivalence relations - that we use for IMCP. We consider only graphs on Cartesian powers of intervals in the linear ordering. These order theoretic equivalence relations involve not only (A,<), but also B.

We cut down the generality further by using $(Q,<,Z^+)$ for (A,<,B). Here Q is the set of all rationals, and Z^+ is the set of all positive integers.

We also use the extended system $(Q[-\infty,\infty],<,Z^+(\infty))$, where $Q[-\infty,\infty] = Q \cup \{-\infty,\infty\}$, and $Z^+(\infty) = Z^+ \cup \{\infty\}$, for additional generality.

The bulk of the paper, sections 3-5, is based only on three order theoretic equivalence relations, which we present in the $(Q[-\infty,\infty],<,Z^+(\infty))$ context. In sections 4,5, we study IMCP where R is the first of these three (order equivalence), and where R' is the third of these three (upper Z+(∞) order equivalence). Order equivalence is the most basic nontrivial order theoretic equivalence relation. It involves $(Q[-\infty,\infty],<)$, and not $Z^+(\infty)$.

In section 6.1, we introduce the order theoretic equivalence relations in the $(Q[-\infty,\infty],<,Z^+(\infty))$ context, with an eye toward their use in IMCP. These are equivalence relations on $Q[-\infty,\infty]^*$ (the finite sequences from $Q[-\infty,\infty]$)

satisfying two fundamental conditions. We also define the standard order theoretic equivalence relations, by imposing an addition condition. See Definitions 6.1.1, 6.1.2.

We give a complete treatment of IMCP where R is order equivalence and R' is any given standard order theoretic equivalence relation. In particular, we show that the Invariant Maximal Clique Theorem holds for a given standard order theoretic equivalence relation R' if and only if R' is either upper $Z^+(\infty)$ order equivalence, or its subset strong $Z^+(\infty)$ order equivalence. See Theorem 6.2.9.

In obtaining these results, we go well beyond the usual axioms of mathematics (ZFC). We also show that such transcendental reasoning is in fact required.

Before giving definitions of the three relevant equivalence relations, we present four statements which are proved here using hypotheses that go well beyond the usual ZFC axioms. The terminology is also revisited in full detail in section 2. Throughout the paper, k,n denote positive integers, unless indicated otherwise. We use Q[0,n] for $Q \cap [0,n]$. The abbreviation used appears right after each title below.

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (Q). IMCT(Q). Every order invariant graph on Q^k has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (extended). IMCT(extended). Every order invariant graph on $Q[-\infty,\infty]^k$ has an upper $Z^+(\infty)$ order invariant maximal clique.

For IMCT and IMCT(extended), we know that the use of hypotheses that go well beyond the usual ZFC axioms is necessary. We do not know whether this is necessary for IMCT(Q). Thus we gain some important generality by using the extended rationals $Q[-\infty,\infty]$, and the extended positive integers, $Z^+(\infty)$. This allows us to give a specific familiar setting, $Q[-\infty,\infty]$, where we know that going beyond the usual ZFC axioms is required. We do show that for some specific positive integers n, IMCT for Q[0,n] can be proved only by going well beyond ZFC. However, the n we give here is $2\uparrow 10$, which is the exponential stack consisting of 10 2's. We do not know if we can reduce $2\uparrow 10$ down to $2\uparrow 2 = 4$, or even $2\uparrow 1 = 2$. See section 6.4.

The large cardinal hypotheses used in this paper have already been extensively studied, and are discussed in section 3. In particular, the system

$$\text{SRP} = \text{ZFC} + \{\text{there exists an ordinal with the k-SRP}\}_k$$

plays a special role. SRP stands for the "stationary Ramsey property", as discussed in section 3.

Sections 3-5 use the following three successively stronger notions of "equivalence". The first two notions are the most fundamentally transparent.

Let $x,y \in Q[-\infty,\infty]^k$. x,y are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

x,y are $Z^+(\infty)$ order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$ then $x_i,y_i \in Z^+(\infty)$.

Mere order equivalence is far too weak to be used for IMCP. E.g., the graph on Q[0,1] with no edges has no order invariant maximal clique.

The use of the far stronger $Z^+(\infty)$ order equivalence leads to some interesting results and open questions (see section 6.2). See the discussion below of IMCA(J).

Note that both of these equivalence relations are "essentially binary", respect $(Q[-\infty,\infty],<,Z^+(\infty))$, and can be viewed as length preserving equivalence relations on $Q[-\infty,\infty]^*$ = the set of all nonempty finite sequences from $Q[-\infty,\infty]$. This is evident from the following. Firstly, order equivalence is defined by

$$(\forall i,j)(x_i < x_j \leftrightarrow y_i < y_j).$$

Secondly, $Z^+(\infty)$ order equivalence is defined by

$$(\forall i,j)((x_i < x_j \leftrightarrow y_i < y_j) \wedge (x_i \in Z^+(\infty) \leftrightarrow y_i \in Z^+(\infty))).$$

Most of the main results of the paper use a strengthening of $Z^+(\infty)$ order equivalence called upper $Z^+(\infty)$ order equivalence.

We say that x,y are upper $Z^+(\infty)$ order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$, then every $x_j \geq x_i$ and every $y_j \geq y_i$ lies in $Z^+(\infty)$.

Upper $Z^+(\infty)$ order equivalence can also be defined in a similar way:

$$(\forall i,j)((x_i < x_j \leftrightarrow y_i < y_j) \wedge (x_j \geq x_i \rightarrow x_j \in Z^+(\infty)) \wedge (y_j \geq y_i \rightarrow y_j \in Z^+(\infty)))$$

Upper $Z^+(\infty)$ order equivalence is fundamental to this paper. The notion is very robust, and can be viewed in different ways. In section 2, some alternative definitions are given, and proved to be equivalent.

We also give a version of IMCT using order theoretic equivalence relations generally, without mentioning any specific ones like upper $Z^+(\infty)$ order equivalence. This is i,ii of Theorem 6.2.9. This exploits the fact that all counterexamples given in the paper for IMCT work for J = $[0,3]^2$.

The key way in which upper $Z^+(\infty)$ order equivalence is stronger than $Z^+(\infty)$ order equivalence can be seen in how the three pairs of two dimensional vectors

(0,1),(0,2)
(3/2,1),(3/2,2)
(5/2,1),(5/2,2)

are related. All three pairs are $Z^+(\infty)$ order equivalent, whereas only the first pair are upper $Z^+(\infty)$ order equivalent. It turns out that the order theoretic equivalence relations strengthening $Z^+(\infty)$ order equivalence are determined by how they handle these three pairs. There are only five ways: only the first of the three pairs are equivalent, in which case we have $Z^+(\infty)$ order equivalence; only the third of the three pairs are equivalent, in which case we have lower $Z^+(\infty)$ order equivalence; only the first and third of the three pairs are equivalent, in which case we have upper/lower $Z^+(\infty)$ order equivalence; all three pairs are equivalent, in which case we have $Z^+(\infty)$ order equivalence; none of the three pairs are equivalent in which case we have strong $Z^+(\infty)$ order equivalence. See section 6.1.

In section 6.2, we explore Invariant Maximal Clique Theorems using these five standard order theoretic equivalence relations. IMCT holds for upper Z+(∞) order equivalence and it subset, strong $Z^+(\infty)$ order equivalence, and fails for the other three of these five. We also show that for all of the standard order theoretic equivalence relations not included in Z+(∞) order equivalence, the Invariant Maximal clique Theorem on $[0,3]^2$ is false (refutable in RCA_0).

For the remainder of the Introduction, we will always assume that J is an interval in Q[-∞,∞]. As indicated above, the following most straightforward version of the Invariant Maximal Clique Problem is easily refutable for any J with more than one point, in all dimensions, using graphs with no edges.

INVARIANT MAXIMAL CLIQUE ERROR (J). Every order invariant graph on J^k has an order invariant maximal clique.

The following alternative uses $Z^+(\infty)$ order equivalence instead of upper $Z^+(\infty)$ order equivalence.

INVARIANT MAXIMAL CLIQUE ALTERNATIVE (J). IMCA(J). Every order invariant graph on J^k has a $Z^+(\infty)$ order invariant maximal clique.

We prove that IMCA holds on J = [0,2], using axioms going well beyond ZFC (a subtle ordinal suffices), and we do not know if ZFC is sufficient. We prove in RCA_0 that IMCA fails on [0,4], or any interval containing at least four positive integers. See section 6.2.

Section 4.1 contains an analysis of the following parameterized statement.

INVARIANT MAXIMAL CLIQUE THEOREM (J). IMCT(J). Every order invariant graph on J^k has an upper $Z^+(\infty)$ order invariant maximal clique.

We prove the following complete characterization of the J for which the Invariant Maximal Clique Theorem (J) holds, in section 4.1 (relying on the main result of section 4.2). We show that this characterization theorem from section 4.2 can only be proved by using much more than the usual axioms for mathematics.

INVARIANT MAXIMAL CLIQUE CHARACTERIZATION. IMCC. Let J be an interval in Q[-∞,∞]. IMCT(J) holds if and only if no positive integer is the smallest element of J or J contains at most 2 positive integers.

We show that IMCT, IMCT(extended), IMCC are provably equivalent to Con(SRP) over ACA' (Theorem 5.9.3).

A major open question left here is the status of IMCT(Q). As far as we know, IMCT(Q) is provable in ZFC, or even in RCA_0.

In section 4.1, we present many RCA_0 and ACA_0 implications between IMCT(J) and IMCT(J'). In section 4.2, we prove IMCT(extended), first in SRP^+, and then in WKL_0 + Con(SRP). In sections 5.1 - 5.7, we prove Con(SRP) from IMCT over ZFC. In section 5.9, we reduce the base theory ZFC to the base theory ACA'. Since by Theorem 4.2.11, WKL_0 + Con(SRP) proves IMCT(extended), this completes the proof that IMCT is equivalent to Con(SRP) over ACA'.

Note that by Theorem 5.9.3, IMCT is provably equivalent to a Π^0_1 sentence over ACA'. However, we can establish this directly through basic considerations from mathematical logic in the following way. In fact, the following direct argument is presented in Lemmas 4.3.18, 4.3.19, as a step towards proving Theorems 4.3.20, 4.3.21, which lead to Theorem 5.9.3.

By an elementary model theoretic exercise, for each order invariant graph G on $Q[0,n]^k$, IMCT asserts the existence of a countable model of an effectively generated sentence δ(G) in first order predicate calculus with equality, effectively constructed from G. Hence by Gödel's completeness theorem, IMCT for G is provably equivalent, over WKL_0, to the formal consistency of δ(G). Therefore IMCT is provably equivalent, over WKL_0, to a Π^0_1 sentence - *in virtue of its logical form*. See Lemmas 4.3.18 and 4.3.19.

This construction is also carried out in a similar but more delicate way to the extended rationals. Specifically, for each order invariant graph G on $Q[-\infty,\infty]^k$, the IMCT(extended) asserts the existence of a countable model of an effectively generated sentence γ(G) in first order predicate calculus with equality, effectively constructed from G. See Lemmas 4.2.9, 4.2.10, and Theorems 4.2.11, 4.2.12.

By combining the well known representation of Π^0_1 sentences by Diophantine equations given by the solution to Hilbert's

Tenth Problem (see [DPR61, [Ma70], [Da73], [DMR76]], [Ma93]), and Lemmas 4.2.10, 4.3.19, we obtain the following.

THEOREM 1.1. There is a polynomial P with integer coefficients such that the following is provable in WKL_0. IMCT holds if and only if P has no integral zero. The forward direction is provable in RCA_0. The same is true for IMCT(extended).

We can derive very strong metamathematical conclusions from Theorem 1.1 concerning IMCT, such as

a. The truth value of IMCT is the same for all models of ZFC (or even WKL_0) with standard integers.
b. If IMCT fails, then it can be refuted in RCA_0.

The above is also true of IMCT(extended).

We caution the reader that there is no way at present of constructing any kind of remotely reasonable polynomial P for Theorem 1.1.

The reversal in section 5 suffices to show that for some particular k = n, IMCT for $[0,n]^k$ is not provable in ZFC (provided ZFC is consistent). In fact, according to Theorem ..., 2↑10 suffices, where 2↑10 is an exponential stack of 10 2's. More generally, we show that for all k,

ACA' + IMCT on $[0,k]^k$ proves
Con(ZFC + there is an ordinal with the ($\lfloor k_6/10^4 \rfloor$-8)-SRP).

According to Theorem 5.9.4, the IMCT for $[0,n]^k$ forms a hierarchy as k,n go to infinity, in the following sense.

ACA' + {IMCT for $[0,n]^k$: k,n ≥ 1} is logically equivalent to ACA' + {Con(ZFC + there exists an ordinal with the k-SRP)}.

The important question remains as to whether we can give an intelligible explicitly Π^0_1 form of the Invariant Maximal Clique Theorem. We have recently been able to achieve this in a satisfactory way, and give an account of this recent work in section 6.3. Some of these forms are explicitly Π01 sentences. They involve shifting from maximal cliques to the dual context of independent dominators. This work will appear in [Fr∞].

We conclude with open problems, most of which are already mentioned in the text.

2. GRAPHS, CLIQUES, RATIONALS, ORDER INVARIANCE.

We now introduce the definitions and background information supporting sections 4-6.

DEFINITION 2.1. A (simple) graph is a pair G = (V,E), where V is a set and E is an irreflexive symmetric binary relation on V - viewed as a subset of V^2.

DEFINITION 2.2. The vertices of G are the elements of V = V(G). The edges of G are the elements of E = E(G).

DEFINITION 2.3. We say that x,y are adjacent in G if and only if x E y. E is called the edge (or adjacency) relation.

DEFINITION 2.4. We say that S is a clique in G if and only if $S \subseteq V$, where any two distinct elements of S are adjacent.

DEFINITION 2.5. A maximal clique is a clique which is not a proper subset of a clique.

DEFINITION 2.6. We write G[A] for the subgraph of G induced by $A \subseteq V$. I.e., G[A] is the graph $(A,E \cap A^2)$.

The following two Theorems are well known.

THEOREM 2.1. Every graph has a maximal clique. This is provable in EFA for finite graphs, and RCA_0 for countable graphs.

THEOREM 2.2. In every graph, every clique is contained in a maximal clique. This is provable in EFA for finite graphs, and ACA_0 for countable graphs.

THEOREM 2.3. "In every countable graph, every clique is contained in a maximal clique" is equivalent to ACA_0 over RCA_0.

Proof: ACA_0 is equivalent to "the set of values of every function $f:N \to N$ exists" (known by us when we formulated ACA_0). See [Si99], Lemma III.1.3. Assume RCA_0 and the statement in quotes.

Let G be the graph on $N^2 \cup N$, where $x,y \in N^2 \cup N$ are adjacent if and only if $x \neq y \wedge (x \in N^2 \rightarrow x \in f) \wedge (y \in N^2 \rightarrow y \in f) \wedge \neg(x \in f \wedge y$ is the second coordinate of $x) \wedge \neg(y \in f \wedge x$ is the second coordinate of $y)$. It is easy to verify that this defined an irreflexive and symmetric relation on $N^2 \cup N$, as required.

Obviously f forms a clique in G. We can characterize the cliques $S \supseteq f$ in G. It is necessary and sufficient that $S \cap N^2 = f$ and no element of S is a value of f. Hence any maximal clique in G, which contains f, must consist of f together with the set of all non values of f. Hence the set of values of f exists. QED

We prove the following well known set theoretic result as background information.

THEOREM 2.4. In ZF, "every graph has a maximal clique" is provably equivalent to the axiom of choice.

Proof: ZFC proves "every graph has a maximal clique" using Zorn's Lemma. For the converse, we argue in ZF. Let R be an equivalence relation on a set A. Let G be the graph on A where $x,y \in A$ are adjacent if and only if x,y are not R equivalent. Let S be a maximal clique in G. Then S must pick exactly one element from each R equivalence class. QED

We now bring invariance into the picture. Since equivalence relations are reflexive (as well as symmetric and transitive), their field is recoverable.

DEFINITION 2.7. Let R be an equivalence relation. We say that $S \subseteq V$ is R invariant if and only if for all R equivalent $x,y \in V$, $x \in S \rightarrow y \in S$.

It is important to see how the space V is used in Definition 2.7.

Note that in Definition 2.7, we do not require that the field of R be related to V. In applications, we generally have $V \subseteq fld(R)$. We don't want to require $V = fld(R)$ because we sometimes want to use the same equivalence relation R for many different V's. This consolidates parameters in a way that we exploit, particularly in section 6.1. Also see the beginning of section 4.

DEFINITION 2.8. Let R be an equivalence relation and $G = (V,E)$ be a graph. We say that G is R invariant if and only if $E \subseteq V^2$ is R invariant.

DEFINITION 2.9. Let R,G be as in Definition 2.8. An R invariant maximal clique is an R invariant $S \subseteq V$ which is a maximal clique in G.

Linear orderings give us a very rich supply of equivalence relations on tuples that we use for invariance conditions.

Let $(A,<_A)$ be a linear ordering, and B be a set. We do not assume that $B \subseteq A$.

DEFINITION 2.10. We say that $x,y \in A^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i <_A x_j \leftrightarrow y_i <_A y_j$.

DEFINITION 2.11. We say that $x,y \in A^k$ are B order equivalent if and only if x,y are order equivalent, where for all i, $x_i \neq y_i \rightarrow x_i,y_i \in B$.

We give three equivalent definitions of upper B order equivalence.

1. $x,y \in A^k$ are upper B order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$ then every $x_j \geq_A x_i$, every $x_j \geq_A y_i$, every $y_j \geq_A x_i$, and every $y_j \geq_A y_i$ lies in B.

2. $x,y \in A^k$ are upper E order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$ then every $x_j \geq_A x_i$ and every $y_j \geq_A y_i$ lies in B.

3. $x,y \in A^k$ are upper E order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$ then every $x_j \geq_A x_i$ lies in E and y_i lies in B.

The first two definitions have the advantage of being nicely symmetric. The second definition is the official definition, highlighted in section 1, that we most commonly use. The third definition more directly supports the operational idea that y is obtained from x by replacing certain extended nonnegative integer coordinates of x with extended nonnegative integers. All definitions of course use the requirement of order equivalence.

THEOREM 2.5. The three definitions of upper B order equivalence are equivalent.

Proof: Obviously $1 \rightarrow 2 \rightarrow 3$. Assume 3. Then x,y are order equivalent. Fix $x_i \neq y_i$. Then $x_i,y_i \in B$. Suppose $y_j \geq_A y_i$. We want $y_j \in B$. By order equivalence, we have $x_j \geq_A x_i$, and so $x_j \in B$. If $x_j = y_j$ then we are done. Assume $x_j \neq y_j$. Then $y_j \in B$.

Suppose $y_j \geq_A x_i$. We want $y_j \in B$. If $x_j \neq y_j$ then $y_j \in E$. If $x_j = y_j$ then since $x_j \geq_A x_i$, we have $x_j,y_j \in B$.

Suppose $x_j \geq_A y_i$. We want $x_j \in B$. If $x_j \neq y_j$ then $x_j \in B$. If $x_j = y_j$ then since $y_j \geq_A y_i$, we have $y_j,x_j \in B$, from the previous paragraph. QED

THEOREM 2.6. B order equivalence is an equivalence relation. Upper B order equivalence is an equivalence relation.

Proof: Both relations are obviously reflexive and symmetric. Let x,y and y,z be B order equivalent. Let $x_i \neq z_i$. Then $x_i \neq z_i \vee y_i \neq z_i$. Hence $z_i \in B$.

For transitivity, we use the second definition of upper B order equivalence. Let x,y and y,z be upper B order equivalent. Let $x_i \neq z_i$. Then $x_i \neq y_i \vee y_i \neq z_i$. Hence every $z_j \geq_A z_i$ lies in B. QED

DEFINITION 2.12. We say that $S \subseteq A^k$ is order (B order, upper B order) invariant if and only if for all order (B order, upper B order) equivalent $x,y \in A^k$, $x \in S \rightarrow y \in S$.

DEFINITION 2.13. We say that a graph on A^k is order invariant if and only if its edge relation, as a subset of A^{2k}, is an order invariant.

DEFINITION 2.14. Let $(A,<_A)$ and $(A',<_{A'})$ be two infinite linear orderings. Let G be an order invariant graph on A^k. Then G transfers canonically to a unique order invariant graph $G\#(<_{A'})$ on A'^k, where for all finite $C \subseteq A$ and finite $D \subseteq A'$, both of the same cardinality, the unique order preserving bijection from C onto D is a graph isomorphism from G[C] onto $G\#(<_{A'})[D]$.

DEFINITION 2.15. We use Q for the set of all rationals and Z^+ for the set of all positive integers. We attain some important extra generality by using the extended rationals $Q([-\infty,\infty]) = Q \cup \{\infty\}$, and the extended positive integers $Z^+(\infty) = Z^+ \cup \{\infty\}$.

DEFINITION 2.16. We say that $J \subseteq Q[-\infty,\infty]$ is an interval if and only if $(\forall x < y < z)(x,z \in J \rightarrow y \in J)$. Every interval

J in Q[-∞,∞] can be written as Q<a,b>, where $-\infty \le a \le b \le \infty$, and < is among (,[, and > is among),]. We allow real numbers a,b here. This representation is unique if J is nonempty. For nonempty J, a,b are called the left and right endpoints of J, respectively.

We apply the above abstract definitions mainly to the intervals $J \subseteq Q[-\infty,\infty]$, which are viewed as linear orderings under the usual <.

DEFINITION 2.14. EFA, or exponential function arithmetic, is the same as the system $I\Sigma_0$(exp) in [HP93]. It is based on S,+,•,exp,<,=, and induction for Σ_0 formulas in its language. SEFA, or superexponential function arithmetic, is the same as the system $I\Sigma_0$(superexp) in [HP93]. It is based on S,+,•,exp,superexp,<,=, and induction for Σ_0 formulas in its language. RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0 are our five principal systems of Reverse Mathematics. See [Si99]. ACA is ACA_0 with full induction. ACA' lies strictly between WKL_0 and ATR_0, and consists of ACA_0 together with "for all $x \subseteq \omega$ and $n \ge 1$, the n-th Turing jump of x exists". ZF is the usual Zermelo Frankel set theory, and ZFC is ZF together with the axiom of choice.

3. THE STATIONARY RAMSEY PROPERTY.

All results in this section are quoted from [Fr01]. All of these results, with the exception of Theorem 3.1, iv ↔ v → vi, are credited in [Fr01] to James Baumgartner.

DEFINITION 3.1. We say that $C \subseteq \lambda$ is unbounded if and only if for all $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta \ge \alpha$.

DEFINITION 3.2. We say that C is closed if and only if for all limit ordinals $x < \lambda$, if the sup of the elements of C below x is x, then $x \in C$.

DEFINITION 3.3. We say that $A \subseteq \lambda$ is stationary if and only if it intersects every closed unbounded subset of λ.

DEFINITION 3.4. For sets A, let S(A) be the set of all subsets of A. For integers $k \ge 1$, let $S_k(A)$ be the set of all k element subsets of A.

DEFINITION 3.5. Let $k \ge 1$. We say that λ is k-SRP if and only if λ is a limit ordinal, and for every $f:S_k(\lambda) \to \{0,1\}$, there exists a stationary $E \subseteq \lambda$ such that f is constant on $S_k(E)$.

Here SRP stands for "stationary Ramsey property."

The k-SRP is a particularly simple large cardinal property. To put it in perspective, the existence of an ordinal with the 2-SRP is stronger than the existence of higher order indescribable cardinals, which is stronger than the existence of weakly compact cardinals, which is stronger than the existence of cardinals which are, for all k, strongly k-Mahlo (see Theorem 3.1 below, and [Fr01], Lemmas 1.11).

Our main results are stated in terms of the stationary Ramsey property. In particular, we use the following extensions of ZFC based on the SRP.

DEFINITION 3.6. SRP^+ = ZFC + "for all k there exists an ordinal with the k-SRP". SRP = ZFC + $\{$there exists an ordinal with the k-SRP$\}_k$. We also use k-SRP for the formal system ZFC + $(\exists\lambda)(\lambda$ is a cardinal with the k-SRP).

For technical reasons, we will need to consider some large cardinal properties that rely on regressive functions.

DEFINITION 3.7. We say that $f:S_k(\lambda) \to \lambda$ is regressive if and only if for all $A \in S_k(\lambda)$, if $\min(A) > 0$ then $f(A) < \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, $f(B) = f(C)$.

DEFINITION 3.8. We say that $f:S_k(\lambda) \to S(\lambda)$ is regressive if and only if for all $A \in S_k(\lambda)$, $f(A) \subseteq \min(A)$. (We take $\min(\emptyset) = 0$, and so $f(\emptyset) = \emptyset$). We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, we have $f(B) \cap \min(B \cup C) = f(C) \cap \min(B \cup C)$.

DEFINITION 3.9. Let $k \geq 1$. We say that α is purely k-subtle if and only if

i) α is an ordinal;
ii) For all regressive $f:S_k(\alpha) \to \alpha$, there exists $A \in S_{k+1}(\alpha\backslash\{0,1\})$ such that f is constant on $S_k(A)$.

DEFINITION 3.9. We say that λ is k-subtle if and only if

i) λ is a limit ordinal;
ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

DEFINITION 3.10. We say that λ is k-almost ineffable if and only if

i) λ is a limit ordinal;
ii) For all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \subseteq \lambda$ of cardinality λ.

DEFINITION 3.11. We say that λ is k-ineffable if and only if

i) λ is a limit ordinal;
ii) For all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous stationary $A \subseteq \lambda$.

THEOREM 3.1. Let $k \geq 2$. Each of the following implies the next, over ZFC.
i. there exists an ordinal with the k-SRP.
ii. there exists a (k-1)-ineffable ordinal.
iii. there exists a (k-1)-almost ineffable ordinal.
iv. there exists a (k-1)-subtle ordinal.
v. there exists a purely k-subtle ordinal.
vi. there exists an ordinal with the (k-1)-SRP.
Furthermore, i,ii are equivalent, and iv,v are equivalent. There are no other equivalences. ZFC proves that the least ordinal with properties i - vi form a decreasing ($\geq$) sequence, with equality between i,ii, equality between iv,v, and strict inequality for the remaining consecutive pairs.

Proof: i $\leftrightarrow$ ii is from [Fr01], Theorem 1.28, iv $\leftrightarrow$ v is from [Fr01], Corollary 2.17. The strict implications ii $\to$ iii $\to$ iv $\to$ vi are from [Fr01], Theorem 1.28. Same references apply for comparing the least ordinals. QED

DEFINITION 3.12. We follow the convention that for integers $p \leq 0$, a p-subtle, p-almost ineffable, p-ineffable ordinal is a limit ordinal, and that the ordinals with the 0-SRP are exactly the limit ordinals. An ordinal is called subtle, almost ineffable, ineffable, if and only if it is 1-subtle, 1-almost ineffable, 1-ineffable.

DEFINITION 3.13. SRP+ is the formal system ZFC + $(\forall k)$(there exists an ordinal with the k-SRP). SRP is the formal system ZFC + $\{$there exists an ordinal with the k-SRP$\}_k$. For each k, we write SRP[k] for the formal system ZFC + "there exists an ordinal with the k-SRP".

4. PROOFS OF INVARIANT MAXIMAL CLIQUE THEOREMS.

We use k,n,m,r,s,t for nonnegative integers, and p,q for rationals, unless indicated otherwise. We suppress outermost universal quantifiers in statements.

In this paper we focus on

INVARIANT MAXIMAL CLIQUE THEOREM (J). IMCT(J). Every order invariant graph on J^k has an upper $Z^+(\infty)$ order invariant maximal clique.

where J is an interval in Q[-∞,∞]. Note that we have quantified over all k.

If J ⊆ Q then we can equivalently reformulate Invariant Maximal Clique Theorem (J) with Z^+ instead of $Z^+(\infty)$.

In order to avoid all possible ambiguities in IMCT(J), we restate it as follows.

INVARIANT MAXIMAL CLIQUE THEOREM (J). IMCT(J). For all order invariant graph G on Jk, there exists S such that
i. S is a maximal clique in G.
ii. S ⊆ Jk is upper Z+(∞) order invariant.

We prefer to use a single notion of invariance for all intervals J ⊆ Q[-∞,∞]. We call this "consolidation of parameters". Of course, this is a matter of choice, as obviously we can equivalently reformulate the Invariant Maximal Clique Theorem (J) in the following way.

INVARIANT MAXIMAL CLIQUE THEOREM (J). IMCT(J). Every order invariant graph on J^k has an upper $J \cap Z^+(\infty)$ order invariant maximal clique.

Of particular interest is

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $[0,n]^k$ has an upper Z^+ order invariant maximal clique.

to which we give preferred status, as well as to the special cases

INVARIANT MAXIMAL CLIQUE THEOREM (Q). IMCT(Q). Every order invariant graph on Q^k has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (extended).
IMCT(extended). Every order invariant graph on $Q[-\infty,\infty]^k$ has an upper $Z^+(\infty)$ order invariant maximal clique.

Note that we have chosen to use Z+ for the first two of these last three. However, we could abandon consolidation of parameters, and equivalently formulate the Invariant Maximal Clique Theorem as

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $[0,n]^k$ has an upper {1,...,n} order invariant maximal clique.

Assuming IMCT(extended) and IMCT(Q), we resolve IMCT(J) for all intervals $J \subseteq Q[-\infty,\infty]$. See the Invariant Maximal Clique Characterization, or IMCC, in section 4.1. We also establish a number of implications between IMCT(J) and IMCT(J'), all within ACA_0 (and many within RCA_0). Note that IMCT(extended) = $IMCT(Q[-\infty,\infty])$, and IMCT(Q) = $IMCT(Q\{-\infty,\infty))$. We derive IMCT from IMCT(extended), within RCA_0.

We note our current lack of understanding as to the provability in ZFC, or even in RCA_0, of a number of instances of IMCT(J). In particular, we do not know IMCT(A) is provable in ZFC, or even in RCA_0.

In section 4.2, we prove IMCT(extended) using SRP^+, and reduce this hypothesis down to the weaker WKL_0 + Con(SRP). This result, combined with the results of section 4.2, establish the provability of IMCT, IMCT(Q), IMCC, and various IMCT(J), from SRP^+ or from the weaker WKL_0 + Con(SRP). All refutations of IMCT(J) are proved within RCA_0.

In section 5, we reverse IMCT, establishing its equivalence with Con(SRP) over ACA'. Using sections 4.1, 4.2, this also establishes the equivalence of IMCT(extended), IMCC, and Con(SRP), over ACA'. See Theorem 5.9.?.

4.1. IMCT(J) BASED ON IMCT(EXTENDED).

DEFINITION 4.1.1. All occurrences of J,J' refer to intervals in $Q[-\infty,\infty]$. For all J, we write IMCT(J) for the Invariant Maximal Clique Theorem (J). We also write IMCT(J,k) for IMCT using J^k.

Note that in the statement IMCT(J,k), k is fixed in advance, rather than being subject to quantification.

It is noteworthy that all counterexamples that we give to IMCT(J) are in fact counterexamples to IMCT(J,2). Since we give a complete determination of the truth values of IMCT(J) here, we obtain as a consequence the transfer principle

$$\mathrm{IMCT}(J,2) \rightarrow \mathrm{IMCT}(J)$$

which we regard as interesting. We shall also see that this transfer principle is not provable in SRP (assuming SRP is consistent). See Theorem 4.1.12.

Recall the three featured Invariant Maximal Clique Theorems from section 4.1.

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (Q). IMCT(Q). Every order invariant graph on Q^k has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (extended). IMCT(extended). Every order invariant graph on $Q[-\infty,\infty]^k$ has an upper $Z^+(\infty)$ order invariant maximal clique.

In the IMCT(J) language, the above are $(\forall n \in Z^+)(\mathrm{IMCT}(Q[0,n]))$, IMCT(Q), $\mathrm{IMCT}(Q[-\infty,\infty])$, respectively.

Our first aim is to prove IMCT(extended) → IMCT ∧ IMCT(Q), within RCA_0.

DEFINITION 4.1.2. We say that h:J → J' is a strong order preserving bijection if and only if h:J → J' is an order preserving bijection which maps $J \cap Z^+(\infty)$ onto an initial segment of $J' \cap Z^+(\infty)$. We say that h:J → H' is a very strong order preserving bijection if and only if h:J → J' is an order preserving bijection which maps $J \cap Z^+(\infty)$ onto $J' \cap Z^+(\infty)$.

THEOREM 4.1.1. (RCA0). Let h:J → J' be a strong order preserving bijection. Then IMCT(J') → IMCT(J). Let h:J → J' be a very strong order preserving bijection. Then IMCT(J) ↔ IMCT(J').

Proof: For the first claim, let J,J',h be as given. Assume IMCT(J'). Let G be an order invariant graph on J^k. Let S ⊆

J'^k be an upper $Z^+(\infty)$ order invariant maximal clique in G#(J'). Since h is an order preserving bijection, we see that $h^{-1}(S)$ is a maximal clique in G, where h acts on coordinates.

We claim that $h^{-1}(S)$ is upper $Z^+(\infty)$ order invariant. Let x,y $\in h^{-1}(S)$ be upper $Z^+(\infty)$ order equivalent. Then h(x),h(y) $\in$ S are upper $Z^+(\infty)$ order equivalent. To see this, since x,y are order equivalent, h(x),h(y) are order equivalent. Suppose $h(x)_i \neq h(y)_i$. Then $x_i \neq y_i$, and so every $x_j \geq x_i$ and every $y_j \geq y_i$ lies in $Z^+(\infty)$. Hence every $h(x_j) \geq h(x_i)$ and every $h(y_j) \geq h(y_i)$ lies in $Z^+(\infty)$. I.e., every $h(x)_j \geq h(x)_i$ and every $h(y)_j \geq h(y)_i$ lies in $Z^+(\infty)$. So h(x),h(y) are upper $Z^+(\infty)$ order equivalent. Since S is $Z^+(\infty)$ order invariant, $h(x) \in S \leftrightarrow h(y) \in S$. Hence $x \in h^{-1}(S) \leftrightarrow y \in h^{-1}(S)$. This establishes the claim and the implication.

For the second claim, note that if h is a very strong order preserving bijection, then h,h^{-1} are strong order preserving bijections. Apply the first claim. QED

LEMMA 4.1.2. (RCA_0). Let max(J) = p $\in Z^+(\infty)$. Then IMCT(J) $\rightarrow$ IMCT(J\{p}).

Proof: Let J,p be as given, and assume the IMCT(J). Let G be an order invariant graph on $(J\backslash\{p\})^k$. Let G' be the following order invariant graph on J^{k+1}. $(p_1,\ldots,p_{k+1})$, $(q_1,\ldots,q_{k+1}) \in J^{k+1}$ are adjacent in G' if and only if

i. $(p_1,\ldots,p_{k+1}) \neq (q_1,\ldots,q_{k+1})$.
ii. $p_1,\ldots,p_k < p_{k+1} \wedge q_1,\ldots,q_k < q_{k+1} \wedge (p_1,\ldots,p_k) \neq (q_1,\ldots,q_k) \rightarrow (p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are adjacent in G.

Clearly this adjacency relation is irreflexive and symmetric. Hence G' is well defined.

Let S be an upper $Z^+(\infty)$ order invariant maximal clique in G'. We first claim that $(p_1,\ldots,p_{k+1}) \in S \wedge p_1,\ldots,p_k < p_{k+1} \rightarrow (p_1,\ldots,p_k,p) \in S$. To see this, let $(p_1,\ldots,p_{k+1}) \in S \wedge p_1,\ldots,p_k < p_{k+1} \wedge (p_1,\ldots,p_k,p) \notin S$. Let $(q_1,\ldots,q_{k+1}) \in S$, where $(p_1,\ldots,p_k,p),(q_1,\ldots,q_{k+1})$ are not adjacent in G'. Since $(p_1,\ldots,p_k,p) \neq (q_1,\ldots,q_{k+1})$, we see that $q_1,\ldots,q_k < q_{k+1} \wedge (p_1,\ldots,p_k) \neq (q_1,\ldots,q_k) \wedge (p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are not adjacent in G. It is now immediate that $(p_1,\ldots,p_{k+1}),(q_1,\ldots,q_{k+1})$ are not adjacent in G'. This contradicts that S is a clique.

Let $S' = \{(p_1,\ldots,p_k): p_1,\ldots,p_k < p \wedge (p_1,\ldots,p_k,p) \in S\}$. Then $S' \subseteq (J\setminus\{p\})^k$.

We claim that S' is a maximal clique in G. Let $(p_1,\ldots,p_k) \neq (q_1,\ldots,q_k)$ be from S'. Then $(p_1,\ldots,p_k,p) \neq (q_1,\ldots,q_k,p)$ are from S, where $p_1,\ldots,p_k,q_1,\ldots,q_k < p$. Since S is a clique in G', $(p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are adjacent in G. This establishes that S' is a clique in G.

Let $(p_1,\ldots,p_k) \in (J\setminus\{p\})^k\setminus S'$. Then $(p_1,\ldots,p_k,p) \in J^k\setminus S$. Let $(q_1,\ldots,q_{k+1}) \in S$, where $(p_1,\ldots,p_k,p) \neq (q_1,\ldots,q_{k+1})$ are not adjacent in G'. Then $q_1,\ldots,q_k < q_{k+1} \wedge (p_1,\ldots,p_k) \neq (q_1,\ldots,q_k) \wedge (p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are not adjacent in G. By the first claim above, $(q_1,\ldots,q_k,p) \in S$. Since $q_1,\ldots,q_k < p$, we have $(q_1,\ldots,q_k) \in S'$. This establishes that S' is a maximal clique in G.

Finally, we show that S' is upper $Z^+(\infty)$ order invariant. Let $(p_1,\ldots,p_k),(q_1,\ldots,q_k) \in (J\setminus\{p\})^k$ be upper $Z^+(\infty)$ order equivalent. We claim that $(p_1,\ldots,p_k,p),(q_1,\ldots,q_k,p)$ are upper $Z^+(\infty)$ order equivalent. Let $p_i \neq q_i$, and let $p_j \geq p_i$, $q_j \geq q_i$, where we take $p_{k+1} = q_{k+1} = p$. Then $1 \leq i \leq k$, and so $p_i = q_i$, and we must have $p_j,q_j \in Z^+(\infty)$ since $p \in Z^+(\infty)$.

Since S is upper $Z^+(\infty)$ order invariant, $(p_1,\ldots,p_k,p) \in S \leftrightarrow (q_1,\ldots,q_k,p) \in S \leftrightarrow (p_1,\ldots,p_k) \in S' \leftrightarrow (q_1,\ldots,q_k) \in S'$. This establishes that S' is an upper $Z^+(\infty)$ order invariant maximal clique in G. QED

LEMMA 4.1.3. (RCA_0). Let $p = \min(J)$. $IMCT(J) \rightarrow IMCT(J\setminus\{p\})$.

Proof: Let J,p be as given, and assume the IMCT(J). Let G be an order invariant graph on $(J\setminus\{p\})^k$. Let G' be the following order invariant graph on J^{k+1}. $(p_1,\ldots,p_{k+1})$, $(q_1,\ldots,q_{k+1}) \in J^{k+1}$ are adjacent in G' if and only if

i. $(p_1,\ldots,p_{k+1}) \neq (q_1,\ldots,q_{k+1})$.
ii. $p_1,\ldots,p_k > p_{k+1} \wedge q_1,\ldots,q_k > q_{k+1} \wedge (p_1,\ldots,p_k) \neq (q_1,\ldots,q_k) \rightarrow (p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are adjacent in G.

Clearly this adjacency relation is irreflexive and symmetric. Hence G' is well defined.

Let S be an upper $Z^+(\infty)$ order invariant maximal clique in G'. We claim that $(p_1,\ldots,p_{k+1}) \in S \wedge p_1,\ldots,p_k > p_{k+1} \rightarrow (p_1,\ldots,p_k,p) \in S$. To see this, let $(p_1,\ldots,p_{k+1}) \in S \wedge p_1,\ldots,p_k > p_{k+1} \wedge (p_1,\ldots,p_k,p) \notin S$. Let $(q_1,\ldots,q_{k+1}) \in S$, where $(p_1,\ldots,p_k,p),(q_1,\ldots,q_{k+1})$ are not adjacent in G'.

Since $(p_1,...,p_k,p) \neq (q_1,...,q_{k+1})$, we see that $q_1,...,q_k > q_{k+1} \wedge (p_1,...,p_k) \neq (q_1,...,q_k) \wedge (p_1,...,p_k),(q_1,...,q_k)$ are not adjacent in G. It is now immediate that $(p_1,...,p_{k+1}),(q_1,...,q_{k+1})$ are not adjacent in G'. This contradicts that S is a clique.

Let $S' = \{(p_1,...,p_k): p_1,...,p_k > p \wedge (p_1,...,p_k,p) \in S\}$. Then $S' \subseteq (J\backslash\{p\})^k$.

We claim that S' is a maximal clique in G. Let $(p_1,...,p_k) \neq (q_1,...,q_k)$ be from S'. Then $(p_1,...,p_k,p) \neq (q_1,...,q_k,p)$ are from S, where $p_1,...,p_k,q_1,...,q_k > p$. Since S is a clique in G', $(p_1,...,p_k)$, $(q_1,...,q_k)$ are adjacent in G. This establishes that S' is a clique in G.

Let $(p_1,...,p_k) \in (J\backslash\{p\})^k\backslash S'$. Then $(p_1,...,p_k,p) \in Jk\backslash S$. Let $(q_1,...,q_{k+1}) \in S$, where $(p_1,...,p_k,p) \neq (q_1,...,q_{k+1})$ are not adjacent in G'. Then $q_1,...,q_k > q_{k+1} \wedge (p_1,...,p_k) \neq (q_1,...,q_k) \wedge (p_1,...,p_k),(q_1,...,q_k)$ are not adjacent in G. By the above, $(q_1,...,q_k,p) \in S$. Since $q_1,...,q_k > p$, we have $(q_1,...,q_k) \in S'$. This establishes that S' is a maximal clique in G.

Finally, we show that S' is upper $Z^+(\infty)$ order invariant. Let $(p_1,...,p_k)$, $(q_1,...,q_k) \in (J\backslash\{p\})^k$ be upper $Z^+(\infty)$ order equivalent. Then $(p_1,...,p_k,p),(q_1,...,q_k,p)$ are upper $Z^+(\infty)$ order equivalent, since $p_1,...,p_k,q_1,...,q_k > p$. Since S is upper $Z^+(\infty)$ order invariant, $(p_1,...,p_k,p) \in S \leftrightarrow (q_1,...,q_k,p) \in S \leftrightarrow (p_1,...,p_k) \in S' \leftrightarrow (q_1,...,q_k) \in S'$. This establishes the claim. QED

THEOREM 4.1.4. (RCA_0). IMCT(extended) $\rightarrow$ IMCT $\wedge$ IMCT(Q).

Proof: Assume IMCT(extended). Let $n \in Z^+$. There is an order preserving bijection from Q[0,n] onto $Q[-\infty,\infty]$ mapping $0,...,n$ to $-\infty,...,n-1,\infty$. This mapping is a strong order preserving bijection, and so by Lemma 4.1.1, we have IMCT(Q[0,n]). This establishes IMCT.

Note that IMCT($Q[-\infty,\infty]$). By Lemma 4.1.2, IMCT($Q[-\infty,\infty)$). By Lemma 4.1.3, IMCT($Q(-\infty,\infty)$). This is IMCT(Q). QED

LEMMA 4.1.5. (RCA_0). Let $p = \min(J) \in Q\backslash Z^+$. IMCT($J\backslash\{p\}$) $\rightarrow$ IMCT(J).

Proof: Let J,p be as given, and assume IMCT($J\backslash\{p\}$). Write J = [p,u>. We can assume that J has infinitely many points. Let $p < p^* < u$, where $Z^+ \cap [p,p^*] = \emptyset$. Let G be an order invariant graph on J^k. Let G' be the following order invariant graph on J^{k+1}. $(p_1,...,p_{k+1})$, $(q_1,...,q_{k+1}) \in J_{k+1}$ are adjacent in G' if and only if

i. $(p_1,...,p_{k+1}) \neq (q_1,...,q_{k+1})$.
ii. $p_1,...,p_k,q_1,...,q_k \geq p_{k+1} = q_{k+1} \wedge (p_1,...,p_k) \neq (q_1,...,q_k) \rightarrow (p_1,...,p_k),(q_1,...,q_k)$ are adjacent in G.

Clearly this adjacency relation is irreflexive and symmetric. Hence G' is well defined.

Let S be an upper $Z^+(\infty)$ order invariant maximal clique in G'. Let $S' = \{(p_1,...,p_k): p_1,...,p_k \geq p^* \wedge (p_1,...,p_k,p^*) \in S\}$. Then $S' \subseteq [p^*,u>^k$.

We claim that S' is a maximal clique in $G[[p^*,\infty]^k]$. Let $(p_1,...,p_k) \neq (q_1,...,q_k)$ be from S'. Then $(p_1,...,p_k,p^*) \neq (q_1,...,q_k,p^*)$ are from S. Since S is a clique in G', $(p_1,...,p_k),(q_1,...,q_k)$ are adjacent in G. This establishes that S' is a clique in $G[[p^*,u>^k]$.

Let $(p_1,...,p_k) \in [p^*,u>^k\backslash S'$. Then $(p_1,...,p_k,p^*) \in J^k\backslash S$. Let $(q_1,...,q_{k+1}) \in S$, where $(p_1,...,p_k,p^*) \neq (q_1,...,q_{k+1})$ are not adjacent in G'. Then $q_1,...,q_k \geq q_{k+1} = p^* \wedge (p_1,...,p_k) \neq (q_1,...,q_k) \wedge (p_1,...,p_k),(q_1,...,q_k)$ are not adjacent in G. Hence $(q_1,...,q_k,p^*) \in S$, $(q_1,...,q_k) \in S'$. This establishes that S' is a maximal clique in G.

Finally, we claim that S' is upper $Z^+(\infty)$ order invariant. Let $(p_1,...,p_k)$, $(q_1,...,q_k) \in [p^*,\infty]^k$ be upper $Z^+(\infty)$ order equivalent. Then $(p_1,...,p_k,p^*)$, $(q_1,...,q_k,p^*)$ are upper $Z^+(\infty)$ order equivalent, since $p_1,...,p_k,q_1,...,q_k \geq p^*$. Since S is upper $Z^+(\infty)$ order invariant, $(p_1,...,p_k,p^*) \in S \leftrightarrow (q_1,...,q_k,p^*) \in S \leftrightarrow (p_1,...,p_k) \in S' \leftrightarrow (q_1,...,q_k) \in S'$. This establishes the claim.

We have derived the $IMCT(G[[p^*,u>^k])$. Let $h:[p^*,u> \rightarrow J$ be an order preserving bijection that is the identity on $[p^*,u> \cap Z^+(\infty) = J \cap Z^+(\infty)$, using the choice of p^*. In particular, h is very strong. Apply Lemma 4.1.2 to obtain IMCT(J). QED

We write $J < J'$ to indicate that J is a proper initial segment of J'.

LEMMA 4.1.6. (ACA_0). Let $J < J'$, where $J'\backslash J$ is disjoint from $Z^+(\infty)$. $IMCT(J) \rightarrow IMCT(J')$.

Proof: Let J,J' be as given, and assume IMCT(J). Let G be an order invariant graph on J'^k. Let S be an upper $Z^+(\infty)$ invariant maximal clique in $G[J^k]$. Then S is a clique in G. Let S' ⊇ S be a maximal clique in G. We first claim that S' ∩ J^k = S. Suppose x ∈ S' ∩ J^k, x ∉ S. Let y ∈ S, x,y not adjacent in $G[J^k]$. This contradicts that S' is a clique in G.

We now claim that S' is upper $Z^+(\infty)$ invariant. To see this, let x,y ∈ S' be $Z^+(\infty)$ order equivalent. If x or y has a coordinate in J'\J, then x or y has highest coordinate outside $Z^+(\infty)$, in which case x = y. Otherwise, x,y ∈ J^k, in which case x ∈ S ↔ y ∈ S. By the first claim, it follows that x ∈ S' ↔ y ∈ S'. QED

LEMMA 4.1.7. (ACA_0). Let J < J', where max(J) ∈ $Z^+(\infty)$ is false, and J'\J is disjoint from $Z^+(\infty)$. Then IMCT(J') → IMCT(J).

Proof: Let J,J' be as given. We can obviously find J*, where

i. J* < J < J'.
ii. max(J*) ∈ $Z^+(\infty)$ is false.
iii. J'\J* is disjoint from $Z^+(\infty)$.
iv. there is a strong order preserving bijection h:J* → J'.

By Lemma 4.1.6 applied to J*,J, we have IMCT(J*) → IMCT(J). By Lemma 4.1.1, we have IMCT(J') → IMCT(J*). QED

LEMMA 4.1.8. (RCA_0). If $|J \cap Z^+| \leq 1$ then IMCT(J) holds. Suppose min(J) ∈ Z^+. If $|J \cap Z^+| \leq 2$ then IMCT(J) holds.

Proof: Suppose $|J \cap Z^+| \leq 1$. Then for all x,y ∈ J^k, x,y are upper $Z^+(\infty)$ order equivalent. Hence all subsets of J^k are upper $Z^+(\infty)$ order invariant. This establishes the first claim.

Let min(J) ∈ Z^+. Suppose $|J \cap Z^+| \leq 2$. We claim that x,y ∈ J^k are upper $Z^+(\infty)$ order equivalent if and only if they are either identical, or x = (min(J),...,min(J)) and y = (min(J)+1,...,min(J)+1). To see this, let x,y ∈ Jk be upper $Z^+(\infty)$ order equivalent, and let $x_i \neq y_i$. Then $x_i,y_i \in Z^+$, and so we can assume x_i = min(J) and y_i = min(J)+1. Since every x_j is ≥ x_i, by order equivalence we have every y_j is ≥ y_i. By upper $Z^+(\infty)$ order equivalence, every $y_j \geq y_i$ lies in $Z^+(\infty)$. Hence x,y are both constant sequences. This establishes the claim.

Let G be an order invariant graph on J^k. If (min(J),...,min(J)), (min(J)+1,...,min(J)+1) are adjacent in G, then any maximal clique in G containing (min(J),...,min(J)), (min(J)+1,...,min(J)+1) is upper $Z^+(\infty)$ order invariant. If (min(J),...,min(J)), (min(J)+1,...,min(J)+1) are not adjacent in G, then any maximal clique in G containing (min(J)+1/2,...,min(J)+1/2) is upper $Z^+(\infty)$ order invariant. Both claims use the order invariance of G. QED

LEMMA 4.1.9. (RCA_0). Suppose $\min(J) \in Z^+$, $|J \cap Z^+| \geq 3$. Then IMCT(J,2) fails.

Proof: Let G be the order invariant graph on J^2, where $(p,q),(p',q') \in J^2$ are not adjacent if and only if

i. $p < q \wedge p' < q' \wedge q = q' \wedge p \neq p'$; or
ii. $q < p' < p = q'$; or
iii. $q' < p < p' = q$; or
iv. $p = q \wedge p' = q'$.

Note that this defines a reflexive and symmetric relation. Hence its negation defines an irreflexive, symmetric relation. Therefore G is well defined, and obviously order invariant on J^2.

Let min(J) = n. Then $n,n+1,n+2 \in J$. Let S be an upper $Z^+(\infty)$ order invariant maximal clique in G. Suppose $(n,n+1) \in S$. By invariance, $(n,n+2),(n+1,n+2) \in S$. But (n,n+2),(n+1,n+2) are not adjacent in G by clause i. This contradicts that S is a clique in G. Hence $(n,n+1) \notin S$.

Let $(p,q) \in S$, where (p,q),(n,n+1) are not adjacent in G. Apply the definition of adjacent to (p,q),(n,n+1). Clauses ii,iii both require something < n, which is impossible. Hence clause i applies, and so $p < q = n+1 \wedge p \neq n$. Hence $(p,n+1) \in S$, $n < p < n+1$. By clause ii, (n+1,n),(p,n+1) are not adjacent in G. Since S is a clique in G, $(n+1,n) \notin S$. By invariance, $(n+2,n+1) \notin S$.

Let $(r,s) \in S$, where (r,s),(n+2,n+1) are not adjacent in G. Then clause iii applies, and so $n+1 < r < n+2 = s$. Hence $(r,n+2) \in S$, $n+1 < r < n+2$.

By invariance, $(p,n+2) \in S$, $n < p < n+1$. Since (p,n+2),(r,n+2) are not adjacent in G (clause i), we have contradicted that S is a clique. QED

Let J,J' be intervals in Q[-∞,∞] with endpoints from Q[-∞,∞].

DEFINITION 4.1.3. We make the following definitions.

J ⇔ J' if and only if RCA_0 proves IMCT(J) ↔ IMCT(J').
J ⇒ J' if and only if RCA_0 proves IMCT(J) → IMCT(J').
J ⇔* J' if and only if ACA_0 proves IMCT(J) ↔ IMCT(J').
J ⇒* J' if and only if ACA_0 proves IMCT(J) → IMCT(J').

We also allow J' to be a statement instead of an interval.

THEOREM 4.1.10. For all n ∈ Z^+(∞), [0,n] ⇔ (0,n] ⇒* <0,n+(1/2)>. Here <,> represent any independent choice from parentheses and brackets. For all p ∈ Q, [1,p] ⇔ p < 3, [1,p) ⇔ p ≤ 3. RCA_0 proves IMCT([0,2)), IMCT([0,3/2]).

Proof: The equivalence is by Lemmas 4.1.3, 4.1.5. The implication is by Lemmas 4.1.3, 4.1.5, 4.1.6, 4.1.7. The second claim is by Lemmas 4.1.8, 4.1.9. The third claim is obvious since any x,y ∈ J^k are upper Z^+(∞) order equivalent. QED

Theorem 4.1.10 encapsulates all of the implications known to us between the IMCT(J) for intervals J ⊆ Q[-∞,∞], that are provable in ACA_0 - in addition to implications given directly by Theorem 4.1.1.

INVARIANT MAXIMAL CLIQUE CHARACTERIZATION. IMCC. (Assuming ACA_0 + IMCT(extended)). Let J be an interval in Q[-∞,∞]. IMCT(J) holds if and only if no positive integer is the smallest element of J, or J contains at most 2 positive integers.

Proof: We will assume IMCT(extended), and work in ACA_0. We first claim that for all n ∈ Z^+(∞), IMCT(Q<0,n>), IMCT(Q<0,n+(1/2)>). To see this, we get [-∞,∞] ⇒ [-∞,n+1] ⇒ [-∞,n+1) ⇒ [0,,n+(1/2)) ⇒ <0,n+(1/2>, and [-∞,∞] ⇒ <0,n], and [-∞,∞] ⇒ [-∞,∞) ⇒ [-∞,n) ⇒ <0,n). Here we do not need to be careful about ⇒ versus ⇒*, since we have ACA_0.

Let J be as given. Suppose min(J) does not exist. There is a very strong order preserving bijection from J onto some (0,p) or (0,p], p ∈ Q[0,∞]. If p < 2 then clearly IMCT(J). If p ≥ 2 then there is a very strong order preserving bijection from J onto Q(0,n>) or Q(0,n+(1/2)>) for some n ∈ Z+(∞). By the first claim, IMCT(J).

Suppose min(J) ∉ Z^+. If min(J) = ∞ then IMCT(J) is trivial. If min(J) ∉ Q ∪ {-∞} then there is a very strong order preserving bijection from J onto some J' where min(J') does not exist. Therefore we can use the previous paragraph.

If min(J) ∈ Q ∪ {-∞}, then since min(J) ∉ Z^+, there is a very strong order preserving bijection from J onto some [0,n],[0,n),[0,n+(1/2], where n ∈ Z ∪ {∞}. We again use the first claim.

Suppose |J ∩ Z^+| ≤ 2. By Lemma 4.1.8, if min(J) ∈ Z^+ then IMCT(J). Suppose min(J) ∈ Z^+ is false. Then min(J) does not exist or min(J) ∉ Z^+, and these cases have already been handled above.

Conversely, suppose min(J) exists ∧ min(J) ∈ Z^+ ∧ |J ∩ Z^+| ≥ 3. Then IMCT(J) fails by Lemma 4.1.9. QED

THEOREM 4.1.11. The following is provable in RCA_0. If k' ≥ k ≥ 1, then IMCT(J,k') → IMCT(J,k).

Proof: Assume k' ≥ k ≥ 1 and IMCT(J,k'). Let G be an order invariant graph on J^k. Take G' to be the order invariant graph on $J^{k'}$, where x,y ∈ $J^{k'}$ are adjacent if and only if $(x_1,...,x_k),(y_1,...,y_k)$ are adjacent in G. Let S' ⊆ J_k' be an upper $Z^+(\infty)$ order invariant maximal clique in G'. Let S = $\{(x_1,...,x_k): (x_1,...,x_k,...,x_k) \in S'\}$.

It is clear that S is a clique in G. To see that S is maximal, let x ∈ $J^k\backslash S$. Then $(x_1,...,x_k,...,x_k) \notin S'$. Let $(x_1,...,x_k,...,x_k),(y_1,...,y_{k'})$ be adjacent in G', where $(y_1,...,y_{k'}) \in S'$. Then $(x_1,...,x_k),(y_1,...,y_k)$ are adjacent in G, where $(y_1,...,y_k) \in S$.

To see that S is upper $Z^+(\infty)$ order invariant, let $(x_1,...,x_k),(y_1,...,y_k) \in S$ be order equivalent, and assume $x_i \neq y_i \wedge x_j \geq x_i \wedge y_j \geq y_i$. Since $(x_1,...,x_k,...,x_k),(y_1,...,y_k,...,y_k) \in S'$, we have $x_j,y_j \in Z^+$. QED

We also have the following form of iMCC with dimensions.

INVARIANT MAXIMAL CLIQUE CHARACTERIZATION (dimensional). (Assuming ACA_0 + IMCT(extended)). Let J be an interval in Q[-∞,∞] and k ≥ 1. IMCT(J,k) holds if and only if

i. k = 1; or
ii. no positive integer is the smallest element of J; or
iii. J contains at most 2 positive integers.
In particular, IMCT(J,2) → IMCT(J).

Proof: Let J,k be as given. To prove IMCT(J,1), let G be an order invariant graph on J. If |J| ≤ 1 then ICTM(J,1) is trivial. Assume |J| ≥ 2. If G has no edges, use S = {p}, p ∈ J\Z+. If G is complete, use S = J. Hence IMCT(J,1).

Suppose ii or iii. By IMCC, we have IMCT(J), and so IMCT(J,k).

Now suppose i,ii,iii all fail. Then ii,iii both fail for J, and k ≥ 2. By Lemma 4.1.2, IMCT(J,2) fails. By Lemma 4.11, IMCT(J,k) fails. The final claim is now clear using IMCC. QED

THEOREM 4.1.12. The statement "for all intervals J ⊆ Q[-∞,∞], IMCT(J,2) → IMCT(J)" is not provable in SRP, provided SRP is consistent.

Proof: Suppose the statement in quotes is provable in SRP. By Theorem 4.2.8, SRP proves IMCT(J,2), for J = Q[-∞,∞]. Hence SRP proves IMCT(extended). By Theorem 5.9.1, SRP is inconsistent. QED

In section 4.2, we will prove IMCT(extended) from appropriate large cardinal hypotheses. Thus we will have a proof of the IMCC from appropriate large cardinal hypotheses - in particular, from SRP^+ or the weaker ACA_0 + Con(SRP).

4.2. IMCT(EXTENDED) USING LARGE CARDINAL HYPOTHESES.

In this section we prove the

INVARIANT MAXIMAL CLIQUE THEOREM (extended). Every order invariant graph on $Q[-\infty,\infty]^k$ has an upper $Z^+(\infty)$ order invariant maximal clique.

using certain large cardinal hypotheses.

DEFINITION 4.2.1. We use the abbreviation IMCT(extended,k) for IMCT(extended) in dimension k.

THEOREM 4.2.1. RCA_0 proves IMCT(extended,1).

Proof: Let G be an order invariant graph on $Q[-\infty,\infty]$. Then G $= \emptyset$ or $G = \{(x,y): x,y \in Q[-\infty,\infty] \wedge x \neq y\}$. In the first case, use $\{0\}$. In the second case use $Q[-\infty,\infty]$. QED

We now fix $k \geq 2$ and λ to be the least ordinal with the (k+1)-SRP. We derive IMCT(extended,k) in ZFC.

We focus on $\lambda \times Q[0,1)$ with the ordering

$$(\alpha,p) <_\lambda (\beta,q) \leftrightarrow \alpha < \beta \vee (\alpha = \beta \wedge p < q).$$

It is obvious that $(\lambda \times Q[0,1),<_\lambda)$ is a dense linear ordering with no endpoints. We follow the convention that $<_\lambda$ holds only if $x,y \in \lambda \times Q[0,1)$. We define $x \leq_\lambda y \leftrightarrow x <_\lambda y \vee x = y$.

For any set X, we think of X^k as the set of functions from $\{1,...,k\}$ into X.

DEFINITION 4.2.2. We define X^{k-} as the set of all partial functions from $\{1,...,k\}$ into X.

DEFINITION 4.2.3. We say that $f:S_k(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ is regressive if and only if for all $x \in S_k(\lambda)$ and $y \in f(x)$, the first coordinates of all elements of y are $< \min(x)$. We say that $E \subseteq \lambda$ is f-homogenous if for all $x,y \in S_k(E)$, $f(x \cap \min(x \cup y)) = f(y \cap \min(x \cup y))$.

DEFINITION 4.2.4. We say that $E \subseteq \lambda$ is pointed if and only if the limit ordinal $\max(E) \in E$ is the sup of $E \cap \max(E)$.

LEMMA 4.2.2. λ is the least k-ineffable ordinal, and a strongly inaccessible cardinal. Let $C \subseteq \lambda$ be closed and unbounded, and let $f_1,...,f_m:S_{k-1}(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ be regressive. There exists a pointed $E \subseteq C$, where for all i, E is f_i-homogenous.

Proof: λ is the least k-ineffable ordinal by Theorem 3.1. λ is a strongly inaccessible cardinal by [Fr01], Lemma 1.10 and that λ is a subtle ordinal. Let $C,f_1,...,f_n$ be as given. Set C to be the set of uncountable cardinals $< \lambda$, which is closed and unbounded by the first claim. Define $f:S_k(\lambda) \to S(\lambda)$ by $f(x) = <f_1(x),...,f_n(x)>$. Here we can use any convenient one-one map from $S((\lambda \times Q[0,1))^{k-})^m$ into $S(\lambda)$ which, for each uncountable cardinal $\kappa < \lambda$, maps $S((\kappa \times Q[0,1))^{k-})^m$ into $S(\kappa)$. Now f may not be regressive, but clearly f is regressive on $S_{k-1}(C)$. So we take the values of f to be the empty set, off of $S_{k-1}(C)$. By k-ineffability, let $D \subseteq \lambda$ be stationary, where D is f-homogenous. Then for all i, D is f_i-homogenous. The set of limits of elements of D is closed and unbounded in λ. Let $\mu \in D$ be a limit of elements of D. Take $E = D \cap \mu+1$. QED

LEMMA 4.2.3. Let $B \subseteq (\lambda \times Q[0,1))^k$. There exists pointed $E \subseteq \lambda$ such that B is upper $E \times \{0\}$ order invariant.

Proof: Let B be as given. We define finitely many functions $f:S_{k-1}(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ as follows. The $x \in (\lambda \times Q[0,1))^{k-}$ fall naturally into finitely many kinds. The kind of x is determined first by its domain, and second by the order type of the ordinal component of its coordinates, listed from left to right. Write these pairs as $\sigma_1,\ldots,\sigma_n$, without repetition.

We define $f_1,\ldots,f_n:S_k(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ as follows. Let $x \in S_k(\lambda)$. To evaluate $f_i(x)$, let $y \in \lambda^{k-}$ be unique with type σi, where the coordinates of y form an initial segment of the elements of x. Set $f_i(x) = \{z \in (\min(x) \times Q[0,1))^{k-}: \text{dom}(z) = \{1,\ldots,k\}\backslash\text{dom}(x) \wedge y \times \{0\} \cup z \in B\}$.

By Lemma 4.2.2, let $E \subseteq \lambda$ be pointed and f_i-homogeneous for all $1 \leq i \leq n$. Let $u,v \in (\lambda \times Q[0,1))^k$ be upper $E \times \{0\}$ order equivalent. We want to show that $u \in B \leftrightarrow v \in B$.

Let $u',v' \in \lambda^{k-}$ be the restriction of u,v, respectively, to its coordinates in $E \times \{0\}$, where all coordinates higher in $<_\lambda$, are in $E \times \{0\}$. Then u',v' arise in the evaluation of some $f_i(x),f_i(y)$, where $x,y \in S_{k-1}(E)$. Hence $u \in B \leftrightarrow v \in B$. QED

DEFINITION 4.2.5. For $\alpha < \lambda$, we define $<_\lambda\alpha = \alpha \times Q[0,1)$. We define $<_\lambda\alpha\hat{} = <_\lambda\alpha \cup \{(\alpha,0)\}$.

LEMMA 4.2.4. Let G be a graph on $(\lambda \times Q[0,1))^k$. There exists $S \subseteq V(G)$ such that for all $\alpha < \lambda$, $S \cap (<_\lambda\alpha\hat{})^k$ is a maximal clique in $G[(<_\lambda\alpha\hat{})^k]$.

Proof: Let G be as given. We build S by transfinite recursion along λ. Let S_0 be the maximal clique in $G[(<_\lambda 0\hat{})^k]$, which is $\{(0,0),\ldots,(0,0)\}$. Suppose S_α is a maximal clique in $G[(<_\lambda\alpha)^k]$. Take $S_{\alpha+1}$ to be a maximal clique in $G[(<_\lambda\alpha+1\hat{})^k]$ extending S_α. Now suppose that for all $\alpha < \gamma < \lambda$, S_α is a maximal clique in $G[(<_\lambda\alpha\hat{})^k]$, where γ is a limit ordinal, and we have for all $\beta < \delta < \gamma$, $S_\alpha \subseteq S_\beta$. Let $S_\gamma' = \cup_{\alpha<\lambda}S_\alpha$. Then S_γ' is a maximal clique in $G[<_\lambda\gamma]$. Let S_γ be the maximal clique in $G[(<_\lambda\gamma\hat{})^k]$ extending S_γ. Finally, define $S = \cup_\alpha<_\lambda S_\alpha$. QED

LEMMA 4.2.5. Let G be a graph on $(\lambda \times Q[0,1))^k$. There exists a pointed $E \subseteq \lambda$ and a maximal clique S in $G[(<_\lambda\max(E)\hat{})k]$ such that S is upper $E \times \{0\}$ order invariant.

Proof: Let G be as given. Let S be as given by Lemma 4.2.4. By Lemma 4.2.3, let $E \subseteq \lambda$ be pointed, where S is upper $E \times \{0\}$ order invariant. By Lemma 4.2.4, $S \cap (<_\lambda\max(E)\hat{})^k$ is as required. QED

LEMMA 4.2.6. Let G be an order invariant graph on $Q[-\infty,\infty]^k$. Suppose that there exists a dense linear ordering $(A,<_A)$ containing endpoints $-\infty,\infty$, and $\infty \in E \subseteq A$ is unbounded above in $A\backslash\{\infty\}$, where some maximal clique in $G\#(<_A)$ is upper E order invariant. Then some maximal clique in G is upper $Z^+(\infty)$ order invariant.

Proof: Let $G,A,<_A$ be as given. Let E and the maximal clique S* be as given.

We construct finite sets $A_1' \subseteq A_2' \subseteq \ldots \subseteq A$, and $x_1 < x_2 < \ldots \in E\backslash\{-\infty,\infty\}$ such that for all $i \geq 1$,

i. $x_i,-\infty,\infty \in A_i'$.
ii. for all $x < y$ from $A_i'\backslash\{-\infty,\infty|$, there exists $u < x < z < y < w$ from $A_{i+1}'\backslash\{-\infty,\infty|$.
iii. for all $v \in A_i'^k\backslash S^*$, there exists $w \in A_{i+1}'k \cap S^*$ such that v,w are not adjacent in $G\#(<_A)$.

It is obvious that this construction can be carried out sequentially, since A is dense, has endpoints $-\infty,\infty$, $E \subseteq A\backslash\{\infty\}$ is unbounded above in $A\backslash\{\infty\}$, and S* is a maximal clique in $G\#(<_A)$.

Set $A' = \cup_i A_i'$, $E' = \{-\infty,\infty,x_1,x_2,\ldots\}$, and $S' = S^* \cap A'^k$. Since $S' \subseteq S^*$, S' is a clique in $G\#(<_A)$.

It is clear that $(A',<_{A'})$ is dense, has endpoints $-\infty,\infty$. It is also clear that $E'\backslash\{\infty\}$ is unbounded in $A'\backslash\{\infty\}$.

Suppose $v \in A'^k\backslash S'$. Let $v \in A_i'^k$. By iii, let $w \in A_{i+1}'k \cap S^*$, v,w not adjacent in $G\#(<_A)$. Then $w \in S'$, v,w are not adjacent in $G\#(<_{A'})$.

Suppose v,w ∈ A'^k are upper E' order equivalent, and v ∈ S′. Since S* is upper E order invariant, we have v,w ∈ S*. Hence v,w ∈ S′.

Let h be any isomorphism from $(A',<',-\infty,\infty,x_1,x_2,...)$ onto $(Q,<,-\infty,\infty,1,2,...)$, which obviously exists. Since S' is a maximal clique in G#(<A'), we see that h[S'] is a maximal clique in G (where h acts on coordinates). Also since S' is upper E' order invariant, we see that h[S'] is upper $Z^+(\infty)$ order invariant. QED

THEOREM 4.2.7. The following is provable in ZFC. For all k ≥ 1, if there exists an ordinal with the (k+1)-SRP then the Invariant Maximal Clique Theorem (extended) holds for dimension k.

Proof: Note that the argument from Lemmas 4.2.1 - 4.2.5 used only that k ≥ 2, and λ is the least ordinal with the (k+1)-SRP. Let G be an order invariant graph on $Q[-\infty,\infty]^k$. Let E.S be as given by Lemma 4.2.5, using the graph $G\#(<_\lambda)$. To apply Lemma 4.2.6, we use $A = (<_\lambda max(E)^\wedge)^k$ under $<_\lambda$, with $-\infty = (0,0)$, and $\infty = (max(E),0)$, with E × {0} and S ∩ $(<_\lambda max(E)^\wedge)^k$. Now apply Lemma 4.2.6. The case k = 1 is by Lemma 4.2.1. QED

THEOREM 4.2.8. SRP proves IMCT(extended) for each fixed dimension k ≥ 1. SRP^+ proves IMCT(extended).

Proof: The first claim is immediate from Theorem 4.2.7. For the second claim, we can couch the proof above of Theorem 4.2.7 as a proof in SRP^+ by treating k as a variable, rather than as fixed. We know that λ exists because of SRP^+. QED

We now invoke some elementary logical considerations. These will allow us to pinpoint concrete aspects of the Invariant Maximal Clique Theorem (extended), and sharpen Theorems 4.2.7, 4.2.8.

Note that any order invariant graph G on $Q[-\infty,\infty]^k$ can be viewed as a finitary object, defined by quantifier free cases. In particular, adjacency in G of $(x_1,...,x_k)$ and $(y_1,...,y_k)$ can be written as a quantifier free formula in <.

Below, we use the most naïve notion of countable model of a sentence, in terms of relativization.

LEMMA 4.2.9. There is an effective procedure for creating a ∀...∀∃...∃ sentence $\gamma(G)$ of first order predicate calculus with equality (logic), where G is an order invariant graph on $Q[-\infty,\infty]^k$, such that the following is provable in RCA_0. (Here the vertex set $Q[-\infty,\infty]k$ is read off of G). For all order invariant graphs G on $Q[-\infty,\infty]^k$, the Invariant Maximal Clique Theorem (extended) for G holds if and only if $\gamma(G)$ has a countable model.

Proof: Fix order invariant G on $Q[-\infty,\infty]^k$. We use the constant symbols $-\infty,\infty$, the binary relation symbol <, the k-ary relation symbol S, and the 1-ary relation symbol T. $\gamma(G)$ is the conjunction of

i. < is a dense linear ordering with left endpoint $-\infty$, and right endpoint ∞.
ii. $T(\infty)$.
iii. $x < \infty \rightarrow (\exists y < \infty)(x < y \wedge T(y))$.
iv. $S(x_1,\ldots,x_k) \wedge S(y_1,\ldots,y_k) \rightarrow (x_1,\ldots,x_k),(y_1,\ldots,y_k)$ are adjacent in G.
v. $\neg S(x_1,\ldots,x_k) \rightarrow (\exists y_1,\ldots,y_k)(S(y_1,\ldots,y_k) \wedge$ $(x_1,\ldots,x_k),(y_1,\ldots,y_k)$ are not adjacent in G).
vi. Let $x_1,\ldots,x_k,y_1,\ldots,y_k$ be such that for all $1 \leq i \leq k$, $x_i \neq y_i \rightarrow$ every $x_j \geq x_i$ has $T \wedge$ every $y_j \geq y_i$ has T. Then $S(x_1,\ldots,x_k) \leftrightarrow S(y_1,\ldots,y_k)$.

$\gamma(G)$ is easily seen to be in class ∀...∀∃...∃.

Suppose that IMCT holds for G. Let S be a maximal clique in G that is upper $Z^+(\infty)$ order invariant. Then $\gamma(G)$ has the obvious countable model $(Q[-\infty,\infty],-\infty,\infty,<,S,Z^+(\infty))$.

Suppose that $\gamma(G)$ has a countable model $M = (A,-\infty,\infty,<,S,T)$. Let $a_0 < a_1 < \ldots < \infty$ be an infinite unbounded sequence from T. Let $h:A \rightarrow Q[-\infty,\infty]$ be an order preserving bijection mapping each a_i to i. Then h[S] is a maximal clique in G that is upper $Z^+(\infty)$ order invariant, where h acts on coordinates.

The second claim follows immediately from the proof of the first. QED

LEMMA 4.2.10. There is a Π^0_1 formula P(k) with only the free variable k, such that the following holds.
i. WKL_0 proves $(\forall k)(P(k) \rightarrow IMCT(extended,k))$.
ii. RCA_0 proves $(\forall k)(IMCT(extended,k) \rightarrow P(k))$.
iii. WKL_0 proves $IMCT(extended) \leftrightarrow (\forall k)(P(k))$.

Proof: By Lemma 4.2.9, RCA_0 proves

IMCT(extended,k) ↔ (∀ order invariant graphs G on $Q[-\infty,\infty]^k$)(γ(G) has a countable model).

We claim that WKL_0 proves

(∀ order invariant graphs G on $Q[-\infty,\infty]^k$)(γ(G) is formally consistent) → IMCT(extended,k).

We argue in WKL_0. Assume the left side. Then (∀ order invariant graphs G on $Q[-\infty.\infty]^k$)(γ(k,G) has a countable model) using the familiar argument for completeness (without soundness) for predicate calculus, that is readily formalized in WKL_0.

We now claim that RCA_0 proves

IMCT(extended,k) → (∀ order invariant graphs G on $Q[-\infty,\infty]^k$)(γ(G) is formally consistent).

We argue in RCA_0. Assume the left side. Let k ≥ 1 and G be an order invariant graph on $Q[-\infty,\infty]^k$. Then γ(G) has a countable model. We would like to conclude that γ(G) is formally consistent. There is a standard argument using cut elimination that relies on an induction that may not be available in RCA_0.

Recall that the sentence γ(G) is ∀...∀∃...∃ in <,-∞,∞,S,T, where the number of quantifiers depends on k only. Add Skolem functions so that the theory is put into purely universal form. By the Invariant Maximal Clique Theorem (extended), this purely universal theory has a model by interpreting the Skolem functions in the obvious way (which only relies on RCA_0). Now if there is an inconsistency in γ(G) then there is an inconsistency in this universal form, and we can apply Herbrand's theorem to get a quantifier free inconsistency. But this is impossible since a truth definition for quantifier free formulas can be given for this model, within RCA_0, using a relative primitive recursion.

Claim iii follows immediately from claims i,ii. QED

We now prove IMCT(extended) using only Con(SRP). For k ≥ 1, let k* be the numeral for k.

THEOREM 4.2.11. The following are provable in WKL_0.

i. $(\forall k \geq 1)$(Con(ZFC + "there exists an ordinal with the (k*+1)-SRP") → IMCT(extended,k).
ii. Con(SRP) → IMCT(extended).

Proof: We argue in WKL_0. Let $k \geq 1$, and assume Con(ZFC + "there exists an ordinal with the (k*+1)-SRP"). Suppose the IMCT(extended,k) is false. By Lemma 4.2.10, ¬P(k). Hence RCA0 proves ¬P(k*). By Lemma 4.2.10, RCA_0 refutes IMCT(extended,k*). By Theorem 4.2.7, ZFC + "there exists an ordinal with the (k*+1)-SRP" is inconsistent. This is a contradiction. The second claim follows immediately from the first. QED

We can draw the following interesting recursion theoretic consequence of Lemma 4.2.9.

THEOREM 4.2.12. The following is provable in ACA_0.
i. If IMCT(extended) holds then the IMCT(extended) holds with Δ^0_2 upper $Z^+(\infty)$ invariant maximal cliques. In fact, there is a Δ^0_2 sequence of upper $Z^+(\infty)$ maximal cliques enumerated by the relevant order invariant G.
ii. If Con(SRP) then the conclusions of i hold.

Proof: We use the sentences $\gamma(G)$ constructed in Lemma 4.2.9. By a standard recursion theoretic model construction, we can build a sequence of models of the $\gamma(G)$, indexed by order invariant G, where the sequence is Δ^0_2, and we can even use complete diagrams. These can be uniformly relatively effectively isomorphed onto maximal cliques in the G's on $Q[-\infty,\infty]^k$ that are upper Z+(∞) order invariant. The result is a sequence of maximal cliques recursive in the Turing jump. The second claim follows from the first and Theorem 4.2.11 ii. QED

4.3. IMCT, IMCT(Q) USING LARGE CARDINAL HYPOTHESES.

According to Theorem 4.1.4, IMCT and IMCT(Q) are derivable from IMCT(extended), in RCA_0. Hence by Theorem 4.2.12, IMCT and IMCT(Q) are provable in WKL_0 + Con(SRP).

In section 4.2, we fixed the dimension $k \geq 1$, and derived IMCT(extended,k) using the existence of an ordinal with the k-SRP.

We derive IMCT(Q,k) using only the weaker assumption of the existence of a (k-1)-subtle ordinal. We follow the convention that a 0-subtle ordinal as any limit ordinal.

DEFINITION 4.3.1. IMCT(k) is IMCT on the $[0,n]^k$. I.e., for all n, every order invariant graph on $[0,n]^k$ has an upper {1,...,n} order invariant maximal clique.

We derive IMCT(k) also using a (k-1)-subtle ordinal. We also derive IMCT([0,n]) using an (n-1)-subtle ordinal.

DEFINITION 4.3.2. IMCT(k,n) is IMCT on $[0,n]^k$. I.e., every order invariant graph on $[0,n]^k$ has an upper {1,...,n} order invariant maximal clique.

We then combine these last two results to give a derivation of IMCT(k,n) using a min(k-1,n-1)-subtle ordinal.

Note that here we can replace $Z^+(\infty)$ by {1,...,n} without change. We do not claim any optimality of these results concerning IMCT(extended,k), IMCT(Q,k), IMCT(k), and IMCT(k,n). Our results are, however, optimal for IMCT(extended) and IMCT. The equivalence, over ACA', of IMCT(extended) and IMCT with Con(SRP), is established in section 5.9.

In this paper, we show that for large k,n, IMCT(k,n) is not provable (assuming ZFC is consistent). See Theorems 5.8.39 and 5.9.3. In particular, k,n ≥ 2↑10 is sufficient, where 2↑10 is the exponential stack of 10 2's.

We expect that an elaboration of the methods of this paper should provide

i. small k such that IMCT(k) is provable in SRP^+ but not in ZFC (assuming ZFC is consistent).
ii. small n such that IMCT([0,n]) is provable in SRP^+ but not in ZFC (assuming ZFC is consistent).
iii. even small k and small n such that IMCT(k,n) is provable in SRP^+ but not in ZFC (assuming ZFC is consistent).

For example, we suspect that perhaps, say, IMCT(4,4), which reads

> Every order invariant graph on $Q[0,4]^4$ has an upper {1,2,3,4} order invariant maximal clique.

is provable in SRP^+ but not in ZFC (assuming ZFC is consistent). We conjecture that IMCT(2,2) is provable in RCA_0. Such results are beyond the scope of this paper.

We hope that the provability computations in this section should provide some useful guidance for this prospective investigation.

We now prove IMCT(Q,k) and IMCT(Q). We follow the treatment given in section 4.2 for IMCT(extended), making modifications as needed. We freely use notation and terminology from section 4.2.

THEOREM 4.3.1. RCA_0 proves IMCT(Q,1).

Proof: Let G be an order invariant graph on Q. Then $G = \emptyset$ or $G = \{(x,y): x,y \in Q \wedge x \neq y\}$. In the first case, use {0}. In the second case use Q. QED

We now fix $k \geq 2$ and λ to be the least (k-1)-subtle ordinal. This is smaller than the least ordinal with the k-SRP. Recall that the least ordinal with the (k+1)-SRP was used in section 4.2, which is the least k-ineffable ordinal, which is already larger than the least k-subtle ordinal. We derive IMCT(Q,k).

In section 4.2, we used the linear ordering $<_\lambda$ on $(\lambda \times Q[0,1))^k$.

DEFINITION 4.3.1. Because Q does not contain its left endpoint, it is convenient to now use $<_\lambda'$ on $(\lambda \times' Q[0,1)^k$, where $\lambda \times' Q[0,1)$ is $\lambda \times Q[0,1)$ without the left endpoint (0,0). This causes no difficulties. We also define $<_\lambda'\alpha = \alpha \times' Q[0,1)$, and for $0 < \alpha < \lambda$, $<_\lambda'\alpha\hat{} = <_\lambda'\alpha \cup \{(\alpha,0)\}$.

LEMMA 4.3.2. λ is a strongly inaccessible cardinal. Let $C \subseteq \lambda$ be closed and unbounded, and let $f_1,\ldots,f_m:S_{k-1}(\lambda) \rightarrow S((\lambda \times' Q[0,1))^{k-})$ be regressive. There exists $E \subseteq C$ of order type ω, where for all i, E is f_i-homogenous.

Proof: We adapt the proof of Lemma 4.2.2. λ is a strongly inaccessible cardinal by [Fr01], Lemma 1.10. Let $C,f_1,\ldots,f_m$ be as given. Set C to be the set of uncountable cardinals < λ, which is closed and unbounded by the first claim. Define $f:S_{k-1}(\lambda) \rightarrow S(\lambda)$ by $f(x) = <f_1(x),\ldots,f_m(x)>$. Here we can use any convenient one-one map from $S((\lambda \times' Q[0,1))^{k-})^m$ into $S(\lambda)$ which, for each uncountable cardinal $\kappa < \lambda$, maps $S((\kappa \times' Q[0,1))^{k-})^m$ into $S(\kappa)$. Now f may not be regressive, but clearly f is regressive on $S_{k-1}(C)$. So we take the values of f to be the empty set, off of $S_{k-1}(C)$. According to [Fr01], Lemma 1.6, there exists f-homogenous $E \subseteq C$ of order type ω. Then for all i, E is f_i-homogenous. QED

LEMMA 4.3.3. Let $B \subseteq (\lambda \times' Q[0,1))^k$. There exists $E \subseteq \lambda$ of order type ω such that B is upper $E \times \{0\}$ order invariant.

Proof: Let B be as given. We define finitely many functions $f:S_{k-1}(\lambda) \to S((\lambda \times' Q[0,1))^{k-})$ as follows. The $x \in (\lambda \times' \{0\})^{k-}$ with at most k-1 distinct coordinates, fall naturally into finitely many kinds. The kind of x is determined first by its domain, and second by the order type of the ordinal component of its coordinates, listed from left to right. Write these pairs as $\sigma_1,\ldots,\sigma_m$, without repetition.

We define $f_1,\ldots,f_n:S_{k-1}(\lambda) \to S((\lambda \times' Q[0,1))^{k-})$ as follows. Let $x \in S_{k-1}(\lambda)$. To evaluate $f_i(x)$, let $y \in \lambda^{k-}$ be unique with type σi, where the coordinates of y form an initial segment of the elements of x. Set $f_i(x) = \{z \in (\min(x) \times Q[0,1))^{k-}: \mathrm{dom}(z) = \{1,\ldots,k\}\backslash\mathrm{dom}(x) \wedge y \times \{0\} \cup z \in B\}$.

By Lemma 4.3.2, let $E \subseteq \lambda$ be of order type ω and f_i-homogeneous for all $1 \leq i \leq n$. We can assume that $0 \notin E$. Let $u,v \in (\lambda \times' Q[0,1))^k$ be upper $E \times \{0\}$ order equivalent, where $u,v \notin (E \times \{0\})^k$. We claim that $u \in B \leftrightarrow v \in B$.

Let $u' \in \lambda^{k-}$ be the restriction of u to its coordinates in $E \times \{0\}$, where all coordinates higher in $<_\lambda'$, are in $E \times \{0\}$. Let $v' \in \lambda^{k-}$ be the restriction of u to its coordinates in $E \times \{0\}$, where all coordinates higher in $<_\lambda'$, are in $E \times \{0\}$. Then u',v' arise in the evaluation of some $f_i(x),f_i(y)$, where $x,y \in S_{k-1}(E)$. Hence $u \in B \leftrightarrow v \in B$.

It remains to show that for order equivalent $u,v \in (E \times \{0\})^k$, $u \in B \leftrightarrow v \in B$. However, this may not be the case. But we can use the ordinary infinite Ramsey theorem to replace E by a suitable subset of E of order type ω. QED

LEMMA 4.3.4. Let G be a graph on $(\lambda \times' Q[0,1))^k$. There exists $S \subseteq V(G)$ such that for all $\alpha < \lambda$, $S \cap (<_\lambda'\alpha)^k$ is a maximal clique in $G[<_\lambda'\alpha]$.

Proof: See the proof of Lemma 4.2.4. We do not use the $<_\lambda\alpha$^ or $<_\lambda'\alpha$^ here. QED

LEMMA 4.3.5. Let G be a graph on $(\lambda \times' Q[0,1))^k$. There exists $E \subseteq \lambda$ of order type ω and a maximal clique S in $G[(<_{\lambda'}\sup(E))^k]$ such that S is upper $E \times \{0\}$ order invariant.

Proof: Let G be as given. Let S be as given by Lemma 4.3.4. By Lemma 4.3.3, let $E \subseteq \lambda$ be of order type ω, where S is upper $E \times \{0\}$ order invariant. QED

LEMMA 4.3.6. Let G be an order invariant graph on Q^k. Suppose that there exists a dense linear ordering $(A,<_A)$, with no endpoints, and $E \subseteq A$ with no upper bound, where some maximal clique in $G\#(<_A)$ is upper E order invariant. Then some maximal clique in G is upper $Z^+(\infty)$ order invariant.

Proof: Let $G,A,<_A$ be as given. Let E and the maximal clique S* be as given.

We construct finite sets $A_1' \subseteq A_2' \subseteq \ldots \subseteq A$, and $x_1 < x_2 < \ldots \in E$ such that for all $i \geq 1$,

i. $x_i \in A_i'$.
ii. for all $x < y$ from A_i', there exists $u < x < z < y < w$ from A_{i+1}'.
iii. for all $v \in A_i'^k \backslash S^*$, there exists $w \in A_{i+1}'k \cap S^*$ such that v,w are not adjacent in $G\#(<_A)$.

It is obvious that this construction can be carried out sequentially, since A is dense, has no endpoints, $E \subseteq A$ is unbounded above in A, and S* is a maximal clique in $G\#(<_A)$.

Set $A' = \cup_i A_i'$, $E' = \{x_1,x_2,\ldots\}$, and $S' = S^* \cap A'^k$. Clearly S' is a clique in $G\#(<_{A'})$.

It is clear that $(A',<_{A'})$ is dense, and has no endpoints. It is also clear that E' is unbounded above in A'.

Suppose $v \in A'^k \backslash S'$. Let $v \in A_i'^k$. By iii, let $w \in A_{i+1}'k \cap S^*$, v,w not adjacent in $G\#(<_{A'})$. Then $w \in S'$, v,w are not adjacent in $G\#(<_{A'})$.

Suppose $v,w \in A'^k$ are upper E' order equivalent, and $v \in S'$. Since S* is upper E order invariant, we have $v,w \in S^*$. Hence $v,w \in S'$.

Let h be any isomorphism from $(A',<',x_1,x_2,\ldots)$ onto $(Q,<,1,2,\ldots)$, which obviously exists. Since S' is a maximal clique in $G\#(<_{A'})$, we see that h[S'] is a maximal clique in G (where h acts on coordinates). Also since S' is upper E' order invariant, we see that h[S'] is upper $Z^+(\infty)$ order invariant. QED

THEOREM 4.3.7. The following is provable in ZFC. For all $k \geq 1$, if there exists a (k-1)-subtle ordinal then IMCT(Q,k) holds.

Proof: Note that the argument from Lemmas 4.3.1 - 4.3.6 used only that $k \geq 2$, and λ is the least (k-1)-subtle ordinal. Let G be an order invariant graph on Q^k. Let E,S be as given by Lemma 4.2.5, using the graph $G\#(<_\lambda')$. To apply Lemma 4.2.6, we use $A = <_\lambda'\text{sup}(E)$ under $<_\lambda'$, with $E \times \{0\}$ and $S \cap (<_\lambda'\text{sup}(E))^k$. Now apply Lemma 4.2.6. The case $k = 1$ is by Lemma 4.2.1. QED

THEOREM 4.3.8. SRP proves IMCT(Q,k) for each fixed k. SRP^+, and WKL_0 + Con(SRP) prove IMCT(Q).

Proof: This is already clear from Theorem 4.1.4, where we derived IMCT(Q) from IMCT(extended) within RCA0, together with Theorem 4.2.12. QED

We do not know how to carry out a rewriting of IMCT(Q) in terms of satisfiability in predicate calculus, as we did in Lemmas 4.2.9, 4.2.10 for IMCT(extended). In fact, we do not know if IMCT(Q) is provably equivalent to a Π^0_1 sentence over ZFC.

We now treat IMCT(k).

THEOREM 4.3.9. RCA_0 proves IMCT(1).

Proof: See the proof of Theorem 4.3.1. QED

Let $k \geq 2$, and λ be the least (k-1)-subtle ordinal.

LEMMA 4.3.10. Let G be a graph on $(\lambda \times Q[0,1))^k$. There exists $E \subseteq \lambda$, $|E| = n$, and a maximal clique S in $G[(<_\lambda\text{max}(E)\hat{})k]$ such that S is upper $E \times \{0\}$ order invariant.

Proof: Lemma 4.3.2 holds with $\times'$ replaced by $\times$, with the same argument. Lemma 4.3.3 holds with $\times'$ replaced by $\times$, although we will need only $|E| = n$, again with the same argument. Instead of using Lemma 4.3.4, we use Lemma 4.2.4 without change. We then obtain the required E, here, in analogy with Lemmas 4.2.5 and 4.3.5. QED

LEMMA 4.3.11. Let G be an order invariant graph on $[0,n]^k$, $n \geq 1$. Suppose that there exists a dense linear ordering $(A,<_A)$ containing endpoints $-\infty,\infty$, and $\infty \in E \subseteq A$ has cardinality n, where some maximal clique in $G\#(<_A)$ is upper E order invariant. Then some maximal clique in G is upper {1,...,n} order invariant.

Proof: Let $G,A,<_A$ be as given. Let E and the maximal clique S* be as given.

We construct finite sets $A_1' \subseteq A_2' \subseteq \ldots \subseteq A$ such that for all $i \geq 1$,

i. $E \cup \{-\infty,\infty\} \subseteq A_1'$.
ii. for all $x < y$ from $A_i'\backslash\{-\infty,\infty\}$, there exists $u < x < z < y < w$ from $A_{i+1}'\backslash\{-\infty,\infty\}$.
iii. for all $v \in A_i'^k\backslash S^*$, there exists $w \in A_{i+1}'k \cap S^*$ such that v,w are not adjacent in $G\#(<_A)$.

It is obvious that this construction can be carried out sequentially, since A is dense with endpoints $-\infty,\infty$, $\infty \in E \subseteq A$ has cardinality n, and S* is a maximal clique in $G\#(<_A)$.

Set $A' = \cup_i A_i'$, and $S' = S^* \cap A'^k$. Since $S' \subseteq S^*$, S' is a clique in $G\#(<_A)$.

It is clear that $(A',<_{A'})$ is dense, with endpoints $-\infty,\infty$.

Suppose $v \in A'^k\backslash S'$. Let $v \in A_i'^k$. By iii, let $w \in A_{i+1}'^k \cap S^*$, v,w not adjacent in $G\#(<_A)$. Then $w \in S'$, v,w are not adjacent in $G\#(<_{A'})$.

Suppose $v,w \in A'^k$ are upper E' order equivalent, and $v \in S'$. Since S* is upper E order invariant, we have $v,w \in S^*$. Hence $v,w \in S'$.

Let h be any isomorphism from $(A',<',-\infty,x_1,x_2,\ldots,x_n,\infty)$ onto $([0,n],<,0,1,2,\ldots,n)$, which obviously exists. Since S' is a maximal clique in $G\#[<_{A'}]$, we see that h[S'] is a maximal clique in G (where h acts on coordinates). Also since S' is upper E order invariant, we see that h[S'] is upper {1,...,n} order invariant. QED

THEOREM 4.3.12. The following is provable in ZFC. For all $k \geq 1$, if there exists a (k-1)-subtle ordinal then the IMCT(k) holds.

Proof: Note that the argument starting from Lemma 4.3.10 used only that $k \geq 2$, and λ is the least (k-1)-subtle ordinal. Let G be an order invariant graph on Q^k. Let E,S be as given by Lemma 4.3.10, using the graph $G\#(<_\lambda')$. To apply

Lemma 4.11, we use A = $<_\lambda$max(E)^ under $<_\lambda$, with E × {0} and S ∩ ($<_\lambda$max(E)^)^k. Now apply Lemma 4.3.10. The case k = 1 is by Lemma 4.3.9. QED

We now come to IMCT([0,n]), n ≥ 1. By Lemma 4.1.8, RCA_0 proves IMCT([0,1]).

Let n ≥ 2, and λ be the least (n-1)-subtle ordinal. Let k ≥ 1. We will prove IMCT([0,n]) in dimension k.

LEMMA 4.3.13. λ is a strongly inaccessible cardinal. Let C ⊆ λ be closed and unbounded, and let $f_1,\ldots,f_m:S_{n-1}(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ be regressive. There exists E ⊆ C of order type ω, where for all i, E is f_i-homogenous.

Proof: By Lemma 4.2.2. QED

LEMMA 4.3.14. Let B ⊆ $(\lambda \times Q[0,1))^k$. There exists E ⊆ λ, |E| = n, such that B is upper E × {0} order invariant.

Proof: Since λ is only (n-1)-subtle, we cannot get E of, say, order type ω here. Let B be as given. We define finitely many functions $f:S_{n-1}(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ as follows. The x ∈ $(\lambda \times Q[0,1))^{k-}$ fall naturally into finitely many kinds. The kind of x is determined first by its domain, and second by the order type of the ordinal component of its coordinates, listed from left to right. Write these pairs as $\sigma_1,\ldots,\sigma_m$, without repetition.

We define $f_1,\ldots,f_m:S_{n-1}(\lambda) \to S((\lambda \times Q[0,1))^{k-})$ as follows. Let x ∈ $S_{n-1}(\lambda)$. To evaluate $f_i(x)$, let y ∈ λ^{k-} be unique with type σ_i, where the coordinates of y form an initial segment of the elements of x. Set $f_i(x) = \{z \in (\min(x) \times Q[0,1))^{k-}: \text{dom}(z) = \{1,\ldots,k\}\backslash\text{dom}(x) \wedge y \times \{0\} \cup z \in B\}$.

By Lemma 4.3.13, let E ⊆ λ have cardinality n and be f_i-homogeneous for all 1 ≤ i ≤ n. Let u,v ∈ $(\lambda \times Q[0,1))^k$ be upper E × {0} order equivalent. We want to show that u ∈ B ↔ v ∈ B.

Let u',v' ∈ λ^{k-} be the restriction of u,v, respectively, to its coordinates in E × {0}, where all coordinates higher in $<_\lambda$, are in E × {0}. If the number of distinct coordinates in u' (or v') is n, then they must comprise all of E × {0}, in which case by the upper E × {0} order equivalence, they must comprise all of E × {0} for v' (or u'). It follows that u' = v'. So we can assume that the number of distinct coordinates in u',v', respectively, is less than n. Hence u',v' arise in the evaluation of some $f_i(x),f_i(y)$, where $x,y \in S_{n-1}(E)$. Hence $u \in B \leftrightarrow v \in B$. QED

LEMMA 4.3.15. Let G be a graph on $(\lambda \times Q[0,1))^k$. There exists $E \subseteq \lambda$, $|E| = n$, and maximal clique S in $G[(<_\lambda max(E)^\wedge)^k]$ such that S is upper $E \times \{0\}$ order invariant.

Proof: By Lemmas 4.3.14 and 4.2.4. Lemma 4.2.4 is true for arbitrarily ordinals λ. QED

THEOREM 4.3.16. The following is provable in ZFC. For all $n \geq 1$, if there exists an (n-1)-subtle ordinal then the IMCT([0,n]) holds.

Proof: Note that the argument from Lemmas 4.3.13 - 4.3.15 used only that $n \geq 2$, and λ is the least (n-1)-subtle ordinal. Let G be an order invariant graph on $Q[0,n]^k$. Let E.S be as given by Lemma 4.3.15, using the graph $G\#(<_\lambda)$. To apply Lemma 4.3.11, we use $A = <_\lambda max(E)^\wedge$ under $<_\lambda$, with $-\infty = (0,0)$, and $\infty = (max(E),0)$, with $E \times \{0\}$ and $S \cap <_\lambda max(E)^\wedge$. Now apply Lemma 4.3.11. The case k = 1 is by Lemma 4.1.8. QED

We now combine the treatments of IMCT(k) and IMCT([0,n]).

THEOREM 4.3.17. The following is provable in ZFC. For all $n,k \geq 1$, if there exists a min(k-1,n-1)-subtle ordinal then IMCT(k,n) holds.

Proof: Immediate from Theorems 4.3.12 and 4.3.16. QED

We now represent IMCT(k,n) as the satisfiability of a sentence in predicate calculus, as was done for IMCT(extended,k) by Lemma 4.2.9.

LEMMA 4.3.18. There is an effective procedure for creating a $\forall\ldots\forall\exists\ldots\exists$ sentence $\delta(G)$ of first order predicate calculus with equality (logic), where G is an order invariant graph on $Q[0,n]^k$, such that the following is provable in RCA_0. (Here the vertex set $Q[0,n]^k$ is read off of G). For all order invariant graphs G on $Q[0,n]^k$, the Invariant Maximal Clique Theorem for G holds if and only if $\gamma(G)$ has a countable model.

Proof: Fix order invariant G on $Q[0,n]^k$. We use the constant symbols 0,1,...,n, the binary relation symbol <, and the k-ary relation symbol S. $\gamma(G)$ is the conjunction of

i. < is a dense linear ordering with left endpoint 0, and right endpoint n.
ii. 0 < 1 < ... < n.
iii. $S(x_1,...,x_k) \wedge S(y_1,...,y_k) \rightarrow (x_1,...,x_k),(y_1,...,y_k)$ are adjacent in G.
iv. $\neg S(x_1,...,x_k) \rightarrow (\exists y_1,...,y_k)(S(y_1,...,y_k) \wedge (x_1,...,x_k),(y_1,...,y_k)$ are not adjacent in G).
v. Let $x_1,...,x_k,y_1,...,y_k$ be such that for all $1 \leq i \leq k$, $x_i \neq y_i \rightarrow$ every $x_j \geq x_i$ is among $1,...,n \wedge$ every $y_j \geq y_i$ among $1,...,n$. Then $S(x_1,...,x_k) \leftrightarrow S(y_1,...,y_k)$.

$\gamma(G)$ is easily seen to be in class $\forall...\forall\exists...\exists$.

Suppose that IMCT holds for G. Let S be a maximal clique in G that is upper 1,...,n order invariant. Then $\gamma(G)$ has the obvious countable model (Q[0,n],<,S,0,1,...,n).

Suppose that $\delta(G)$ has a countable model M = (A,<,S,0',...,n'). Let h:A $\rightarrow$ Q[0,n] be an order preserving bijection mapping each i' to i. Then h[S] is a maximal clique in G that is upper {1,...,n} order invariant, where h acts on coordinates. QED

LEMMA 4.3.19. There is a Π^0_1 formula P(k,n) with only the free variables k,n such that the following holds.
i. WKL0 proves $(\forall k,n)(P(k,n) \rightarrow \text{IMCT}(k,n))$.
ii. RCA0 proves $(\forall k,n)(\text{IMCT}(k,n) \rightarrow P(k,n))$.
iii. WKL0 proves IMCT $\leftrightarrow (\forall k,n)(P(k,n))$.

Proof: See the proof of Lemma 4.2.10. QED

We now prove IMCT using only Con(SRP). For $k \geq 1$, let k* be the numeral for k.

THEOREM 4.3.20. The following are provable in WKL_0.
i. For all $k \geq 1$, Con(ZFC + "there exists a (k*-1)-subtle ordinal") $\rightarrow$ IMCT(k).
ii. For all $n \geq 1$, Con(ZFC + "there exists a (n*-1)-subtle ordinal") $\rightarrow$ IMCT([0,n]).
iii. For all $k,n \geq 1$, Con(ZFC + "there exists a min(k*-1,n*-1)-subtle ordinal") $\rightarrow$ IMCT(k,n).
iv. Con(SRP) $\rightarrow$ IMCT.

Proof: See the proof of Theorem 4.2.11. QED

We can draw the following interesting recursion theoretic consequence of Lemma 4.3.18.

THEOREM 4.3.21. The following is provable in ACA_0.
i. If IMCT then IMCT holds with Δ^0_2 maximal cliques. In fact, there is a Δ^0_2 sequence of upper Z^+ order invariant maximal cliques enumerated by the relevant order invariant G.
ii. If Con(SRP) then the conclusions of i holds.
iii. If Con(SRP) then the conclusions of i hold for IMCT(Q).

Proof: For i,ii, see the proof of Theorem 4.2.12. For iii, we use Theorem 4.2.12, and the proof of Theorem 4.14 that RCA_0 proves IMCT(extended) → IMCT(Q). There we proved IMCT(Q) for a given order invariant graph G on some Q^k based on IMCT(extended) for an effectively obtained graph G' on $Q[-\infty,\infty]^{k+2}$, where we can obtain an upper Z^+ invariant maximal clique in G uniformly effectively from any upper $Z^+(\infty)$ invariant maximal clique in G'. This establishes iii using Theorem 4.2.12. QED

5. REVERSAL OF THE INVARIANT MAXIMAL CLIQUE THEOREMS.

In this section 5, we reverse IMCT.

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant maximal clique.

In light of Theorem 4.1.4, this also reverses the Invariant Maximal Clique Theorem (extended).

DEFINITION 5.1. For positive integers p, define k_p to be the floor of loglog...logk, where there are p log's. This may be undefined, since we may get to a negative number or zero before finishing the evaluation. We define $2\uparrow p$ to be an exponential stack of p 2's, $p \geq 1$.

DEFINITION 5.2. For positive integers r, we define SRP_r = ZFC + "there exists an ordinal with the r-SRP".

In sections 5.1 - 5.8, we fix positive integers $n \geq k$ such that $k_6 \geq 10^5$, and assume IMCT(k,n). We construct a model of Con(SRP[$\lfloor k_6/10^4 \rfloor$-8)) over the base theory ZFC. In section 5.9, we show how to reduce the base theory ZFC to ACA'.

As a consequence, we see that ACA' proves IMCT → Con(SRP). When combined with Theorem 4.2.11 and 4.3.20, we obtain the equivalence of IMCT and IMCT(extended) with Con(SRP), over ACA'.

In particular, we have the unprovability of IMCT(2↑10,2↑10) in ZFC, assuming that ZFC is consistent, via Gödel's second incompleteness theorem.

Instead of assuming IMCT(k,n), we could instead assume IMCT(k,k), in light of the following monotonicity. However, the roles played by the dimension k and the endpoint n (of [0,n]), are conceptually different, that it is better to retain both k and n. In addition, we anticipate future results concerning the strength of IMCT(k,n) for various k,n.

THEOREM 5.1. The following is provable in RCA_0. Let k' ≥ k and n' ≥ n. Then IMCT(k',n') → IMCT(k,n).

Proof: Assume IMCT(k',n'). By Theorem 4.1.11, IMCT(k,n'). By Theorem 4.1.1, IMCT(k,n). QED

In this paper, we make no attempt to obtain sharp results concerning the strength of the various IMCT(k,n). It may be that IMCT(4,4), or even IMCT(3,3) are already unprovable in ZFC (assuming ZFC is consistent). We conjecture that IMCT(2,2) is provable in RCA_0.

5.1. THE GRAPH G(k,n).

Unless otherwise indicated, we use i,j,k,n,m,r for positive integers, p,q for elements of Q, and x,y,z for elements of Q^k.

We now fix integers n ≥ k so that $k_6 \geq 10^5$. Note that as a consequence, k,n are very large integers.

DEFINITION 5.1.1. For $x \in Q^k$, we write x = (x[1],...,x[k]), and |x| for the maximum coordinate of x. If $x_1,...,x_p \in Q^k$, we write $|x_1,...,x_p| = \max(|x_1|,...,|x_p|)$.

DEFINITION 5.1.2. We use $p^{<j>}$ for p,...,p, where there are j p's, j ≥ 1.

DEFINITION 5.1.3. We define W = {x ∈ $Q[0,n]^k$: x[5],...,x[k] < x[1] < x[2] < x[3] ∧ x[4] < x[2]}. The coordinate x[3] plays a technical role related to details in section 5.3.

DEFINITION 5.1.4. We let $G(k,n)$ be the unique order invariant graph $(Q[0,n]^k,E)$ such that the following conditions hold. We will prove below that $G(k,n)$ does in fact exist and is unique.

Let $x,y \in Q[0,n]^k$. Conditions 2-6,8,9 are used in section 5.2. Condition 1 is used in section 5.3. Condition 7 is used in section 5.4.

Condition 1. Suppose $x,y \in W \wedge x \neq y$. Then $x \mathrel{E} y$ if and only if

i. $x[1] = y[1] \wedge x[2] = y[2] \wedge x[3] = y[3] \wedge x[5] = y[5] \wedge \ldots \wedge x[k] = y[k] \rightarrow x[4] = y[4]$; and
ii. $x[1] = y[1] \wedge x[2] = y[2] \wedge x[3] = y[3] \wedge x[4] = y[4] \rightarrow x[5] = y[5] \wedge \ldots \wedge x[k] = y[k]$.

Condition 2. Suppose $|x| > |y|$, and x is of the form $(p,\ldots,p)$. Then $x \mathrel{E} y$.

Condition 3. Suppose $|x| > |y|$, and x is of the form $(x[1],x[2],x[3],\ldots,x[3])$, $x[1] > x[2],x[3]$. Then $x \mathrel{E} y \leftrightarrow x[2] < x[3]$.

Condition 4. Suppose $|x| > |y|$, and x is of the form $(p^{<2>},z[1],\ldots,z[r],p,z[r],\ldots,z[r])$, $r \leq k-3$, all z's $< p$. Then $x \mathrel{E} y \leftrightarrow y \neq (z[1],\ldots,z[r]],\ldots,z[r])$.

Condition 5. Suppose $|x| > |y|$, and x is of the form $(p^{<3>},z[1],\ldots,z[r],p,w[1],\ldots,w[r],\ldots,w[r])$, $2r+4 \leq k$, all z's and w's $< p$. Then $x \mathrel{E} y \leftrightarrow y \neq (z[1],\ldots,z[r],\ldots,z[r]) \wedge y \neq (w[1],\ldots,w[r])$.

Condition 6. Suppose $|x| > |y|$, and x is of the form $(p^{<4>},z[1],\ldots,z[i-1],p,z[i+1]\ldots,z[r],p,z[j],\ldots,z[j])$, $1 \leq i \neq j \leq r \leq k-6$, all z's $< p$. Then $x \mathrel{E} y \leftrightarrow y$ is not among the $(z[1],\ldots,z[i-1],q,z[i+1],\ldots,z[r],\ldots,z[r])$, where $q < z[j]$.

NOTE: The fifth p replaces $z[i]$ here. In the two extreme cases, the fifth p replaces $z[1]$, or the fifth p replaces $z[\log k]$.

The next condition is ad hoc and specifically designed for use in section 5.4.

A particular formula ρ of L(k,n) is given by Lemma 5.4.1. At this point we need to know only that ρ has no S, no quantifiers, and no constants, with variables among $v[1],...,v[k_4],v[k_1+1],...,v[4k_1]$.

Condition 7. Suppose |x| > |y|, and x is of the form $(p^{<5>},z[1],...,z[5],w[1],...,w[k_4],...,w[k_4])$, all z's, w's < p. Then x E y ↔ y is not among the $(w[k_1+1],...,w[4k_1],...,w[4k_1])$ with $w[k_1+1],...,w[4k_1] < z[1] \wedge \rho(w[1],...,w[k_4],w[k_1+1],...,w[4k_1])$.

Condition 8. Suppose no condition 1-7 applies to x,y, but some condition 1-7 applies to y,x. Then apply the first such condition to y,x.

Condition 9. None of conditions 1-8 apply. Then x E y ↔ x ≠ y.

LEMMA 5.1.1. Every x,y ∈ $Q[0,n]^k$ falls under exactly one of conditions 1-9. There is a unique graph G = G(k,n) obeying conditions 1-9. G is order invariant.

Proof: Let x,y ∈ $Q[0,n]^k$. Then at least one condition applies to x,y, because of the escape condition 9.

Conditions 1-7 are mutually exclusive by first counting the number of occurrences of the largest coordinate in x. These counts are 1,k,1,3,4,6,5, respectively. Also, conditions 1 and 3 are mutually exclusive since in condition 3 the first coordinate of x is largest, but this is not the case in condition 1. Also condition 8 is obviously exclusive from condition 1-7. And then condition 9 is obviously exclusive from conditions 1-8.

This establishes that conditions 1-9 define a unique binary relation E on $Q[0,n]^k$.

We now establish irreflexivity and symmetry. Let x = y. Then condition 9 applies, in which case x ¬E y.

Suppose x E y. Hence x ≠ y.

case 1. Condition 1 applies to x,y. Since condition is symmetric, we see that condition 1 applies to y,x, and we have y E x.

case 2. Some condition 2-7 applies to x,y. Then no condition 1-7 applies to y,x. Hence condition 8 applies to y,x. The output under condition 8 at y,x is the same as the output under the applicable condition 2-7 at x,y. Hence y E x.

case 3. Condition 8 applies to x,y. Then some condition 1-7 applies to y,x. The result of applying condition 8 to x,y is the same as the result of applying the applicable condition 1-7 to x,y, which is "true". Hence y E x.

case 4. Condition 9 applies to x,y. Since condition 8 does not apply to x,y, we see that conditions 1-7 do not apply to y,x. Therefore condition 8 does not apply to y,x. Hence condition 9 applies to y,x. Since $x \neq y$, we have y E x.

G is order invariant since all of the conditions involve only order. QED

Through the end of section 5.5, we fix the unique $G = ([0,n]^k,E) \in GR(k)$. We have already fixed $n \geq k$, $k_6 \geq 10^5$. Note that n,k must necessarily be very large integers.

By the Invariant Maximal Clique Theorem, we fix a maximal clique $S \subseteq Q[0,n]^k$ which is upper Z^+ order invariant. This is the same as $S \subseteq Q[0,n]^k$ being upper $Z^+(\infty)$ order invariant, and the same as $S \subseteq Q[0,n]^k$ being upper {1,...,n} order invariant. We refer to the upper {1,...,n} order invariance of $S \subseteq Q[0,n]^k$ as "invariance".

Through section 5.4, adjacency, and clique refer to the graph G.

5.2. REPRESENTATION BY S FORMULAS.

LEMMA 5.2.1. For all $1 \leq m \leq n$, $(m,...,m) \in S$.

Proof: Let $1 \leq m \in n$. Note that $(n,...,n) \notin W$, and so by condition 2, (n,...,n),x are adjacent if $|x| = n \wedge x \neq (n,...,n)$. By condition 2, (n,...,n),x are adjacent if $|x| < n$. Hence (n,...,n) is adjacent to all $x \neq (n,...,n)$. Since S is a maximal clique, $(n,...,n) \in S$. By invariance, $(m,...,m) \in S$. QED

LEMMA 5.2.2. Let $0 \leq p,q < n$. Then $p < q \leftrightarrow (n,p,q,...,q) \in S$.

Proof: Let $0 \leq p,q < n$. Note that $(n,p,q,...,q) \notin W$.

Suppose (n,p,q,...,q) ∈ S. Then condition 3 applies to (n,p,q,...,q),(1,...,1). If p ≥ q then by condition 3, (n,p,q,...,q),(1,...,1) are not adjacent. By Lemma 5.2.1, (1,...,1) ∈ S. Since S is a clique, (n,p,q,...,q) ∉ S. Hence p < q. this establishes half of the equivalence.

Suppose (n,p,q,...,q) ∉ S. Let (n,p,q,...,q),y be not adjacent, where y ∈ S. Then they are not equal, and |y| ≤ n. If |y| = n then they fall under condition 9, and are adjacent. Hence |y| < n, and so condition 3 applies. Hence p ≥ q. This establishes the other half of the equivalence. QED

LEMMA 5.2.3. Let r ≤ k-3 and z[1],...,z[r] < n. Then (n,n,z[1],...,z[r],n,z[r],...,z[r]) ∈ S ↔ (z[1],...,z[r],...,z[r]) ∉ S.

Proof: Let r,z[1],...,z[r] be as given. Note that (n,n,z[1],...,z[r],...,z[r]) ∉ W.

Suppose (n,n,z[1],...,z[r],...,z[r]) ∈ S. By condition 4, (n,n,z[1],...,z[r],n,z]r],...,z[r]),(z[1],...,z[r],...,z[r]) are not adjacent. Since S is a clique, we have (z[1],...,z[r],...,z[r]) ∉ S. This establishes half of the equivalence.

Now suppose (n,n,z[1],...,z[r],n,z[r],...,z[r]) ∉ S. Let (n,n,z[1],...,z[r],n,z[r],...,z[r]),y be not adjacent, where y ∈ S. They are not equal, and |y| ≤ n. If |y| = n then condition 9 applies, and therefore they are adjacent. Hence |y| < n, and condition 4 applies. By condition 4, y = (z[1],...,z[r],...,z[r]). This establishes the other half of the equivalence. QED

LEMMA 5.2.4. Let 2r+4 ≤ k and z[1],...,z[r],w[1],...,w[r]] < n. Then ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]) ∈ S ↔ ¬((z[1],...,z[r],...,z]r]) ∈ S ∨ (w[1],...,w[r],...,w[r]) ∈ S).

Proof: Let r,z[1],...,z[r],w[1],...,w[r] be as given. Note that ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]) ∉ W.

Suppose ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]) ∈ S. By condition 5, ($n^{<3>}$,z[1],...,z[r],n,w[1],..., w[r],..., w[r]),(z[1],...,z[r],...,z[r]) are not adjacent, and ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]),(w[1],...,w[r],...,w[r]) are not adjacent. Since S is a clique, we have (z[1],...,z[r],...,z[r]), (w[1],...,w[r],...,w[r]) ∉ S. This establishes half of the equivalence.

Now suppose ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]) ∉ S. Let ($n^{<3>}$,z[1],...,z[r],n,w[1],...,w[r],...,w[r]),y be not adjacent, where y ∈ S. They are not equal, and |y| ≤ n. If |y| = n then condition 9 applies, and therefore they are adjacent. Hence |y| < n, and condition 5 applies. By condition 5, y = (z[1],...,z[r],...,z[r]) ∨ y = (w[1],...,w[r],...,w[r]). This establishes the other half of the equivalence. QED

LEMMA 5.2.5. Let 1 ≤ i ≠ j ≤ r ≤ k-6 and z[1],...,z[r] < n. Then ($n^{<4>}$,z[1],...,z[i-1],n,z[i+1],...,z[r],n, z[j],..., z[j]) ∈ S ↔ ¬(∃q < z[j])((z[1],...,z[i-1],q,z[i+1],..., z[r],...,z[r]) ∈ S).

Proof: Let i,j,r,z[1],...,z[r] be as given. Note that ($n^{<4>}$,z[1],...,z[i-1],n,z[i+1],...,z[r],n,z[j],...,z[j]) ∉ W.

Suppose ($n^{<4>}$,z[1],...,z[i-1],n,z[i+1],...,z[r],n,z[j], ...,z[j]) ∈ S. By condition 6, ($n^{<4>}$,z[1],...,z[i-1], n,z[i+1],...,z[r],n,z[j],...,z[j]),(z[1],...,z[i-1],q, z[i+1],...,z[r],...,z[r]), q < z[j], are not adjacent. Since S is a clique, we have z[1],...,z[i-1],q,z[i+1],...,z[r],...,z[r]) ∉ S, provided q < z[j]. This establishes half of the equivalence.

Now suppose ($n^{<4>}$,z[1],...,z[i-1],n,z[i+1],...,z[r],n, z[j],...,z[j]) ∉ S. Let ($n^{<4>}$,z[1],...,z[i-1],n,z[i+1], ...,z[r],n,z[j],...,z[j]),y be not adjacent, where y ∈ S. They are not equal, and |y| ≤ n. If |y| = n then condition 9 applies, and therefore they are adjacent. Hence |y| < n, and condition 6 applies. By condition 6, y is some (z[1],...,z[i-1],q,z[i+1],...,z[r],...,z[r]), where q < z[j]. This establishes the other half of the equivalence. QED

LEMMA 5.2.6. Let z[1],...,z[5],w[1],...,w[k_4] < n. Then ($n^{<5>}$,z[1],...,z[5],w[1],...,w[k_4],...,w[k_4]) ∈ S ↔ ¬(∃w[k_1+1],...,w[4k_1] < z[1])((z[1],...,z[5],w[k_1+1], ...,w[4k_1],...,w[4k_1]) ∈ S ∧ ρ(w[1],...,w[k_4],w[k_1+1],...,w[4k_1])).

Proof: Let z[1],...,z[5],w[1],...,w[k4] < n. Note that ($n^{<5>}$,z[1],...,z[5],w[1],...,w[k_4],...,w[k_4]) ∉ W.

Suppose $(n<5>,z[1],...,z[5],w[1],...,w[k_4],...,w[k_4]) \in S$. By condition 7, $(n^{<5>},z[1],...,z[5],w[1],...,w[k_4], ...,w[k_4]),(w[k_1+1],...,w[4k_1],...,w[4k_1])$ are not adjacent, provided $w[k_1+1],...,w[4k_1] < z[1] \wedge \rho(w[1],...,w[k_4],w[k_1+1],...,w[4k_1])$. Since S is a clique, we have $(w[k_1+1],...,w[4k_1],...,w[4k_1]) \notin S$, provided $w[k_1+1],...,w[4k_1] < z[1] \wedge \rho(w[1],...,w[k_4],w[k_1+1],...,w[4k_1])$. This establishes half of the equivalence.

Now suppose $(n^{<5>},z[1],...,z[5],w[1],...,w[k_4],...,w[k_4]) \notin S$. Let $(n^{<5>},z[1],...,z[5],w[1],...,w[k_4],...,w[k_4]),y$ be not adjacent, where $y \in S$. They are not equal, and $|y| \leq n$. If $|y| = n$ then condition 9 applies, and therefore they are adjacent. Hence $|y| < n$, and condition 7 applies. By condition 7, $y = (w[k_1+1],...,w[4k_1],...,w[4k_1])$, where $w[k_1+1],...,w[4k_1] < z[1] \wedge \rho(w[1],...,w[k_4],w[k_1+1],...,w[4k_1])$. This establishes the other half of the equivalence. QED

DEFINITION 5.2.1. We introduce the language L(k,n) as follows.

i. constants 0,1,...,n.
ii. variables v[1],...,v[k].
iii. the binary relation symbol <.
iv. the k-ary relation symbol S.

DEFINITION 5.2.2. The terms are the constants and the variables. The atomic formulas are

a. $s < t$, where s,t are terms.
b. $S(\alpha[1],...,\alpha[k])$, where $\alpha[1],...,\alpha[k]$ are terms.

DEFINITION 5.2.3. The formulas of L(k,n) are given by

c. every atomic formula of L(k,n) is a formula of L(k).
d. if φ is a formula of L(k,n) then $(\neg\varphi)$ is a formula of L(k).
e. if φ,ψ are formulas of L(k,n) then $(\varphi \vee \psi)$ is a formula of L(k,n).
f. if φ is a formula of L(k,n) and $1 \leq i \leq k$, then $(\exists v[i] < t)(\varphi)$ is a formula of L(k,n), where t is a term of L(k,n) other than v[i].

DEFINITION 5.2.4. The intended interpretation of L(k,n) is as follows. The variables range over Q[0,n]. < is as usual on Q[0,n]. Each constant $0 \leq m \leq n$ is interpreted as m. S is interpreted as the maximal clique that we have fixed, as a k-ary relation.

We use the following usual Tarskian semantics.

DEFINITION 5.2.5. The assignments are the $x \in Q[0,n]^k$. We write $\varphi<x>$ to indicate that the formula φ of L(k,n) holds under the assignment x.

DEFINITION 5.2.6. For terms α, we write $\alpha<x>$ to be the value of α at x. Specifically, $v[i]<x> = x[i]$, $m<x> = m$.

DEFINITION 5.2.7. We define

$(v[i] < v[j])<x> \leftrightarrow x[i] < x[j]$.
$S(\alpha[1],\ldots,\alpha[k])<x> \leftrightarrow (\alpha[1]<x>,\ldots,\alpha[k]<x>) \in S$.
$(\neg\varphi)<x> \leftrightarrow$ not $\varphi<x>$.
$(\varphi \vee \psi)<x> \leftrightarrow \varphi<x>$ or $\psi<x>$.
$(\exists v[i] < t)(\varphi)<x> \leftrightarrow$ there exists $p < t<x>$ such that $\varphi<x[x_{i1},\ldots,x_{i-1},p,x_{i+1},\ldots,x_k>$.

DEFINITION 5.2.8. The S formulas are the atomic formulas of the form $S(\alpha[1],\ldots,\alpha[k])$.

DEFINITION 5.2.9. Let φ be a formula of L(k,n). We define the complexity $\#(\varphi)$ as the sum of

i. the total number of occurrences of $\neg,\vee,\exists$ in φ.
ii. the largest i such that v[i] occurs in φ (free or bound); 0 if no v[i] occurs in φ.
iii. the least $m \geq 0$ such that every subformula $S(\alpha[1],\ldots,\alpha[k])$ has $\alpha[m] = \ldots = \alpha[k]$.

DEFINITION 5.2.10. Let $1 \leq p \leq n$. We say that φ is p-valid if and only if for all $x \in Q[0,p)^k$, $\varphi<x>$. We say that φ,ψ are p-equivalent if and only if for all $x \in Q[0,p)^k$, $\varphi<x> \leftrightarrow \psi<x>$.

DEFINITION 5.2.11. For terms α, define α^* as follows. For $1 \leq i \leq k$, $v[i]^* = v[i]$. For $2 \leq m \leq r$, $m^* = m-1$. $0^* = 0$, $1^* = 1$.

LEMMA 5.2.7. Let $S(\alpha[1],\ldots,\alpha[k])$ be an S formula whose constants are among $n,n-1,\ldots,n-r$, $r \leq k-2$. Then $S(\alpha[1],\ldots,\alpha[k])$ and $S(\alpha[1]^*,\ldots,\alpha[k]^*)$ are (n-r-1)-equivalent.

Proof: Since $n \geq k$, the constants are among $n,n-1,...,2$. We are moving $n,n-1,...,n-r$ down to $n-1,n-2,...,n-r-1$, while fixing the other coordinates, all of which are $< n-r-1$. So we obtain the desired equivalence by invariance. QED

DEFINITION 5.2.12. For $r \geq 0$, let $P(r)$ be the following statement. Suppose $\#(\varphi) = r$, $16^r \leq \log k$, and φ has no constants. Then φ is $(n-2r)$-equivalent to an S formula ψ with $\#(\psi) \leq 16^r$, with the same free variables, and constants among $n,...,n-2r$.

We will prove that $P(r)$ holds for all integers $r \geq 0$.

LEMMA 5.2.8. $P(0)$.

Proof: There are no φ such that $\#(\varphi) \leq 0$ and φ has no constants. QED

Assume $P(r)$, $r \geq 0$. We will prove $P(r+1)$.

LEMMA 5.2.9. Let $\#(v[i] < v[j]) \leq r+1 \wedge 16^{r+1} \leq \log k$. Then $v[i] < v[j]$ is $(n-2(r+1))$-equivalent to an S formula ψ with $\#(\psi) \leq 16^{r+1}$, with the same free variables, and constants among $n,...,n-2(r+1)$.

Proof: Let i,j,r be as given. Then $\max(i,j) \leq r+1$. By Lemma 5.2.2, $v[i] < v[j]$ is n-equivalent to $S(n,v[i],v[j],...,v[j])$, which has complexity $\leq \max(i,j)+3 \leq r+4 \leq 16^{r+1}$. QED

LEMMA 5.2.10. Let $\#(S(\alpha[1],...,\alpha[k])) \leq r+1 \wedge 16r+1 \leq \log k$, where $S(\alpha[1],...,\alpha[k])$ has no constants. Then $S(\alpha[1],...,\alpha[k])$ is $(n-2(r+1))$-equivalent to an S formula ψ with $\#(\psi) \leq 16^{r+1}$, with the same free variables, and constants among $n,...,n-2(r+1)$.

Proof: Set $\psi = S(\alpha[1],...,\alpha[k])$. QED

LEMMA 5.2.11. Let $\#(\neg\varphi) \leq r+1 \wedge 16^{r+1} \leq \log k$, where $\neg\varphi$ has no constants, $16^{r+1} \leq \log k$. Then $\neg\varphi$ is $(n-2(r+1))$-equivalent to an S formula ψ with $\#(\psi) \leq 16^{r+1}$, with the same free variables, and constants among $n,...,n-2(r+1)$.

Proof: Let φ,r be as given. Then $\#(\varphi) \leq r$. Applying $P(r)$, let φ be $(n-2r)$-equivalent to an S formula ψ with $\#(\psi) \leq 16^r$, with the same free variables, and constants among $n,...,n-2r$. Write

For all $x_1,\ldots,x_k < n-2r$, $\varphi<x_1,\ldots,x> \leftrightarrow$
$S(\alpha[1],\ldots,\alpha[16^r],\ldots,\alpha[16^r])< x_1,\ldots,x_k>$.

By Lemma 5.2.7, for all $x_1,\ldots,x_k < n-2r-1$, $\varphi<x_1,\ldots,x_k> \leftrightarrow$
$S(\alpha[1]^*,\ldots,\alpha[16^r]^*,\ldots,\alpha[16^r]^*)<x_1,\ldots,x_k>$.

We have $r \leq 16^{r+1} \leq \log k$. By Lemma 5.2.3, for all
$x[1],\ldots,x[k] < n-2(r+1)$, $\neg\varphi<x_1,\ldots,x_k> \leftrightarrow$
$\neg S(\alpha[1]^*,\ldots,\alpha[16^r]^*,\ldots,\alpha[16^r]^*)<x_1,\ldots,x_k> \leftrightarrow$
$S(n,n,\alpha[1]^*,\ldots,\alpha[16^n]^*,\ldots,\alpha[16^n]^*)<x_1,\ldots,x_k>$. Also,
$\#(S(n,n,\alpha[1]^*,\ldots,\alpha[16n]^*,\ldots,\alpha[16n]^*)) \leq 16^r+2 \leq 16^{r+1}$. QED

LEMMA 5.2.12. Let $\#(\varphi \vee \rho) \leq r+1 \wedge 16^{r+1} \leq \log k$, where $\varphi \vee \rho$ has no constants. Then $\varphi \vee \rho$ is $(n-2(r+1))$-equivalent to an S formula ψ with $\#(\psi) \leq 16^{r+1}$, with the same free variables, and constants among $n,\ldots,n-2(r+1)$.

Proof: Let φ,ρ,r be as given. Then $\#(\varphi),\#(\rho) \leq r$. Applying $P(r)$, let φ be $(n-2r)$-equivalent to an S formula σ with $\#(\sigma) \leq 16^r$, with the same free variables, and constants among $n,\ldots,n-2r$. Let ρ be $(n-2r)$-equivalent to an S formula τ with $\#(\tau) \leq 16^n$, with the same free variables, and constants among $n,\ldots,n-2r$. Write

For all $x_1,\ldots,x_k < n-2r$, $\varphi<x_1,\ldots,x_k> \leftrightarrow$
$S(\alpha[1],\ldots,\alpha[16^r]\ldots,\alpha[16^r])<x_1,\ldots,x_k>$.
For all $x_1,\ldots,x_k < n-2r$, $\rho<x_1,\ldots,x_k> \leftrightarrow$
$S(\beta[1],\ldots,\beta[16^r],\ldots,\beta[16^r])<x_1,\ldots,x_k>$.

By Lemma 5.2.7, using $n-2r \geq 2$,

For all $x_1,\ldots,x_k < n-2r-1$, $\varphi<x_1,\ldots,x_k> \leftrightarrow$
$S(\alpha[1]^*,\ldots,\alpha[16^r]^*,\ldots,\alpha[16^r]^*)<x_1,\ldots,x_k>$.
For all $x_1,\ldots,x_k < n-2r-1$, $\rho<x_1,\ldots,x_k> \leftrightarrow$
$S(\beta[1]^*,\ldots,\beta[16^r]^*,\ldots,\beta[16^r]^*)<x_1,\ldots,x_k>$.

We have $16^{r+1} \leq \log k$. By Lemma 5.2.4,

For all $x_1,\ldots,x_k < n-2r-1$, $(\varphi \vee \rho)<x_1,\ldots,x_k> \leftrightarrow$
$(S(\alpha[1]^*,\ldots,\alpha[16^r]^*,\ldots,\alpha[16^r]^*) \vee$
$S(\beta[1]^*,\ldots,\beta[16^r]^*,\ldots,\beta[16^r]^*))<x_1,\ldots,x_k> \leftrightarrow$
$\neg S(n^{<3>},\alpha[1]^*,\ldots,\alpha[16^r]^*,n,\beta[1]^*,\ldots,\beta[16^r]^*,\ldots,\beta[16^r]^*)<x_1,\ldots,x_k>$.

We now apply Lemmas 5.2.7 to $\neg S(n^{<3>},\alpha[1]^*,\ldots,\alpha[16^r]^*,n,\beta[1]^*,\ldots,\beta[16^r]^*,\ldots,\beta[16^r]^*)<x_1,\ldots,x_k>$, using $n-2r-1 \geq 2$.

For all $x_1,...,x_k < n-2(r+1)$, $(\varphi \vee \rho)<x_1,...,x_k> \leftrightarrow \neg S((n-1)^{<3>},\alpha[1]**,...,\alpha[16^r]**,n-1,\beta[1]**,...,\beta[16^r]**,..., \beta[16^r]**)<x_1,...,x_k>$.

We apply Lemma 5.2.3 to $\neg S((n-1)<3>,\alpha[1]**,...,\alpha[16^r]**,n-1,\beta[1]**,...,\beta[16^r]**,...,\beta[16^r]**)<x_1,...,x_k>$, using $2(16^r)+4 \leq k-2$. This is clear from $16^{r+1} \leq \log k$.

For all $x_1,...,x_k < n-2(r+1)$, $(\varphi \vee \rho)<x_1,...,x_k> \leftrightarrow$ $\neg S((n-1)^{<3>},\alpha[1]**,...,\alpha[16^r]**,n-1,\beta[1]**,...,\beta[16^r]**, ...,\beta[16^r]**)<x_1,...,x_k> \leftrightarrow$ $S(n,n,(n-1)^{<3>},\alpha[1]**,...,\alpha[16r]**,n-1,\beta[1]**,...,\beta[16r]**, ...,\beta[16^r]**)<x_1,...,x_k>$. Also $\#(S(n,n,(n-1)^{<3>},\alpha[1]**,..., \alpha[16^r]**,n-1,\beta[1]**,...,\beta[16^r]**,...,\beta[16^r]**)) \leq 2(16^r)+6+r+1 \leq 16^{r+1}$. QED

LEMMA 5.2.13. Let $\#((\exists v[i] < v[j])(\varphi)) \leq r+1 \wedge 16^{r+1} \leq \log k$, where $(\exists v[i] < v[j])(\varphi)$ has no constants. Then $(\exists v[i] < v[j])(\varphi)$ is $(n-2(r+1))$-equivalent to an S formula ψ with $\#(\psi) \leq 16^{r+1}$, with the same free variables, and constants among $n,...,n-2(r+1)$.

Proof: Let i,j,r,φ be as given. Then $\#(\varphi) \leq r$. Since $\#((\exists v[i] < v[j])(\varphi)) \geq \max(i,j)+1$, we have $1 \leq i \neq j \leq r$. By P(r), let φ be $(n-2r)$-equivalent to an S formula ρ with $\#(\rho) \leq 16^r$, with the same free variables, and constants among $n,...,n-2r$.

For all $x_1,...,x_k < n-2r$, $\varphi<x_1,...,x_k> \leftrightarrow$ $S(\alpha[1],...,\alpha[16^r],...,\alpha[16^r])<x_1,...,x_k>$.

By Lemma 5.2.7 and $n-2r \geq 2$,

For all $x_1,...,x_k < n-2r-1$, $\varphi<x_1,...,x_k> \leftrightarrow$ $S(\alpha[1]*,...,\alpha[16^r]*,...,\alpha[16^r]*)<x_1,...,x_k>$.

By Lemma 5.2.5, $16^r \leq k-7$, and also Lemma 5.2.7, $n-2r-1 \geq 2$,

For all $x_1,...,x_k < n-2r-1$,

$(\exists v[i] < v[j])(\varphi)<x_1,...,x_k> \leftrightarrow$
$(\exists v[i] < v[j])(S(\alpha[1]*,...,\alpha[16^r]*,...,\alpha[16^r]*))<x_1,...,x_k>$
$\leftrightarrow$
$(\exists p < v[j])(S(\alpha[1]*,...,\alpha[i-1]*,p,\alpha[i+1]*,...,\alpha[16r], ...,\alpha[16r]*)<x_1,...,x_{i-1},p,x_{i+1},...,x_k> \leftrightarrow$
$\neg S(n^{<4>},\alpha[1]*,...,\alpha[i-1]*,n,\alpha[i+1]*,...,\alpha[16^r]*,n,\alpha[j]*, ...,\alpha[j]*)<x_1,...,x_k> \leftrightarrow$

$\neg S((n-1)^{<4>},\alpha[1]^{**},\ldots,\alpha[i-1]^{**},n-1,\alpha[i+1]^{**},\ldots,\alpha[16^r]^{**},n-1,\alpha[j]^{**},\ldots,\alpha[j]^{**})<x_1,\ldots,x_k>$.

By Lemma 5.2.3 and $16^r+6 \leq k-2$,

For all $x_1,\ldots,x_k < n-2(r+1)$,

$\neg S((n-1)^{<4>},\alpha[1]^{**},\ldots,\alpha[i-1]^{**},n-1,\alpha[i+1]^{**},\ldots,\alpha[16^r]^{**},n-1,\alpha[j]^{**},\ldots,\alpha[j]^{**})<x_1,\ldots,x_k> \leftrightarrow$
$S(n,n,(n-1)^{<4>},\alpha[1]^{**},\ldots,\alpha[i-1]^{**},n-1,\alpha[i+1]^{**},\ldots,\alpha[16^r]^{**},n-1,\alpha[j]^{**},\ldots,\alpha[j]^{**})<x_1,\ldots,x_k>$.

Note that $S(n,n,(n-1)^{<4>},\alpha[1]^{**},\ldots,\alpha[i-1]^{**},n-1,\alpha[i+1]^{**},\ldots,\alpha[16^r]^{**},n-1,\alpha[j]^{**},\ldots,\alpha[j]^{**})$ has the same free variables as $(\exists v[i] < v[j])(\varphi)$, with constants among $n,\ldots,n-2(r+1)$, and complexity at most $\leq 2(16^r) + 8 + r + 1 \leq 16^{r+1}$. QED

LEMMA 5.2.14. Suppose $\#(\varphi) = r \leq k_2/4$, where φ has no constants. Then φ is $(n-2r)$-equivalent to an S formula ψ with $\#(\psi) \leq 16^r$, with the same free variables, and constants among $n,\ldots,n-2r$.

Proof: By Lemmas 5.2.8 - 5.2.13, P(r) holds for all integers $r \geq 0$. Note that $r \leq k_2/4 \leq (\log\log k)/4 \rightarrow 16^r \leq \log k$. Let $\#(\varphi) = r \leq (\log\log k)/8$ and φ has no constants. Apply P(r). QED

LEMMA 5.2.15. Suppose $\#(\varphi) = r \leq k_2/4$, where φ has no constants. Then φ is r-equivalent to an S formula ψ with $\#(\psi) \leq 16^r$, with the same free variables, and constants among $r,\ldots,3r$.

Proof: From Lemma 5.2.14 and invariance. QED

LEMMA 5.2.16. Suppose $\#(\varphi) = r \leq k_2/16$, where all constants in φ are $< r$. Then φ is r-equivalent to an S formula ψ with $\#(\psi) \leq 2^{8r}$, with the same free variables, the same constants $< r$, and additional constants among $r,\ldots,9r$.

Proof: Let φ,r be as given. Let φ' result from replacing the constants in φ by new variables among $v[r+1],\ldots,v[2r]$, after changing bound variables if necessary. Then $\#(\varphi') \leq 4r \leq k_2/4$. By Lemma 5.2.15, φ' is 2r-equivalent to an S formula ψ' with $\#(\psi') \leq 2^{8r}$, with the same free variables, and constants among $4r,\ldots,12r$. Let ψ be the result of replacing the new variables with the constants that they replaced. Now apply invariance to reduce the constants 4r,...,12r to r,...,9r. QED

5.3. TUPLING.

We will develop, for some $0 \leq x[1] < x[2] < x[3] \leq n$, a one-one map from tuples $x[5],\ldots,x[k] < x[1]$ into values $x[4] < x[2]$. The map developed may depend on x[1],x[2],x[3]. For this purpose, we use condition 1 on our graph G. Up to now, we have only used conditions 2-6.

In this section, we will often use the ∈ S notation rather than the S( ) notation, because we need to consider $S \cap W$.

LEMMA 5.3.1. Let $0 \leq p_5,\ldots,p_k < i \leq n-4$. Then $\neg S(i,i+2,n,i+1,p_5,\ldots,p_k)$.

Proof: Let $p_4,\ldots,p_k < i \leq n-4$, $S(i,i+2,n,i+1,p_5,\ldots,p_k) \in S$. By invariance, $(i,i+3,n,i+1,p_5,\ldots,p_k),(i,i+3,n,i+2,p5,\ldots,p_k) \in S \cap W$. By condition 1i, $(i,i+3,n,i+1,p_5,\ldots,p_k)$, $(i,i+3,n,i+2,p_5,\ldots,p_k)$ are not adjacent. This contradicts that S is a clique. QED

LEMMA 5.3.2. Let $0 \leq p_5,\ldots,p_k < i \leq n-4$. There exists q such that $(i,i+2,n,q,p_5,\ldots,p_k) \in S \cap W$. There exists q such that $(i,i+2,i+3,q,p_5,\ldots,p_k) \in S \cap W$.

Proof: Let $0 \leq p_5,\ldots,p_k < i \leq n-3$. Note that $(i,i+2,n,i+1,p_5,\ldots,p_k) \in W$. By Lemma 5.3.1, $\neg S(i,i+2,n,i+1,p_5,\ldots,p_k)$. Let $(i,i+2,n,i+1,p_5,\ldots,p_k),y$ be not adjacent, where S(y). Then they are not equal and $|y| \leq n$. Using $|y| \leq n$, if $y \notin W$ then condition 9 applies to $(i,i+2,n,i+1,p_5,\ldots,p_k),y$, and they are adjacent, which is impossible.

We have established that $y \in W$. Condition 1 applies to $(i,i+2,n,i+1,p_5,\ldots,p_k),y$. Write $y = (p_1',\ldots,p_k')$. By condition 1, we have

$$(i = p_1' \wedge i+2 = p_2' \wedge n = p_3' \wedge i+1 \neq p_4' \wedge p_5 = p_5' \wedge \ldots \wedge p_k = p_k') \vee$$
$$(i = p_1' \wedge i+2 = p_2' \wedge n = p_3' \wedge i+1 = p_4' \wedge \neg(p_5 = p_4' \wedge \ldots \wedge p_k = p_k')).$$

Since $y \in W$, we have $p_5',\ldots,p_k' < p_1'$. So if the second conjunction above holds, then $p_5',\ldots,p_k' < i$, and we can apply Lemma 5.3.1 to obtain $y \notin S$. This is a contradiction.

Hence the first conjunction above holds. We have y = $(i,i+2,n,p_4',p_5...,p_k) \in S \cap W$ and $p_4' < i+2$. By invariance, $(i,i+2,i+3,p_4',p_5,...,p_k) \in S$ by invariance. Also $(i,i+2,i+3,q,p_5,...,p_k) \in W$ by inspection, or from y = $(i,i+2,i+3,q,p_5,...,p_k)$. QED

LEMMA 5.3.3. $(\forall i \leq n-4\}(\forall p_5,...,p_k < i)(\exists! q < i+2))(S(i,i+2,i+3,q,p_5,...,p_k))$. $(\forall i \leq n-4)(\forall p_5,...,p_k,p_5',...,p_k' < i)(\forall q < i+2))(S(i,i+2,i+3,q,p_5,...,p_k) \wedge S(i,i+2,i+3,q,p_5',...,p_k') \rightarrow p_5 = p_5' \wedge ... \wedge p_k = p_k')$.

Proof: Let $1 \leq i \leq n-4$ and $p_5,...,p_k < i$. By Lemma 5.3.2, let $(i,i+2,i+3,q,p_5,...,p_k) \in S \cap W$. Then $q < i+2$. For uniqueness, let $(i,i+2,i+3,q',p_5,...,p_k) \in S$, $q \neq q' < i+2$. Then Condition 1 applies to $(i,i+2,i+3,q,p_5,...,p_k),(i,i+2,i+3,q',p_5,...,p_k)$. Therefore these are not adjacent, contradicting that S is a clique.

Let $i \leq n-4$, $p_5,...,p_k,p_5',...,p_k' < i$, $q < i+2$, $S(i,i+2,i+3,q,p_5,...,p_k)$, $S(i,i+2,i+3,q,p_5',...,p_k')$, where $\neg(p_5 = p_5' \wedge ... \wedge p_k = p_k')$. Then condition 1 applies to $(i,i+2,i+3,q,p_5,...,p_k)$, $(i,i+2,i+3,q,p_5',...,p_k')$. By condition 1ii, these are not adjacent, contradicting that S is a clique. QED

5.4. NORMAL FORM.

DEFINITION 5.4.1. We define the rank of an S formula as the least m such that the m-th and all succeeding terms are the same.

We will introduce the crucial formula Γ with free variables $v[1],...,v[k_4]$, as follows. It is clarifying to always display its free variables $v[1],...,v[k_4]$. Γ is "universal" in the sense of Lemma 5.4.1. Also, Γ can be put into a very convenient form as can be seen by Lemma 5.4.7.

DEFINITION 5.4.2. Let X be the set of all S formulas of rank $\leq 2k_4$, with variables among $v[5],...,v[2k_4]$, and no constants.

DEFINITION 5.4.3. We use the lexicographic enumeration $\varphi_1,...,\varphi_{k\_2}$ of X, which is extended by repeating the lexicographically greatest element of X in order to use all subscripts $\leq k_2$.

DEFINITION 5.4.4. We now introduce the formula

Γ(v[1],...,v[k_4]) = ($\exists$v[k_4+1],...,v[2k_4+k_2] < v[1])($\exists$j ≤ k_2)
(φ_j(v[5],...,v[2k_4]) ∧
S(v[1],v[2],v[3],v[4],v[k_4+1],...,v[2k_4+k_2],...,v[2k_4+k_2]) ∧
j is greatest ≤ k_2 with v[2k_4+1] = ... = v[2k_4+j]).

Informally, Γ(v[1],...,v[k_4]) expresses that

φ_j holds at v[5],...,v[2k_4],
where v[k_4+1],...,v[2k_4+k_2] is "coded" by
v[1],v[2],v[3],v[4], and
j is "hard coded" by v[2k_4+1],...,v[2k_4+k_2].

Of course, it is only certain quadruples v[1],v[2],v[3],v[4] that do any legitimate coding according to Lemma 5.3.3.

We now establish the key universality properties of Γ(v[1],...,v[k_4]).

LEMMA 5.4.1. Let i ≤ n-4, $\psi \in$ X, and let variables v[k_4+1],...,v[2k_4] < i be fixed. There exists q < i+2 such that ψ(v[5],...,v[2k_4]) is i-equivalent to Γ(i,i+2,i+3,q,v[5],...,v[k_4]).

Proof: Let i,ψ,v[k_4+1],...,v[2k_4] be as given. Let $\psi = \varphi_j$, j ≤ k_2.

By Lemma 5.3.3, let q < i+2 be such that the fixed v[k_4+1],...,v[2k_4] are the unique rationals < i such that S(i,i+2,i+3,q,v[k_4+1],...,v[2k_4],$0^{<j>}$,1/2,...,1/2).

Let v[5],...,v[k_4] < i. Suppose ψ(v[5],...,v[2k_4]. Set the existentially quantified v[k_4+1],...,v[2k_4+k_2] to be v[k_4+1],...,v[2k_4],$0^{<j>}$,1/2,...,1/2, respectively. Then Γ(i,i+2,i+3,q,v[5],...,v[k_4]).

Suppose Γ(i,i+2,i+3,q,v[5],...,v[k_4]). Let v[k_4+1]',...,v[2k_4+k_2]' < i and j' ≤ k_2 be such that

S(i,i+2,i+3,q,v[k_4+1]',...,v[2k_4+k_2]',...,v[2k_4+k_2]') ∧
j' is greatest ≤ k_2 with v[2k_4+1]' = ... = v[2k_4+j]').

By Lemma 5.3.3, v[k_4+1]',...,v[2k_4+k_2]' is v[k_4+1],...,v[2k_4],$0^{<j>}$,1/2,...,1/2, respectively. Hence j = j', and so $\varphi_{j'}$(v[5],...,v[k_4],v[k_4+1],...,v[2k_4]). QED

LEMMA 5.4.2. Let #(φ) ≤ r ≤ 9k_6 have variables among v[5],...,v[r], with constants < r, and let the variables

$v[j+1],\ldots,v[r]$ be fixed $< r$, where $5 \leq j \leq r$. There exists $q < 9r+3$ such that $\varphi(v[5],\ldots,v[r])$ is r-equivalent to $\Gamma(9r+1,9r+3,9r+4,q,v[5],\ldots,v[j],\ldots,v[j])$.

Proof: Let $\varphi,r,i,j,v[j+1],\ldots,v[r]$ be as given. By Lemma 5.2.16, let ψ be an S formula, $\#(\psi) \leq 2^{8r}$, with variables among $v[5],\ldots,v[r]$, and constants among $r,\ldots,9r$, such that φ,ψ are r-equivalent.

Fix the variables $v[j+1],\ldots,v[r] < r$ in ψ. Note that $2^{8r} \leq 2k_4$. Hence we can now view ψ as $\psi' \in X$, where instead of fixing the variables $v[j+1],\ldots,v[r] < r$, we instead fix the variables $v[k_4+1],\ldots,v[k_4+r-j]$ with the same values. We also fix the next 8r+1 variables to be $r,\ldots,9r$, and then fill out the remaining variables up to $v[2k_4]$ with 9r.

It is now clear that φ, with $v[j+1],\ldots,v[r]$ fixed as given, and ψ' with $v[k_4+1],\ldots,v[2k_4]$ fixed according to the previous paragraph, have remaining free variables among $v[5],\ldots,v[j]$, and are r-equivalent.

We now apply Lemma 5.4.1 to ψ', with i set to 9r+1. Let $q < 9r+3$ be such that $\psi'(v[5],\ldots,v[j])$, with variables fixed as above, is (9r+1)-equivalent to $\Gamma(9r+1,9r+3,9r+4,q,v[5],\ldots,v[j],\ldots,v[j])$. Hence $\varphi(v[5],\ldots,v[r])$ is (9r+1)-equivalent to $\Gamma(9r+1,9r+3,9r+4,q,v[5],\ldots,v[j],\ldots,v[j])$. QED

LEMMA 5.4.3. There exists a formula ρ with no S, no quantifiers, and no constants, with variables among $v[1],\ldots,v[k_4],v[k_1+1],\ldots,v[3k_1]$, such that for all $v[1],\ldots,v[k_4]$, $\Gamma(v[1],\ldots,v[k_4]) \leftrightarrow (\exists v[k_1+1],\ldots,v[3k_1] < v[1])(S(v[k_1+1],\ldots,v[2k_1],\ldots,v[2k_1]) \wedge S(v[2k_1+1],\ldots,v[3k_1],\ldots,v[3k_1]) \wedge \rho)$.

Proof: Using propositional logic, we put $\Gamma(v[1],\ldots,v[k_4])$ in the form

$$\begin{array}{c} \text{there exists } v[k_4+1],\ldots,v[2k_4+k_2] < v[1] \text{ such that} \\ S(\alpha[1],\ldots,\alpha[2k_4+k_2],\ldots,\alpha[2k_4+k_2]) \wedge \\ \rho_1 \rightarrow S(\beta[1,1],\ldots,\beta[1,2k_4],\ldots,\beta[1,2k_4]) \wedge \\ \rho_2 \rightarrow S(\beta[2,1],\ldots,\beta[2,2k_4],\ldots,\beta[2,2k_4]) \wedge \\ \ldots \wedge \\ \rho_{k\_2} \rightarrow S(\beta[k_2,1],\ldots,\beta[k_2,2k_4+k_2],\ldots,\beta[k_2,2k_4+k_2]) \end{array}$$

where for $i < k_2$, ρ_i is $v[2k_4+1] = \ldots = v[2k_4+i] \neq v[2k_4+i+1]$, ρ_{k_2} is $v[2k_4+1] = \ldots = v[2k_4+k_2]$, and the α's and β's are among the variables $v[1],...,v[2k_4+k_2]$. We can put this in the form

$$(\exists v[k_4+1],...,v[2k_4+k_2],v[k_1+1],...,v[k_1+2k_4] < v[1])$$
$$(S(\alpha[1],...,\alpha[2k_4+k_2],...,\alpha[2k_4+k_2]) \wedge$$
$$S(v[k_1+1],...,v[k_1+2k_4],...,v[k_1+2k_4]) \wedge$$
$$\rho_1 \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[1,1],...,\beta[1,2k_4]) \wedge$$
$$\rho_2 \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[2,1],...,\beta[2,2k_4]) \wedge$$
$$... \wedge$$
$$\rho_{k\_2} \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[2,1],...,\beta[2,2k_4]))$$

$$(\exists v[2k_1+1],...,v[2k_1+k_2],v[k_1+1],...,v[k_1+2k_4] < v[1])$$
$$(S(v[2k_1+1],...,v[2k_1+k_2],...,v[2k_1+k_2]) \wedge$$
$$S(v[k_1+1],...,v[k_1+2k_4],...,v[k_1+2k_4]) \wedge$$
$$(v[2k_1+1],...,v[2k_1+k_2]) = (\alpha[1],...,\alpha[2k_4+k_2]) \wedge$$
$$\rho_1 \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[1,1],...,\beta[1,2k_4]) \wedge$$
$$\rho_2 \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[2,1],...,\beta[2,2k_4]) \wedge$$
$$... \wedge$$
$$\rho_{k\_2} \rightarrow (v[k_1+1],...,v[k_1+2k_4]) = (\beta[2,1],...,\beta[2,2k_4]))$$

By the suitable addition of dummy variables, we obtain the required form. QED

We fix ρ according to Lemma 5.4.3.

LEMMA 5.4.4. $S(v[k_1+1],...,v[2k_1],...,v[2k_1]) \wedge S(v[2k_1+1],...,v[3k_1],...,v[3k_1])$ and $S(n^{<3>},n-1,n-1,v[k_1+1],...,v[3k_1],...,v[3k_1])$ are (n-1)-equivalent.

Proof: Let $v[k_1+1],...,v[3k_1] < n-1$. By Lemmas 5.2.3, 5.2.4, and invariance,

$S(v[k_1+1],...,2k_1,...,2k_1) \wedge S(2k_1+1,...,3k_1,...,3k_1) \leftrightarrow$
$\neg S(n,n,v[k_1+1],...,2k_1,...,2k_1) \wedge$
$\neg S(n,n,2k_1+1,...,3k_1,...,3k_1) \leftrightarrow$
$\neg(S(n,n,v[k_1+1],...,v[2k_1],...,v[2k_1]) \vee$
$S(n,n,v[2k_1+1],...,v[3k_1],...,v[3k_1])) \leftrightarrow$
$\neg(S(n-1,n-1,v[k_1+1],...,v[2k_1],...,v[2k_1]) \vee S(n-1,n-1,v[2k_1+1],...,v[3k_1],...,v[3k_1])) \leftrightarrow$
$S(n<3>,n-1,n-1,v[k_1+1],...,v[3k_1],...,v[3k_1])$.

QED

LEMMA 5.4.5. $\Gamma(v[1],...,v[k_4])$ and $(\exists v[k_1+1],...,v[3k_1] < v[1])(S(n^{<3>},n-1,n-1,v[k_1+1],...,v[3k_1],...,v[3k_1]) \wedge \rho)$ are (n-1)-equivalent. $\Gamma(v[1],...,v[k_4])$ and $(\exists v[k_1+1],...,v[3k_1] < v[1])(S((n-1)^{<3>},n-2,n-2,v[k_1+1],...,v[3k_1],...,v[3k_1]) \wedge \rho)$ are (n-2)-equivalent.

Proof: By Lemmas 5.4.3, 5.4.4, and 5.2.7. QED

LEMMA 5.4.6. $\Gamma(v[1],...,v[k_4])$, $(\exists v[k_1+1],...,v[3k_1] < v[1])(S((n-1)^{<3>},n-2,n-2,v[k_1+1],...,v[3k_1],...,v[3k_1]) \wedge \rho)$, and $S(n^{<5>},(n-1)^{<3>},n-2,n-2,v[1],...,v[k_4],...,v[k_4])$ are (n-2)-equivalent.

Proof: By Lemmas 5.2.5 and 5.2.6. QED

LEMMA 5.4.7. For all $i \leq n-2$, $\Gamma(v[1],...,v[k_4])$ and $S((i+2)^{<5>},(i+1)^{<3>},i,i,v[1],...,v[k_4],...,v[k_4])$ are i-equivalent.

Proof: Let $i \leq n-2$ and $v[1],...,v[k_4] < i$. By Lemmas 5.4.3 and 5.4.4, $\Gamma(v[1],...,v[k_4]) \leftrightarrow (\exists v[k_1+1],...,v[4k_1] < v[1])(S((n-1)^{<3>},n-2,n-2,v[k_1+1],...,v[4k_1],...,v[4k_1]) \wedge \rho) \leftrightarrow S(n^{<5>},(n-1)^{<3>},n-2,n-2,v[1],...,v[k_4],...,v[k_4])$. The result follows by invariance, since we can replace n by i+2. QED

LEMMA 5.4.8. Let $\#(\varphi) \leq r \leq 9k_6$, with variables among $v[5],...,v[r]$, with constants < r, and let the variables $v[j+1],...,v[r]$ be fixed < r, where $5 \leq j \leq r$. There exists $q < 9r+3$ such that $\varphi(v[5],...,v[r])$ and $S((9r+7)^{<5>},(9r+6)^{<3>},9r+5,9r+5,9r+1,9r+3,9r+4,q,v[5],...,v[j],...,v[j])$ are r-equivalent.

Proof: Let $\varphi,r,j,v[j+1],...,v[r]$ be as given. By Lemma 5.4.2, let $q < 9r+3$, where $\varphi(v[5],...,v[r])$ and $\Gamma(9r+1,9r+3,9r+4,q,v[5],...,v[j],...,v[j])$ are r-equivalent. We now apply Lemma 5.4.7 with i = 9r+5.

We obtain that $\Gamma(9r+1,9r+3,9r+4,q,v[5],...,v[j],...,v[j])$ and $S((9r+7)^{<5>},(9r+6)^{<3>},9r+5,9r+5,9r+1,9r+3,9r+4,q,v[5],...,v[j],...,v[j])$ are (9r+5)-equivalent.

The Lemma is now established by combining these two equivalences. QED

LEMMA 5.4.9. Let $\#(\varphi) \leq r \leq 9k_6-4$, with constants < r, and let the variables $v[j+1],...,v[r]$ be fixed < r, where $1 \leq j \leq r$. There exists $q < 9r+39$ such that $\varphi(v[1],...,v[r])$ and $S((9r+43)^{<5>},(9r+39)^{<3>},9r+41,9r+41,9r+37,9r+39,9r+40,q,v[1],...,v[j],...,v[j])$ are r-equivalent.

Proof: Let φ,r,v[j+1],...,v[r] be as given. Let φ' be the result of adding 4 to all indices of variables in φ. Then #(φ') = r+4, with variables among v[5],...,v[r+4]. Fix v[j+5],...,v[r+4] < r. By Lemma 5.4.8, let q < 9r+39, where φ'(v[5],...,v[r+4]) and S($(9r+43)^{<5>}$,$(9r+39)^{<3>}$,9r+41,9r+41,9r+37,9r+39,9r+40,q,v[5], ...,v[j+4],...,v[j+4]) are (r+4)-equivalent. Hence φ(v[1],...,v[r]) and S($(9r+43)^{<5>}$,$(9r+39)^{<3>}$,9r+41,9r+41,9r+37,9r+39,9r+40,q,v[1], ...,v[j],...,v[j]) are (r+4)-equivalent. QED

LEMMA 5.4.10. Let #(φ) ≤ r+1 ≤ $9k_6$-4, with constants < r+1, and let the variables v[j+1],...,v[r+1] be fixed < r+1, where 1 ≤ j ≤ r+1. There exists q < 9r+48 such that φ(v[1],...,v[r]) and S($(9r+52)^{<5>}$,$(9r+48)^{<3>}$,9r+50,9r+50,9r+46,9r+48,9r+49,q,v[1], ...,v[j],...,v[j]) are (r+1)-equivalent.

Proof: Immediate from Lemma 5.4.9, replacing r with r+1. QED

LEMMA 5.4.11. Let #(φ) ≤ r ≤ k_6, with constants < r, and let the variables v[j+1],...,v[r] be fixed < r, where 1 ≤ j ≤ r. Let 1 ≤ s ≤ r. There exists q < 9r+39 such that (∀v[1],...,v[j])(S($(72r+52)^{<5>}$,$(72r+48)^{<3>}$,72r+50,72r+50,72r+17,72r+21,72r+22,q,v[1],...,v[j],...,v[j]) ↔ φ(v[1],...,v[r]) ∧ v[1],...,v[j] < s).

Proof: Let φ,r,j,s,v[j+1],...,v[r] be as given. Let φ' = φ ∧ v[1],...,v[r] < s. Then #(φ') ≤ 8r ≤ $9k_6$-4. Then Lemma 5.4.10 is applicable to φ' with r set to this 8r. Let q < 72r+48, where φ'(v[1],...,v[r]) and S($(72r+25)^{<5>}$,$(72r+24)^{<3>}$,72r+23,72r+23,72r+17,72r+21,72r+22, q,v[1],...,v[j],...,v[j]) are (8r+1)-equivalent. The result follows. QED

5.5. LINEARLY ORDERED SET THEORY.

We now shift to a more familiar context for mathematical logicians. We define a linearly ordered set theory LOST(k).

DEFINIITON 5.5.1. The language L(k) of LOST(k) is based on the binary relation symbols ∈,=,<, and the constants 0,1,...,k. We use variables v_1,v_2,... and the usual connectives and quantifiers. For formulas φ of L(k), we define the complexity measure c(φ) as the sum of

i. the total number of occurrences of ¬,∨,∃ in φ.

ii. the largest i such that v[i] occurs in φ (free or bound); 0 if no v[i] occurs in φ.

Thus L(k) differs from L(k,n) in that in L(k), we use ∈,=,<,0,...,k, whereas in L(k,n) we use S,<,0,...,n. Also, in L(k), we use variables v1,v2,..., whereas in L(k,n), we use variables v[1],...,v[k]. Furthermore, all quantifiers in formulas of L(k,n) are required to be bounded, whereas all quantifiers in formulas of L(k) are not bounded.

DEFINIITION 5.5.2. Let φ be a formula of L(k) and $1 \le i \le k$. Define $\varphi^{<i}$ to be the result of bounding all quantifiers in φ to points < i. Officially, in order for $\varphi^{<i}$ to be a formula of L*(k), we must expand $(\forall v < i)$, $(\exists v < i)$ to $(\forall v)(v < i \rightarrow \ldots)$, and $(\exists v)(v < i \wedge \ldots)$.

DEFINITION 5.5.3. The axioms of LOST(k) are as follows.

BASIC. < is a strict linear ordering with least element 0, and no greatest element. $0 < 1 < \ldots < k$. $v_1 \in v_2 \rightarrow v_1 < v_2$.

LIMITED SEPARATION. Let φ be a formula of L(k) with no constants, where $c(\varphi) \le i \le k_6/10^4$. $v_2,\ldots,v_i < i \rightarrow (\exists v_{i+1} < 10^4 i)(\forall v_1)(v_1 \in v_{i+1} \leftrightarrow \varphi^{<i} \wedge v_1 < i)$.

LIMITED INDISCERNIBILITY. Let φ be a formula of L(k), where $c(\varphi) \le k_3$, and all constants are $< k_3$. Let φ' result from φ by replacing all occurrences of constants $i,\ldots,k_3-1$, with occurrences of constants $i,\ldots,k_3-1$, in an order preserving way, where $1 \le i \le k_3$. $v_1,\ldots,v_{k\_3} < i \rightarrow (\varphi^{<k\_3} \leftrightarrow \varphi'^{<k\_3})$.

DEFINITION 5.5.4. We define the structure M(S) = $(Q[0,k]^{14},\in',<',0',\ldots,k')$ in the language of L(k) as follows. Let $x,y \in Q[0,k]^{14}$. Define $x \in' y \leftrightarrow S(y[1],\ldots,y[14],x[1],\ldots,x[14],\ldots,x[14]) \wedge |x| < |y|$. Define $x <' y \leftrightarrow |x| < |y| \vee (|x| = |y| \wedge x$ is lexicographically earlier than y). For $1 \le i \le k$, define $i' = (0,\ldots,0,i) \in Q[0,k]^{14}$.

LEMMA 5.5.1. Basic holds in M(S).

Proof: Obvious. QED

LEMMA 5.5.2. Limited Separation holds in M(S).

Proof: Let φ be a formula of L(k) with no constants, where $c(\varphi) \le i \le k_6/10^4$. We need to interpret the formula $\varphi^{<i}$ of L(k) as a formula $\psi^{<i}$ in L(k,n). The variables in φ are

among $v_1,\ldots,v_i$. We interpret each variable v_j, $1 \le j \le i$, as the fourteen variables v[14j-13],...,v[14j] of L(k,n). We also interpret <,∈ as <',∈'. In this interpretation of $\varphi^{<i}$, the quantifiers are strictly bounded to i, since the interpretation of i is i'. Thus the constants in $\psi^{<i}$ are among 0,i.

We need an estimate on $\#(\psi^{<i})$. Examination shows that there is a linear blowup over $c(\varphi^{<i})$. A multiple of 100 is easily sufficient. I.e., $\#(\psi^{<i}) \le 10^2 i$.

In Limited Separation, we focus on the range of $v_1 < i$ such that $\varphi^{<i}$ holds with $v_2,\ldots,v_i$ fixed < i, in the context of L(k). This corresponds to focusing on the range B of $(v[1],\ldots,v[14]) \in Q[0,i)^{14}$ such that ψ holds with v[15],...,v[14i] fixed < i, in the context of L(k,n).

We now apply Lemma 5.4.11 to ψ, with the r,j,s of Lemma 5.4.11 set to $10^2 i$,14,i, respectively. For Limited Separation, we fix $v_2,\ldots,v_i$ in $\varphi^{<i}$, which corresponds to fixing v[15],...,v[14i] in φ. We obtain

$$(\forall v[1],\ldots,v[14])(S(p_1,\ldots,p_{14},v[1],\ldots,v[14],\ldots,v[14]) \leftrightarrow \psi^{<i}(v[1],\ldots,v[14i]) \wedge v[1],\ldots,v[14] < i$$

where $p_1,\ldots,p_{14}$ are parameters $< 10^4 i$, and $p_1 > i$. Thus we can set $v_{i+1} = (p_1,\ldots,p_{14})$ for this instance of Limited Separation. QED

LEMMA 5.5.3. Limited Indiscernibility holds in M(S). M(S) satisfies LOST(k).

Proof: Let φ be a formula of L(k), where $c(\varphi) \le k_3$, where all constants are $< k_3$. Let $i \ge k_3$. Let φ' be the result of replacing the constants $i,\ldots,k_3-1$ by "new variables" $v_{k\_3+i},\ldots,v_{2k\_3-1}$ in φ. We will later replace the $v_{k\_3+1},\ldots,v$ that appear in φ' by constants $i,\ldots,k_3-1$.

We follow the same interpretation procedure as in the proof of Lemma 5.5.2, to interpret $\varphi'^{<k\_3}$ as a formula $\psi^{<k\_3}$ of L(k,n), $\#(\psi^{<k\_3}) \le 200k_3$, with variables among $v[1],\ldots,v[14(2k_3-1)]$, and constants among $0,\ldots,i-1,14k_3$.

Note that the interpretation of $\varphi^{<k\_3}$ is the same as replacing the "new variables" in $\psi^{<k\_3}$ by the constants from whence they came.

We now apply Lemma 5.2.16 to $\#(\psi^{<k\_3}) \leq 200k_3$ to obtain an S formula ρ, $\#(\rho) \leq 2^{1600k\_3}$, where the variables in ρ are among $v[1],\ldots,v[14(2k_3-1)]$, the constants in ρ are among $0,\ldots,i-1,14k_3,200k_3,\ldots,1800k_3$, and $\psi^{<k\_3},\rho$ are $200k_3$-equivalent.

For Limited Indiscernibility, fix $v_i,\ldots,v_{k\_3} < i$ in φ. This corresponds to fixing $v_1,\ldots,v_{k\_3} < i$ in φ^*, and fixing $v[1],\ldots,v[14k_3] < i$ in $\psi^{<k\_3}$ and in ρ. Also the replacement of constants $i,\ldots,k_3-1$ in φ by constants $i,\ldots,k_3-1$ in an order preserving way corresponds to doing so in $\psi^{<k\_3}$ and in ρ. Since ρ is an S formula, the truth value is preserved under this replacement, using invariance. QED

We can now simplify LOST(k).

DEFINITION 5.5.5. Let LOST(k)' consist of the following axioms.

BASIC. < is a strict linear ordering with least element 0, and no greatest element. $0 < 1 < \ldots < k$. $v_1 \in v_2 \rightarrow v_1 < v_2$.

LIMITED SEPARATION'. Let φ be a formula of L*(n) with no constants, where $c(\varphi) \leq i \leq k_6/10^4$. $v_2,\ldots,v_i < i \rightarrow (\exists v_{i+1} < i+1)(\forall v_1)(v_1 \in v_{i+1} \leftrightarrow \varphi^{<i} \wedge v_1 < i)$.

LIMITED INDISCERNIBILITY. Let φ be a formula of L*(n), where $c(\varphi) \leq k_3$, and all constants are $< k_3$. Let φ' result from φ by replacing all occurrences of constants $i,\ldots,k_3-1$, with occurrences of constants $i,\ldots,k_3-1$, in an order preserving way, where $1 \leq i \leq k_3$. $v_1,\ldots,v_{k\_3} < i \rightarrow (\varphi^{<k\_3} \leftrightarrow \varphi'^{<k\_3})$.

LEMMA 5.5.4. LOST(k)' and LOST(k) are logically equivalent. M(S} satisfies LOST(k)'.

Proof: It is clear that Limited Separation' is provable in LOST(k), since we can reduce $10^4 i$ down to $i+1$. QED

5.6. BOUNDED SEPARATION AND BOUNDED INDISCERNIBILITY.

In this section, we work entirely in the structure (M,S), and show that the system LOST(k)* holds in (M,S). In LOST(k)*, Separation and Indiscernibility no longer have complexity restrictions on the formulas. See Definition 5.6.12.

We need to first develop finite sequence coding. For this, we first develop the natural numbers.

LOST(k)' is a linearly ordered set theory without extensionality. A related situation appears at the beginning of section 5.7 in [Fr12].

DEFINITION 5.6.1. We use $\equiv$ for extensionality equality in the form $x \equiv y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$. We use $\approx$ as a special symbol in certain contexts.

DEFINITION 5.6.2. We write $x \approx \emptyset$ if and only if x has no elements.

DEFINITION 5.6.3. Let $r \geq 1$. We write $x \approx \{y_1,...,y_k\}$ if and only if $(\forall z)(z \in x \leftrightarrow (z = y_1 \vee ... \vee z = y_k))$.

DEFINITION 5.6.4. We write $x \approx <y,z>$ if and only if there exists a,b such that

i) $x \approx \{a,b\}$;
ii) $a \approx \{y\}$;
iii) $b \approx \{y,z\}$.

LEMMA 5.6.1. If $x \approx <y,z> \wedge w \in x$, then $w \approx \{y\} \vee w \approx \{y,z\}$. If $x \approx <y,z> \wedge x \approx <u,v>$, then $y = u \wedge z = v$.

Proof: See Lemma 5.7.2 of [Fr12]. QED

LEMMA 5.6.2. Let $4 \leq i \leq k6/10^4$ and $v_2,v_3 < i$. There exists $v_4 < i$ such that $(\forall v_1)(v_1 \in v_4 \leftrightarrow v_1 \in v_2 \vee v_1 \in v_3)$. In fact, $1 \leq i \leq k_6/10^4$ suffices.

Proof: Let i,v_2,v_3 be as given. Note that $c(v_1 \in v_2 \vee v_1 = v_3) = 4$. By Limited Separation', let $v_4 < 10^4 i$, where $(\forall v_1)(v_1 \in v_4 \leftrightarrow (v_1 \in v_2 \vee v_1 \in v_3) \wedge v_1 < i)$. Then $(\exists v_4)(v_4 < 10^4 i \wedge (\forall v_1)(v_1 \in v_2 \leftrightarrow v_1 \in v_2 \vee v_1 \in v_3))$. By Limited Indiscernibility, $(\exists v_4)(v_4 < i \wedge (\forall v_1)(v_1 \in v_2 \leftrightarrow v_1 \in v_2 \vee v_1 \in v_3))$.

For the last claim, note that we have established

$$(\forall v_2,v_3 < i)(\exists v_4)(v_4 < i \wedge (\forall v_1)(v_1 \in v_2 \leftrightarrow v_1 \in v_2 \vee v_1 \in v_3))^{<k\_3}.$$

Hence by Limited Indiscernibility, we can drop i to any of 1,2,3. QED

LEMMA 5.6.3. Let $i \geq k_6/10^4$ and $r \geq 2$. For all $v_2,\ldots,v_r < i \leq k_6$, there exists $v_{r+1} < i$ such that $(\forall v_1)(v_1 \in v_{r+1} \leftrightarrow v_1 \in v_2 \vee \ldots \vee v_1 \in v_r)$.

Proof: By Lemma 5.6.2. QED

LEMMA 5.6.4. Let $2 \leq i \leq k_6/10^4$. For all $v_2 < i$, there exists $v_3 < i$ such that $(\forall v_1)(v_1 \in v_3 \leftrightarrow v_1 = v_2)$. I.e., $v_3 \approx \{v_2\}$. In fact, we need only $1 \leq i \leq k_6/10^4$.

Proof: Let i,v_2 be as given. Note that $c(v_1 = v_2) = 2$. By Limited Separation', let $v_3 < 10^4 i$, where $(\forall v_1)(v_1 \in v_3 \leftrightarrow v_1 = v_2 \wedge v_1 < i)$. Then $(\exists v_3)(v_3 < 10^4 i \wedge (\forall v_1)(v_1 \in v_2 \leftrightarrow v_1 = v_2))$. By Limited Indiscernibility, $(\exists v_3)(v_3 < i \wedge (\forall v_1)(v_1 \in v_2 \leftrightarrow v_1 = v_2))$. For the second claim, see the proof of Lemma 5.6.2. QED

LEMMA 5.6.5. Let $i \leq k_6/10^4$ and $r \geq 2$. For all $v_2,\ldots,v_r < i$, there exists $v_{r+1} < i$ such that $v_{r+1} \approx \{v_2,\ldots,v_r\}$.

Proof: By Lemmas 5.6.3 and 5.6.4. QED

LEMMA 5.6.6. Let $i \leq k_6/10^4$. For all $v_2,v_3 < i$. There exists $v_4 < i$ such that $v_4 \approx \langle v_2,v_3\rangle$.

Proof: By Lemma 5.6.5 with $r = 1,2$. QED

DEFINITION 5.6.5. Let $r \geq 2$. We inductively define $x \approx \langle y_1,\ldots,y_r\rangle$ as follows. $x \approx \langle y_1,\ldots,y_{r+1}\rangle$ if and only if $(\exists z)(x \approx \langle z,y_3,\ldots,y_{r+1}\rangle \wedge z \approx \langle y_1,y_2\rangle)$. In addition, we define $x \approx \langle y\rangle$ if and only if $x = y$.

LEMMA 5.6.7. Let $i \leq k_6/10^4$ and $r \geq 1$. If $x \approx \langle y_1,\ldots,y_r\rangle$ and $x \approx \langle z_1,\ldots,z_r\rangle$, then $y_r = z_r \wedge \ldots \wedge y_r = z_r$. For all $y_1,\ldots,y_r < i$, there exists $x < i$ such that $x \approx \langle y_1,\ldots,y_r\rangle$.

Proof: The first claim is by external induction on $r \geq 2$, the case $r = 1$ being trivial. The basis case $r = 2$ is by Lemma 5.6.1. Suppose this is true for a fixed $r \geq 2$. Let $x \approx \langle y_1,\ldots,y_{r+1}\rangle$, $x \approx \langle z_1,\ldots,z_{r+1}\rangle$. Let u,v be such that $x \approx \langle u,y_3,\ldots,y_{k+1}\rangle$, $x \approx \langle v,z_3,\ldots,z_{k+1}\rangle$, $u \approx \langle y_1,y_2\rangle$, $v \approx \langle z_1,z_2\rangle$. By induction hypothesis, $u = v \wedge y_3 = z_3 \wedge \ldots \wedge y_{r+1} = z_{r+1}$. By Lemma 5.6.1, since $u = v$, we have $y_1 = z_1 \wedge y_2 = z_2$.

The second claim is also by external induction on $k \geq 2$, the case $k = 1$ being trivial. The basis case $k = 2$ is by Lemma 5.6.6. Suppose this is true for a fixed $r \geq 2$. Let $y_1,\ldots,y_{r+2}$. By Lemma 5.6.7, let $z \approx \langle y_1,y_2\rangle$, $z < i$. By induction hypothesis, let $x \approx <z,y_3,\ldots,y_{k+2}>$, $x < i$. Then $x \approx <y_1,\ldots,y_{k+2}>$. QED

DEFINITION 5.6.6. Let $i \leq k_6/10^4$ and $r \geq 1$. We say that R is an (i,r)-relation if and only if $(\forall x \in R)(\exists y_1,\ldots,y_r < i)(x \approx <y_1,\ldots,y_r>)$.

DEFINIITON 5.6.7. Let R be an (i,r)-relation. We define $R(y_1,\ldots,y_r) \leftrightarrow (\exists x \in R)(x \approx <y_1,\ldots,y_r>)$.

DEFINITION 5.6.8. An (i,r)-partial function is an (i,r+1)-relation such that $(\forall x_1,\ldots,x_r,y,z)(R(x_1,\ldots,x_r,y) \wedge R(x_1,\ldots,x_r,z) \rightarrow y = z)$.

DEFINITION 5.6.9. We say that R is a natural number if and only if

i. $R < 1$.
ii. R is a (1,2)-ary relation.
iii. R is a reflexive linear ordering.
iv. Let $A < 1 \wedge (\exists x \in A)(R(x,x))$. Then $(\exists x \in A)(R(x,x) \wedge (\forall y \in A)(R(y,x) \rightarrow x = y) \wedge (\exists x \in A)(R(x,x) \wedge (\forall y \in A)(R(x,y) \rightarrow x = y))$.

Condition iii asserts that every nonempty subset of the field of R, below 1, has an R least and an R greatest element.

LEMMA 5.6.8. Let R be a natural number. Let $A < k_3-1 \wedge (\exists x \in A)(R(x,x))$. Then $(\exists x \in A)(R(x,x) \wedge (\forall y \in A)(R(y,x) \rightarrow x = y) \wedge (\exists x \in A)(R(x,x) \wedge (\forall y \in A)(R(x,y) \rightarrow x = y))$.

Proof: Use Limited Indiscernibility. QED

Let R,S be natural numbers. A comparison function from R to S is a (4,2)-partial function which maps R onto a proper initial segment of S, or a proper initial segment of R onto S, or R onto S. Each proper initial segment is required to be determined by a point. F is required to be order preserving; i.e., preserve R,S.

LEMMA 5.6.9. Let R,S be natural numbers. There is a comparison function $f < 1$ between R,S. f is extensionally unique.

Proof: Let R,S be as given. By Limited Separation' and Lemma 5.6.8, we see that the partial comparison functions f < 1 between R,S cohere. By Limited Separation', we can form the set of all arguments used in these f < 1. By Limited Separation', this set can be taken to be < 2. We can apply Lemma 5.6.8 to get a largest such argument. From this, we obtain an inclusion largest partial comparison function f < 1 between R,S.

Suppose f is not a comparison function between R,S. Then we can properly extend f with one new argument to a comparison function g < 2 between R,S. By Limited Indiscernibility', we can properly extend f with one new argument to a comparison function g < 1 between R,S. This contradicts the maximality of F.

The extensional uniqueness of f is from Lemma 5.6.8. QED

LEMMA 5.6.10. Let R be a natural number and x < 1, where ¬R(x,x). There is a natural number S obtained by extending R with x at the top. S is extensionally unique. There exists a natural number of any given standard size.

Proof: Using Limited Separation', we can obtain the required S except that we will only have S < w. Then apply Limited Indiscernibility. QED

DEFINITION 5.6.10. Let $i \leq k_6/10^4$. We say that α is an i-finite sequence if and only if y ≈ <R,f>, where

i. y < i.
ii. R is a natural number.
iii. f is an (i,1)-partial function whose domain is the field of R (the x with R(x,x)).

Because of Lemma 5.6.9, we can relate any two i-finite sequences.

LEMMA 5.6.11. Let α be an i-finite sequence. We can extend α by any x < i, delete the last term, or change any term to any x < i.

Proof: Using Limited Separation', we can obtain the required modifications β, except that we only have β < i+1. Then we can apply Limited Indiscernibility to obtain β < i as required. QED

We are now prepared to handle arbitrary formulas $\varphi^{<i}$ of L(k), where $i \leq k_6/104$. Suppose the free variables of $\varphi^{<i}$ are among v[1],...,v[r], r ≥ 1. Then the (i,r)-graph of $\varphi^{<i}$ is the set of all i-finite sequences of length r for which $\varphi^{<i}$

holds. Note that the (i,r)-graph is extensionally unique if it exists.

LEMMA 5.6.12. Let $2 \leq i+1 \leq k_6/10^4$ and φ be a formula of L(k) with free variables among v[1],...,v[r], $r \geq 1$. There is an (i,r)-graph W < i+1 of $\varphi^{<i}$.

Proof: Fix i as given. We prove by induction on φ that for all $r \geq 1$, if the free variables of φ are among $v_1,...,v_r$, then LOST(k)' proves the existence of the (i,r)-graph W of $\varphi^{<i}$, with W < i+1.

1. Let φ be $v_p \in v_q$. Let $r \geq p,q$. Note that the standard integers p,q,r may be much larger than even k,n. This prevents us from directly using Limited Separation'. However, by Lemma 5.6.10, there exist natural numbers of any given of any given standard size, and in particular of sizes p,q,r. We can use these as parameters, and also use only a few quantifiers. In this way, we can form the set of all i-finite sequences of length r in which the p-th term is an element of the q-th term using Limited Separation'.

2. Let φ be $v_p < v_q$, or $v_p = v_q$. Argue as in 1 above.

3. Let $\varphi = \neg\psi$. Let the free variables of φ be among $v_1,...,v_r$. Let W < i+1 be the (i,r)-graph of ψ. We can obviously form the (i,r)-graph W' of φ using Limited Separation', where W' < i+2, with the help of a parameter for a natural number of size r. Thus there exists W' < i+2 with the required property, stated with parameters < i+1, and quantification over [0,i). Hence we can apply Limited Indiscernibility to get W' < i+1 with the same property.

4. Let $\varphi = \psi \vee \rho$. Let the free variables of ψ,ρ be among $v_1,...,v_r$. Argue as in 3 above.

5. Let $\varphi = (\exists v_p)(\psi)$. Let the free variables of φ be among $v_1,...,v_r$, and the free variables of ψ be among $v_1,...,v_s$, where $s \geq p,r$. Let W < i+1 be the (i,s)-graph of ψ. We use natural numbers of sizes s,p,r as parameters. Note that we are treating $\varphi^{<i} = (\exists v_p < i)(\psi)$. This allows us to use Limited Separation' to get the required W' < i+2, as in 3 above. We then apply Indiscernibility to obtain the required W' < i+1.

QED

DEFINITION 5.6.11. We abbreviate $\lfloor k_6/10^4 \rfloor$-3 by t. Then t ≥ 7 because of our assumption that $k_6 \geq 10^5$. The language L(k)* is based on the binary relation symbols ∈,=,<, and the constants 0,1,...,t. We use variables $v_1,v_2,...$ and the usual connectives and quantifiers.

DEFINITION 5.6.12. The axioms of LOST(k)* are as follows.

BASIC. < is a strict linear ordering with least element 0, and no greatest element. 0 < 1 < ... < t. $v_1 \in v_2 \rightarrow v_1 < v_2$.

BOUNDED SEPARATION. Let φ be a formula of L(k)*, with free variables among $v_1,...,v_r$, r ≥ 1, and no constants, and let i < t. $v_2,...,v_r < i \rightarrow (\exists v_{i+1} < i+1)(\forall v_1)(v_1 \in v_{i+1} \leftrightarrow \varphi^{<i} \wedge v_1 < i)$.

BOUNDED INDISCERNIBILITY. Let φ be a formula of L(k)*, with free variables among v1,...,vr, r ≥ 1, where all constants are < t. Let φ' result from φ by replacing all occurrences of constants i,...,t-1, with occurrences of constants i,...,t-1, in an order preserving way, where i < t. $v_1,...,v_r < i \rightarrow (\varphi^{<t} \leftrightarrow \varphi'^{<t})$.

LEMMA 5.6.13. Bounded Separation holds.

Proof: Let i < t. From Lemma 5.6.12, we obtain the (i,r)-graph W < i+1 of φ. Using Lemma 5.6.11, we can build the i-finite sequence α < i for $(v_2,...,v_r)$, where $v_2,...,v_r < i$. We can form the projection W' < i+2 of W on the first coordinate using the parameters W,α, by Limited Separation'. Since W,α < i+1, we can obtain this projection W' < i+1 by Limited Indiscernibility. QED

LEMMA 5.6.14. Let $i+2 \leq k_6/10^4$. The satisfaction relation T for formulas $\varphi^{<i}$ exists, with T < i+1.

Proof: Let i be as given. Bounded Separation and Lemma 5.6.8 gives us all of the internal induction that we need. Prove by internal induction on natural numbers r that the level r satisfaction relation T_r over [0,i), for formulas $\varphi^{<i}$ of complexity ≤ r, exists, with T_r < i+1. Indiscernibility is used here to push back from T_r < i+2 to T_r < i+1 in the induction step. We can then form the satisfaction relation T for formulas $\varphi^{<i}$, with T < i+2, referring to the T_r < i+1, being the level r satisfaction relation over [0,i). Now we have that (∃T < i+2)(T is the satisfaction relation over [0,i)). Apply Limited

Indiscernibility, to obtain ($\exists$T < i+1)(T is the satisfaction relation over [0,i)). QED

LEMMA 5.6.15. LOST(k)* holds in (M,S).

Proof: By Lemma 5.6.13, we have only to prove Bounded Indiscernibility from LOST(k)' + Bounded Separation. Let φ,φ',r,i be as in Bounded Indiscernibility, using t = $\lfloor k_6/10^4\rfloor-3 \geq 7$. Then i < $\lfloor k_6/10^4\rfloor-2$, and so Lemma 5.6.14 is applicable. The constants being moved in φ<t are among i,...,t-1. In fact, we can instead treat φ<t instead of φ, and allow the constants being moved to be among i,...,t. Of course, t is much smaller than k3. The difficulty to be overcome is that the complexity of φ,φ' is arbitrary, as is the number of parameters, r.

We can treat the formulas in question with the satisfaction relation T < t+2 for formulas $\psi^{<t+1}$, applied to $(v_1,...,v_r,i_1,...,i_s)$, where $i \leq i_1 < ... < i_s$ are the constants $\leq$ t appearing in $\varphi^{<t}$, and $v_1,...,v_r < i$. (Constants < i appearing in φ can be absorbed in the v's). Note that T takes care of φ whose complexity is too high for Limited Indiscernibility. However, r remains, and may be too big for Limited Indiscernibility. Hence we use $(\alpha,i_1,...,i_s)$, α < i, where $\alpha \approx \langle v_1,...,v_r\rangle$, via Lemma 5.6.7. Then we can apply Limited Indiscernibility in order to move around $i_1,...,i_s \leq$ t. QED

5.7. TRANSFINITE INDUCTION.

We now connect with the development in [Fr12], Chapter 5. Chapter 5 there is devoted entirely to the reversal of Proposition C of [Fr12], section 5.1. The target large cardinals there are the strongly Mahlo cardinals of finite order, which are considerably smaller than the target cardinals here - the ordinals with the k-SRP. So connecting with the [Fr12], Chapter 5 development must be done with some care, as it cannot be an automatic import.

In [Fr12], section 5.7, we start with a linearly ordered set theoretic structure, which is not internally well founded, and end with a linearly ordered second order structure (no $\in$ relation, but instead with families of relations of each arity) which is internally well founded.

In this section, we largely follow the development from [Fr12], section 5.7. We start with the linearly ordered set theoretic structure M(S), which is also not internally well founded, and also end with a linearly ordered second order structure - also with no ∈ relation, but instead with families of relations in each arity, which is internally well founded.

Many of the Lemmas need to be modified to take into account that only constants 0,...,t are used in our Bounded Separation and Bounded Indiscernibility. We generally refer to the proofs in [Fr12], section 5.7, indicating where they need to be modified. In fact, we have already begun to follow this development, with Definitions 5.6.1 - 5.6.9 and Lemma 5.6.1 - 5.6.7 in our previous section.

DEFINITION 5.7.1. Let $x < t$. We write rk(x) for the least i $\leq$ t such that $x < i$. We write $x \approx \{y: \varphi(y)\}$ if and only if $(\forall y)(y \in x \leftrightarrow \varphi(y)\}$. If there is such an x, then x is unique up to $\equiv$.

The following Lemma goes a long way in establishing that even though we have only finitely many indiscernibles in M(S) and in LOST(k)*, we can still carry out the development in [Fr12], section 5.7. This is also very useful in combination with Lemmas 5.6.5 and 5.6.7, which were established in M(S).

LEMMA 5.7.1. Let $\varphi(y,z,w_1,...,w_r)$ be a formula of L(k)* with all free variables shown, and no constants, with parameters $z,w_1,...,w_r < i \leq t-2$, and all quantifiers bounded to z (using $\leq$ z). There exists $x < i$ such that $x \approx \{y < z: \varphi(y,z,w_1,...,w_r)\}$.

Proof: Let $\varphi(y,z,w_1,...,w_r),i,r,z,w_1,...,w_r$ be as given. By Bounded Separation, Bounded Indiscernibility, and $i+1 < t$, we have

$$(\exists x < i+1)(\forall y)(y \in x \leftrightarrow \varphi{<}i \wedge y < z \wedge y < i).$$
$$(\exists x < i+1)(\forall y < i+1)(y \in x \leftrightarrow \varphi \wedge y < z).$$
$$(\exists x < i)(\forall y < i)(y \in x \leftrightarrow \varphi \wedge y < z).$$
$$(\exists x < i)(\forall y)(y \in x \leftrightarrow \varphi \wedge y < z).$$
$$(\exists x < i)(x \approx \{y < z: \varphi(y,z,w1,...,wr)\}.$$

QED

In light of Lemma 5.7.1, we frequently make critical definitions only for points < t-2.

DEFINITION 5.7.2. Let R,S be r-ary relations. We define $R \equiv' S$ if and only if $(\forall x_1,...,x_r)(R(x_1,...,x_r) \leftrightarrow S(x_1,...,x_r))$.

We define $R \subseteq' S$ if and only if $(\forall x_1,\ldots,x_r)(R(x_1,\ldots,x_r) \rightarrow S(x_1,\ldots,x_r))$.

DEFINITION 5.7.3. A binary relation is defined to be a 2-ary relation. Let R be a binary relation. We "define"

$$\mathrm{dom}(R) \approx \{x: (\exists y)(R(x,y))\}.$$
$$\mathrm{rng}(R) \approx \{x: (\exists y)(R(y,x))\}.$$
$$\mathrm{fld}(R) \approx \{x: (\exists y)(R(x,y) \vee R(y,x)\}.$$

Note that this constitutes a definition of dom(R), rng(R), fld(R) up to $\equiv$.

LEMMA 5.7.1. For all binary relations $R < i \leq t-2$, dom(R) and rng(R) and fld(R) exist $< i$.

Proof: By Lemma 5.7.1. QED

DEFINITION 5.7.4. A pre well ordering is a binary relation $R < t-2$ such that

i) $(\forall x \in \mathrm{fld}(R))(R(x,x))$;
ii) $(\forall x,y,z \in \mathrm{fld}(R))((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$;
iii) $(\forall x,y \in \mathrm{fld}(R))(R(x,y) \vee R(y,x))$;
iv) $(\forall x \subseteq \mathrm{fld}(R))(\neg(x \approx \emptyset) \wedge x < t-2 \rightarrow (\exists y \in x)(\forall z \in x)(R(y,z)))$.

Note that R is a pre well ordering if and only if R is reflexive, transitive, connected, and every nonempty subset of its field (or domain), $< t-2$, has an R least element.

Note that all pre well orderings are reflexive. Clearly for pre well orderings R, $\mathrm{dom}(R) \equiv \mathrm{rng}(R) \equiv \mathrm{fld}(R)$.

LEMAM 5.7.2. Let $R < t-2$ be a binary relation satisfying i)-iii), and iv) for $x < \mathrm{rk}(R)$. Then R is a pre well ordering.

Proof: Let R,i be as given. By Bounded Indiscernibility, we can raise i to t-2 since the parameters are $< i$, and the quantification can be taken over $[0,i)$, or over $[0,t-2)$. QED

DEFINITION 5.7.5. We will often write R(x,y) as $x \leq_R y$, and write $x =_R y$ for $x \leq_R y \wedge y \leq_R x$. We also define $x \geq_R y \leftrightarrow y \leq_R x$, $x <_R y \leftrightarrow x \leq_R y \wedge \neg y \leq_R x$, $x >_R y \leftrightarrow y <_R x$, and $x \neq_R y \leftrightarrow \neg x =_R y$.

DEFINITION 5.7.6. Let R be a pre well ordering and x ∈ fld(R). We "define" the binary relation R|<x by

$$(\forall y,z)(R|{<}x(y,z) \leftrightarrow y \leq_R z <_R x)).$$

where we require that R|<x < rk(R). Note that R|<x is unique up to ≡′.

LEMMA 5.7.3. For all pre well orderings R and x ∈ fld(R), R|<x exists. Any R|<x is a pre well ordering.

Proof: By Lemma 5.7.2. QED

When we write R|<x, we require that x ∈ fld(R).

DEFINITION 5.7.7. Let R,S be pre well orderings. We say that T is an isomorphism from R onto S if and only if

i) T < max(rk(R),rk(S)) is a binary relation;
ii) dom(T) ≡ dom(R), rng(T) ≡ dom(S);
iii) Let T(x,y), T(z,w). Then $x \leq_R z \leftrightarrow y \leq_S w$;
iv) Let $x =_R u$, $y =_S v$. Then T(x,y) ↔ T(u,v).

LEMMA 5.7.4. Let R,S be pre well orderings, and T be an isomorphism from R onto S. Let T(x,y), T(z,w). Then $x <_R z \leftrightarrow y <_S w$, and $x =_R z \leftrightarrow y =_S w$.

Proof: See [Fr12], Lemma 5.7.5. QED

LEMMA 5.7.5. Let R,S be pre well orderings, and a,b ∈ dom(S). Let T be an isomorphism from R onto S|<a, and T* be an isomorphism from R onto S|<b. Then $a =_S b$ and T ≡′ T*.

Proof: Let R,S,a,b,T,T* be as given. Suppose there exists x ∈ dom(R) such that for some y, ¬(T(x,y) ↔ T*(x,y)). By Bounded Separation, we can form the set of all x with this property. Let x be R least with this property. Now see [Fr12], Lemma 5.7.9. QED

DEFINITION 5.7.8. Let R,S be pre well orderings. Let T be an isomorphism from R onto S. Let x ∈ dom(R). We write T|<x for the restriction of T to first arguments $u <_R x$, where T|<x < rk(T). We write T|≤x for the restriction of T to first arguments $u \leq_R x$, where T|≤x < rk(T). Note that T|<x, T|≤x are each unique up to ≡'.

LEMMA 5.7.6. Let R,S be pre well orderings. Let T be an isomorphism from R onto S, and T(x,y). Then some T|<x is an isomorphism from R|<x onto S|<y.

Proof: See [Fr12], Lemma 5.7.10. QED

LEMMA 5.7.7. Let R,S be pre well orderings, T be an isomorphism from R onto S, and T* be an isomorphism from R|<x onto S|<y. Then T* ≡' T|<x and T(x,y).

Proof: See [Fr12], Lemma 5.7.11. QED

DEFINITION 5.7.9. Let T < t-2 be a binary relation. We write T^{-1} for the binary relation given by $T^{-1}(x,y) \leftrightarrow T(y,x)$. Then T^{-1} < t-2 exists. Obviously T^{-1} is unique up to ≡'.

LEMMA 5.7.8. Let R,S be pre well orderings, and T be an isomorphism from R onto S. Then T^{-1} is an isomorphism from S onto R.

Proof: See [Fr12], Lemma 5.7.12. QED

DEFINITION 5.7.9. Let R be a pre well ordering. We can append a new point ∞ on top and form the extended pre well ordering R^+ with $rk(R^+) \leq rk(R)$. The canonical way to do this is to use R itself as the new point, with $rk(R^+) = rk(R)$. This defines R^+ uniquely up to ≡'.

Clearly $R^+|<\infty$ ≡' R.

LEMMA 5.7.9. Let R,S be pre well orderings. Exactly one of the following holds.
1. R,S are isomorphic.
2. R is isomorphic to some S|<y, y ∈ dom(S).
3. Some R|<x, x ∈ dom(R), is isomorphic to S.
In case 2, the y is unique up to $=_S$. In case 3, the x is unique up to $=_R$. In all three cases, the isomorphism is unique up to ≡'.

Proof: See [Fr12], Lemma 5.7.13. The uniqueness claims rely on Lemma 5.7.2. Existence requires that we put together all local isomorphisms. Again, this is appropriately handled using Lemma 5.7.2, which will control the ranks. QED

LEMMA 5.7.10. Let R,S,S* be pre well orderings. Let T be an isomorphism from R onto S, and T* be an isomorphism from S onto S*. Define T**(x,y) ↔ (∃z)(T(x,z) ∧ T*(z,y)). Then T** is an isomorphism from R onto S*.

Proof: See [Fr12], Lemma 5.7.14. Use Lemma 5.7.2 to ensure that T** < max(rk(S),rk(S*)). QED

DEFINITION 5.7.10. Let R,S be pre well orderings. We define

R =** S ↔
R,S are pre well orderings and R,S are isomorphic.

R <** S ↔
R,S are pre well orderings and there exists y ∈ fld(S) such that R and S|<y are isomorphic.

R ≤** S ↔
R <** S ∨ R =** S.

LEMMA 5.7.11. In <**, the y is unique up to $=_S$. <** is irreflexive and transitive on pre well orderings. =** is an equivalence relation on pre well orderings. ≤** is reflexive and transitive and connected on pre well orderings. Let R,S,S* be pre well orderings. (R ≤** S ∧ S <** S*) → R <** S*. (R <** S ∧ S ≤** S*) → R <** S*. R <** S ∨ S <** R ∨ R =** S, with exclusive ∨. R ≤** S ∨ S ≤** R. (R ≤** S ∧ S ≤** R) → R =** S.

Proof: See [Fr12], Lemma 5.7.15. QED

LEMMA 5.7.8. Every nonempty set A < t-2 of pre well orderings has a ≤** least element.

Proof: See [Fr12], Lemma 5.7.16. Let A be a nonempty set of pre well orderings, and fix S ∈ A. We need to form B ≈ {y ∈ dom(S): (∃R ∈ A)(T =** S|<y)} < t-2, which we can do using Lemma 5.7.2. QED

DEFINITION 5.7.11. For x,y < t-2, we define x <# y if and only

there exists a pre well ordering S ≤ y such that
for every pre well ordering R ≤ x, R <** S.

DEFINITION 5.7.12. For x,y < t-2, we define x ≤# y if and only if

for all pre well orderings R ≤ x there exists a
pre well ordering S ≤ y such that R ≤** S.

LEMMA 5.7.9. <# is an irreflexive and transitive relation on D. ≤# is a reflexive and transitive relation on D. Let x,y ∈ D. x ≤# y ∨ y <# x. x <# y → x ≤# y. (x ≤# y ∧ y <# z) → x <# z. (x <# y ∧ y ≤# z) → x <# z. x ≤ y → x ≤# y. x <# y → x < y. x ≤# y ↔ ¬y <# x. x <# y ↔ ¬y ≤# x.

Proof: See [Fr12], Lemma 5.7.17. QED

DEFINITION 5.7.13. For x,y < t-2, we define x =# y if and only if x ≤# y ∧ y ≤# x.

LEMMA 5.7.10. =# is an equivalence relation on our linear ordering. Let x,y ∈ D. x ≤# y ↔ (x <# y ∨ x =# y). x <# y ∨ y <# x ∨ x =# y, with exclusive ∨.

Proof: See [Fr12], Lemma 5.7.18. QED

DEFINITION 5.7.14. Let x < t-2. We say that S is x-critical if and only if

i) S < rk(x);
ii) S is a pre well ordering;
iii) for all pre well orderings R ≤ x, R <** S;
iv) for all y ∈ dom(S), S|<y is ≤** some pre well ordering R ≤ x.

LEMMA 5.7.11. Assume (∀y ∈ x)(y is a pre well ordering), where x < t-2. Then there exists a pre well ordering S < rk(x) such that (∀R ∈ x)(R ≤** S) ∧ (∀u ∈ dom(S))(∃R ∈ x)(S|<u <** R).

Proof: See [Fr12], Lemma 5.7.19. We have to put together pre well orderings < x < t-2. This is accomplished by Lemma 5.7.2, which controls the ranks. QED

LEMMA 5.7.12. Assume (∀y ∈ x)(y is a pre well ordering), rk(x) < t-2. Then there exists a pre well ordering S < rk(x) such that (∀R ∈ x)(R <** S) ∧ (∀R <** S)(∃y ∈ x)(R ≤** y).

Proof: See [Fr12], Lemma 5.7.20. Argue as in Lemma 5.7.11. QED

LEMMA 5.7.13. For all x < t-2, there exists an x-critical S. If S is x-critical then x < S.

Proof: See [Fr12], Lemma 5.7.21. Argue as in Lemma 5.7.11. QED

LEMMA 5.7.14. For all x, all x-critical S are isomorphic. For all x,y, x <# y if and only if (∃R,S)(R is x-critical ∧ S is y-critical ∧ R <** S).

Proof: See [Fr12], Lemma 5.7.21. QED

LEMMA 5.7.15. If y ∈ x < t-2 then x has a <# least element. Every first order property with quantifiers bounded to t-2, with parameters < t-2, that holds of some x < t-2, holds of a <# least x. 0 is a <# least element.

Proof: See [Fr12], Lemma 5.7.24.

LEMMA 5.7.16. If x ≤ y < t-2 then x ≤# y. If x ≤ y ≤ z < t-2 and x =# z, then x =# y =# z.

Proof: See [Fr12], Lemma 5.7.25.

LEMMA 5.7.17. =# is an equivalence relation on [0,t-2) whose equivalence classes are nonempty intervals (not necessarily with endpoints). These are called the intervals of =#. x <# y if and only if the interval of =# in which x lies is entirely below the interval of =# in which y lies. There is no highest interval for =#. The constants 0,1,...,t-2 all lie in different intervals of =#, each entirely higher than the previous.

Proof: See [Fr12], Lemma 5.7.26. QED

We now develop the natural numbers directly in terms of a pre well ordering. This more direct approach than [Fr12], section 5.7, is made possible by the stronger indiscernibility that we have been using.

LEMMA 5.7.18. There is a pre well ordering R < 1 with no greatest element and no limit point.

Proof: By Lemma 5.7.13, let S be 1-critical. Note that if R < 1 is a pre well order, then R^+ < 1 is a pre well order, using R itself as the ∞. Hence S has no greatest element. Now S may (does) have a limit point. However, we can restrict to below the least limit point, forming the required pre well ordering S', but with S' < 2. Now apply Bounded Indiscernibility to replace 2 by 1. QED

DEFINITION 5.7.15. $M^* = (C,<,0,1,...,t-3,Y_1,Y_2,...)$, where the following components are defined below.

i) $(C,<)$ is a linear ordering;
ii) $0 < 1 < ... < t-3$ are elements of C;
iii) for $r \geq 1$, Y_r is a set of r-ary relations on C.

DEFINITION 5.7.16. For $x < t-2$, we write [x] for the equivalence class of x under =#.

By Lemma 5.7.17, each [x] is a nonempty interval in [0,t-2). BY Lemma 5.7.13, [x] has a strict sup < t-2.

DEFINITION 5.7.17. We define $C = \{[x]: 0 \leq x < t-2\}$. We define $[x] < [y] \leftrightarrow x <\# y$. For all $0 \leq i \leq t-3$, define this i to be [i].

DEFINITION 5.7.18. Let $r \geq 1$. We define Y_m to be the set of all m-ary relations R on C, where the following associated m-ary relation R' on [0,t-2) is given by a point R' < t-2.

$$R'(x_1,...,x_r) \leftrightarrow x_1,...,x_r < t-2 \wedge R([x_1],...,[x_r])$$

We now define the language L^2 suitable for M*.

DEFINITION 5.7.19. L^2 is based on the following primitives.

i) The binary relation symbol <;
ii) The constant symbols 0,...,t-3;
iii) The first order variables v_n, $n \geq 1$;
iv) The second order variables $B^m{}_r$, $n,m \geq 1$;

In addition, we use $\forall,\exists,\neg,\wedge,\vee,\rightarrow,\leftrightarrow,=$. Commas and parentheses are also used. "B" indicates "bounded set".

DEFINITION 5.7.19. The atomic formulas of L^2 are of the form

$$s = s'$$
$$s < s'$$
$$B^n{}_m(s_1,...,s_n)$$

where $s,s',s_1,...,s_n$ are first order variables or constant symbols. The formulas of L^2 are built up from the atomic formulas of L^2 in the usual way using the connectives and quantifiers.

Note that there is no epsilon relation in L^2.

The first order quantifiers range over C. The second order quantifiers B^m_r range over Y_m.

LEMMA 5.7.19. Let $m \geq 1$ and $R \subseteq C^m$ be M* definable (with first and second order parameters allowed). Then $\{(x_1,\ldots,x_k): R([x_1],\ldots,[x_k])\}$ is definable over $([0,t-3),<,\in)$ (with parameters allowed). If, furthermore, R is M* definable without parameters, then $\{(x_1,\ldots,x_k): R([x_1],\ldots,[x_k])\}$ is definable over $([0,t-3),<,\in)$ without parameters. If $R \subseteq C^m$ is bounded above and M* definable (with first and second order parameters allowed), then $R \in Y_r$.

Proof: See [Fr12], Lemma 5.7.29. Use Lemma 5.7.2. QED

Recall our assumption on k that $k_6 \geq 10^5$. Hence $\lfloor k_6/10^4 \rfloor \geq 10$.

LEMMA 5.7.20. There exists a structure $M^* = (C^*,<,0,1,\ldots,t-3,Y_1^*,Y_2^*,\ldots)$, $t = \lfloor k_6/10^4 \rfloor - 3 \geq 7$, such that the following holds.
i) $(C^*,<)$ is a linear ordering with no greatest element;
ii) $0 < 1 < \ldots < t-3$, and 0 is least;
iii) For all $k \geq 1$, Y_k^* is a set of k-ary relations on C^* whose field is bounded above;
iv) Let $k \geq 1$, and φ be a formula of L^2 in which the m-ary second order variable B^m_r is not free. Then $(\exists B^m_r \in Y_m^*)(\forall x_1,\ldots,x_k)(B^k_n(x_1,\ldots,x_m) \leftrightarrow (x_1,\ldots,x_m \leq y \wedge \varphi))$;
v) Every nonempty M* definable subset of C* has a < least element;
vi) Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula of L^2 without free second order variables. Let $0 < i_1,\ldots,i_{2r} \leq t-3$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type. Let $y_1,\ldots,y_r \in C^*$, $y_1,\ldots,y_r < i_1,\ldots,i_r$. Then $\varphi(i_1,\ldots,i_r,y_1,\ldots,y_r) \leftrightarrow \varphi(i_{r+1},\ldots,i_{2r},y_1,\ldots,y_r)$.

Proof: See [Fr12], Lemma 5.7.30. Note that Lemma 5.7.20 has only indiscernibles 1,...,t-3, but a stronger form of indiscernibility than [Fr12], Lemma 5.7.30. QED

5.8. ZFC + V = L.

We now connect with the development in [Fr12], section 5.8, which starts with [Fr12], Lemma 5.7.30, and ends with [Fr12], Lemma 5.8.37.

LEMMA 5.7.30 from [Fr12]. There exists a structure $M\hat{} = (C,<,0,1,+,-,\bullet,\uparrow,\log,\omega,c_1,c_2,\ldots,Y_1,Y_2,\ldots)$ such that the following holds.

i) (C,<) is a linear ordering;
ii) ω is the least limit point of (C,<);
iii) ({x: x < ω},<,0,1,+,-,•,↑,log) satisfies TR(Π^0_1,L);
iv) For all x,y $\in$ C, $\neg(x < \omega \wedge y < \omega) \rightarrow x+y = x \bullet y = x-y = x\uparrow = \log(x) = 0$;
v) The c_n, $n \geq 1$, form a strictly increasing sequence of elements of C, all > ω, with no upper bound in C;
vi) For all $k \geq 1$, Y_k is a set of k-ary relations on C whose field is bounded above;
vii) Let $k \geq 1$, and φ be a formula of L^ in which the k-ary second order variable B^k_n is not free, and the variables B^m_r range over Y_r. Then $(\exists B^k_n \in Y_k)(\forall x_1,\ldots,x_k)(B^k_n(x_1,\ldots,x_k) \leftrightarrow (x_1,\ldots,x_k \leq y \wedge \varphi))$;
viii) Every nonempty M^ definable subset of C has a < least element;
ix) Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula of L^ without free second order variables. Let $1 \leq i_1,\ldots,i_{2r}$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and the same min. Let $y_1,\ldots,y_r \in C$, $y_1,\ldots,y_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then $\varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)$.

LEMMA 5.8.37 from [Fr12]. There exists a countable model M^+ of ZFC + V = L + TR(Π^0_1,L), with distinguished elements $d_1,d_2,\ldots$, such that
i) The d's are strictly increasing ordinals in the sense of M^+, without an upper bound;
ii) Let $r \geq 1$, and $i_1,\ldots,i_{2r} \geq 1$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. Let R be a 2r-ary relation M^+ definable without parameters. Let $\alpha_1,\ldots,\alpha_r \leq \min(d_{i\_1},\ldots,d_{i\_r})$. Then $R(d_{i\_1},\ldots,d_{i\_r},\alpha_1,\ldots,\alpha_r) \leftrightarrow R(d_{i\_r+1},\ldots,d_{i\_2r},\alpha_1,\ldots,\alpha_r)$.

First we compare our Lemma 5.7.20 and [Fr12], Lemma 5.7.30.

i. Our Lemma 5.7.20 is based on a stronger strong form of indiscernibility than [Fr12], Lemma 5.7.30. Our indiscernibility does not live in ω, but the indiscernibility in [Fr12], Lemma 5.7.30 naturally lives in ω. Of course, [Fr12], Lemma 5.7.30 makes up for this by explicitly having ω, and asserting that the indiscernibles are > ω.

ii. Our Lemma 5.7.20 does not have any arithmetic, or even ω, whereas [Fr12], Lemma 5.7.30 does. In [Fr12], chapter 5, the arithmetic was both tricky and important to control, because the weaker indiscernibility involved also live naturally in ω, and because in [Fr12] we are deriving 1-

consistency. Here the indiscernibility does not live in ω, and we are only deriving consistency (existence of a model), and cannot derive 1-consistency. From Lemma 5.7.20, it is easily seen that 1,...,t-3 are each limit points.

iii. Our Lemma 5.7.20 is based on only finitely many indiscernibles. In completing the proof of Lemma 5.8.1 below, we will be working with the initial segment of M* up to the last indiscernible, t-3, as well as up to t-4.

We now assume our Lemma 5.7.20, with the structure M* = $(C^*,<,0,1,...,t-3,Y_1^*,Y_2^*,...)$.

DEFINITION 5.8.1. It will be convenient to use the truncation M** = M* = $(C^{**},<,0,1,...,t-4,Y_1^{**},Y_2^{**},...)$, where C** = C* ∩ [0,t-3), and each Y_i^{**} consists of the elements of Y_i^* that have field ⊆ [0,t-3). We use M* to establish properties of M**. In proving properties about M**, it is useful, at one point, to use the further truncation M*** = $(C^{***},<,0,1,...,t-5,Y_1^{***},Y_2^{***},...)$, where C*** = C* ∩ [0,t-4), and each Y_i^{***} consists of the elements of Y_i^* that have field ⊆ [0,t-4).

LEMMA 5.8.1 Each of 1,...,t-3 are limit points. The least limit point ω exists. ω < 1.

Proof: The first claim is by indiscernibility. The second claim is by separation. The third claim is by indiscernibility. QED

LEMMA 5.8.2. M* has impredicative comprehension for binary relations on ω, and therefore lots of internal arithmetic on ω.

Proof: Use Lemma 5.7.20 iv). QED

LEMMA 5.8.3. Let r ≥ 1 and F be an r-ary M** definable function, defined without second order parameters. For all x < t-3, $\{F(y_1,...,y_k): y_1,...,y_k < x\}$ is bounded above in M**. For all x < t-3, the restriction of F to $[0,x)^k$ is an internal function in M**. If F is definable without parameters, and i ≤ t-3, then F maps [0,i)k into [0,i).

Proof: Let F be an M** definable counterexample. From the point of view of M* it uses x < t-3, the parameter t-3 and other parameters < t-3. In M*, we can use indiscernibility to get an M*** definable counterexample. From the point of view of M*, it uses some x' < t-4, the parameter t-4, and other parameters < t-4. By indiscernibility, we can shift the second counterexample living in M*** to the same counterexample living in M** using the same x' < t-4, and the same other parameters < t-4, and also the parameter t-3, again viewed in M*. But this is impossible since t-4 itself serves as an upper bound. The last claim is immediate from separation in M*.

For the third claim, let F be as given, and i < t-3. The case i = t-3 is immediate since t-3 is the strict sup of M**. Note that F is defined in M* with the parameter t-3. Using this definition, M* sees that F maps $[0,t-3)^k$ into [0,t-3). By indiscernibility, M* sees that F maps $[0,i)^k$ into]0,i), i ≤ t-3. QED

It does not appear that Lemma 5.8.3 can be established with M** replaced by M*.

For this reason, we prefer to state all remaining Lemmas of this section within the context of M**, rather than in the context of M*. Note that we have used M*** in the proof of Lemma 5.8.3. However, we will not use M*** any further.

DEFINITION 5.8.2. We write x+1 for the immediate successor of x in <.

Note that x+1 exists since t-3 is a limit point.

LEMMA 5.8.4. Let x,y be given, x > 0. There is a unique strictly increasing internal f with dom(f) = [0,x), rng(f) an interval, and f(0) = y.

Proof: See [Fr12], Lemma 5.8.2. First establish uniqueness of approximations using separation and least element. Then using separation, take the union of approximations, and apply Lemma 5.8.3, and then extend if required, using more least element. QED

DEFINITION 5.8.3. We now define (x,y) <* (z,w) if and only if

i) max(x,y) < max(z,w); or
ii) max(x,y) = max(z,w) and (x,y) lexicographically precedes (z,w).

LEMMA 5.8.5. Every M** definable binary relation R that holds of some (x,y), x,y ∈ C*, holds of a <* least (x,y).

Proof: See [Fr12], Lemma 5.8.3. QED

LEMMA 5.8.6. There is an M** definable binary function $F:C^{**2} \rightarrow C^{**}$, defined without parameters, such that for all $x,y \in C^{**}$, $F(x,y)$ is the strict sup of all $F(z,w)$ with $(z,w) <^* (x,y)$. F is unique.

Proof: See [Fr12], Lemma 5.8.4, and the proof of this Lemma 5.8.4. QED

DEFINITION 5.8.4. In M**, we write P for the F constructed in the proof of Lemma 5.8.6.

LEMMA 5.8.7. For all $x \in C^{**}$, $x \leq P(0,x)$. Let $x,y \in C^{**}$. $x > 0 \rightarrow x,y < P(x,y)$. $x,y \leq P(x,y)$. $P:C^{**2} \rightarrow C^{**}$ is a bijection.

Proof: See [Fr12], Lemma 5.8.5. QED

DEFINITION 5.8.5. For $x_1,\ldots,x_{r+1} \in C^{**}$, we inductively define $P(x_1,\ldots,x_{r+1}) = P(P(x_1,x_2),x_3,\ldots,x_{r+1})$, for $r \geq 1$. Also define $P(x) = x$. This is our mechanism for coding sequences of points of standard finite length as points.

LEMMA 5.8.8. In each arity $r \geq 1$, P is a bijection from C** into C**. For all $r \geq 1$ and x1,...,xr $\in$ C**, $(\forall x_1,\ldots,x_r)(x_1,\ldots,x_r \leq P(x_1,\ldots,x_r))$. For all $r \geq 1$ and $1 \leq i \leq t-3$, we have x1,...,xr $< i \rightarrow P(x_1,\ldots,x_r) < i$.

Proof: For all but the last claim, see [Fr12], Lemma 5.8.6. The last claim follows from indiscernibility. QED

LEMMA 5.8.9. Let $r \geq 1$ and $R \subseteq C^{**r}$. Then R is an internal relation if and only if $\{P(x_1,\ldots,x_r): R(x_1,\ldots,x_r)\}$ is an internal set in M**. The same holds for C**, M**.

Proof: See [Fr12], Lemma 5.8.7. QED

LEMMA 5.8.10. Any M** definable subset of C** that contains 0 and is closed under +1, contains all $x < \omega$.

Proof: See [Fr12], Lemma 5.8.8. QED

DEFINITION 5.8.6. An internal finite sequence is an internal function whose domain is some $[1,x]$, $x < \omega$.

We can use P to code internal finite sequences (from C**) of indefinite length, as a single element of C**.

LEMMA 5.8.11. Let $f:[1,x] \to C^{**}$, $x < \omega$, be internal. There exists a unique internal $g:[1,x] \to C^{**}$ such that for all $1 \leq u < x$,
i) $g(1) = f(1)$;
ii) $g(u+1) = P(g(u),f(u+1))$.
For this g, we have $g(x) \geq \max(f)$.

Proof: See [Fr12], Lemma 5.8.9. QED

We use Lemma 5.8.11 to code finite sequences. Let $f:[1,x] \to C^{**}$, $x < \omega$.

DEFINITION 5.8.7. Define $\#(f) = P(x,g(x))+1$, where g is given by Lemma 5.8.11. For empty f, define $\#(f) = 0$.

LEMMA 5.8.12. For all internal finite sequences $f,f' \in C^{**}$, if $\#(f) = \#(f')$ then $f = f'$.

Proof: See [Fr12], Lemma 5.8.10. QED

LEMMA 5.8.13. $(\forall x)(\exists y > x,\omega)(\forall z,w \leq x)(P(z,w) < y)$.

Proof: See [Fr12], Lemma 5.8.11. QED

LEMMA 5.8.14. $(\forall x)(\exists y > x,\omega)(\forall z,w < y)(P(z,w) < y)$.

Proof: See [Fr12], Lemma 5.8.12. QED

LEMMA 5.8.15. Let f be an internal finite sequence, $\text{rng}(f) \subseteq [0,x]$. Then $\max(f) < \#(f) < P^*(x)$.

Proof: See [Fr12], Lemma 5.8.13. QED

We will need a notation for reverse finite sequence coding.

DEFINITION 5.8.8. Let $y \in C^{**}$ and $1 \leq i,n < \omega$. We define $y[i:n]$ to be the i-th term in the finite sequence of length n coded by y, if this exists; undefined otherwise. I.e., $y[i:n]$ is $f(i)$,

where $i \leq n$ and f is such that
$f:[1,n] \to C$, $\#(f) = y$,
provided f exists;
undefined otherwise.

By Lemma 5.8.8, the choice of f here, if it exists, is unique.

LEMMA 5.8.16. x[i:n] forms an M** definable partial function from C** × $[0,\omega)^2$ into C without parameters. Let f:[1,n] → C** be internal, 1 ≤ n < ω. The maximum value max(f) of f exists. There exists a unique x such that for all i ≤ n, f(i) = x[i:n]. max(f) < x < P*(max(f)).

Proof: See [Fr12], Lemma 5.8.14. QED

M**, with its least element and separation, is a relatively familiar context in which to work, compared with the earlier contexts in this chapter.

In order to construct the constructible hierarchy, we will use the usual language of set theory, L(∈,=).

DEFINITION 5.8.9. We take L(∈,=) to be based on ∈,=, variables v_n, n ≥ 1, and ¬,∧,∀.

By Lemmas 5.8.2 and 5.8.10, we take internal arithmetic for granted, formulated on [0,ω).

In particular, we have access to the internal set GN of all Gödel numbers of formulas of L(∈,=).

DEFINITION 5.8.10. Let R be an internal binary relation. We let R# = P*(y), where y is least such that (∀x ∈ fld(R))(x < y).

The idea is that R# is large enough to accommodate all of the internal finite sequence codes that we need, in the sense of Definition 5.8.7.

We wish to formally define the notion SAT(R,n,x,m).

DEFINITION 5.8.11. The intended meaning of SAT(R,n,x,m) is that

i) R is a binary relation;
ii) n ∈ GN, x < R#;
iii) the subscript of every free variable in the formula φ of L(∈,=) with Gödel number n is ≤ m < ω;
iv) (fld(R),R) satisfies φ at the partial assignment x[1:m],x[2:m],...,x[m:m].

Note that we allow R to be empty.

In order for clause iv) to hold, we require that x[1:m],x[2:m],...,x[m:m] ∈ fld(R).

Note that if m = 0 then the partial assignment in clause iv) is empty.

In order to make this definition over M**, we first need the following.

LEMMA 5.8.17. Let R be an internal binary relation. There exists a unique internal ternary relation SAT_R ⊆ GN × [0,R#) × [0,ω) satisfying the usual Tarski satisfaction conditions.

Proof: See [Fr12], Lemma 5.8.15. QED

DEFINITION 5.8.12. We now define SAT(R,n,x,m) if and only if R is a binary relation, and SAT_R(n,x,m) holds, where SAT_R(n,x,m) is given by Lemma 5.8.15.

DEFINITION 5.8.13. Let R be an internal binary relation. We say that n,x,m is a code over R if and only if

i) n ∈ GN;
ii) 1 ≤ m < ω;
iii) x < R# is greater than all elements of fld(R).

We remark that condition iii) is convenient because x does not interfere with the elements of fld(R).

DEFINITION 5.8.14. If n,x,m is a code over R then we write H(R,n,x,m) for

$$\{y: (\exists z)(z[1:m] = y \wedge z[2:m] = x[2:m] \wedge \ldots \wedge z[m:m] = x[m:m] \wedge SAT(R,n,z,m))\}.$$

Note that in the above definition, we use x[2:m],...,x[m:m] but not x[1:m]. This means that we can easily modify x without changing H(R,n,x,m). We will exploit this freedom below.

We think of H(R,n,x,m) as the internal subset of fld(R) that is coded by the code n,x,m. Informally, the H(R,n,x,m), where n,x,m is a code over R, code exactly the "subsets of fld(R) that are first order definable over R". The case R = ∅ is handled appropriately with this notation.

DEFINITION 5.8.15. We say that n,x,m is a minimal code over R if and only if n,x,m is a code over R such that

i) for all codes n′,x′,m′ over R, if H(R,n′,x′,m′) = H(R,n,x,m) then P(n,x,m) ≤ P(n′,x′,m′);
ii) for all y ∈ fld(R), H(R,n,x,m) ≠ {z: R(z,y)}.

Thus the minimal codes over R code exactly the R definable subsets of fld(R) that are not already of the form {z: R(z,y)}, y ∈ fld(R). Also, by minimality, no two distinct minimal codes over R code the same subset of fld(R).

Minimal codes are preferred codes used in order to ensure the propagation of extensionality as we construct the constructible hierarchy.

LEMMA 5.8.18. Let $\varphi(v_1,...,v_m)$, m ≥ 1, be a formula of L(∈,=) with Gödel number n. Let R be an internal binary relation. Then SAT(R,n,x,m) holds if and only if φ(x[1:m],...,x[m:m]) holds in (fld(R),R). H(R,n,x,m) = {y: φ(y,x[2:m],...,x[m:m]) holds in (fld(R),R)}.

Proof: See [Fr12], Lemma 5.8.16. QED

LEMMA 5.8.19. Let $\varphi(v_1,...,v_m)$, m ≥ 1, be a formula of L(∈,=). Let R be an internal binary relation, and $z_1,...,z_{m-1}$ ∈ fld(R). Then {y: $\varphi(y,z_1,...,z_{m-1})$ holds in (fld(R),R)} is either of the form {y: R(y,x)}, x ∈ fld(R), or of the form H(R,n′,x',m′), for some unique minimal code n′,x',m′ over R, but not both.

Proof: See [Fr12], Lemma 5.8.17. QED

DEFINITION 5.8.16. We say that a binary relation R is adequate if and only if

$$R(0,1) \wedge (\forall x)(\neg R(x,0)).$$

In particular, for adequate R, we have 0,1 ∈ fld(R).

For internal adequate binary relations R, we construct FODO(R) as follows.

DEFINITION 5.8.17. We define FODO(R)(u,v) if and only if either R(u,v), or

i) there exists a minimal code n,x,m over R such that v = P(n,x,m);

ii) $u \in H(R,n,x,m)$.

The reason that we need the adequacy of R is that $\emptyset = \{x: R(x,0)\}$, and so there is no minimal code n,x,m over R with $H(R,n,x,m) = \emptyset$. It will be convenient to have the sets with minimal codes over R be nonempty.

DEFINITION 5.8.18. Let R be an internal binary relation. We say that R is extensional if and only if for all $x,y \in fld(R)$, $(\forall z)(R(z,x) \leftrightarrow R(z,y)) \rightarrow x = y$.

DEFINITION 5.8.19. We say that a binary relation R is sharply extended by a binary relation S if and only if

i) $(\forall x \in fld(S)\backslash fld(R))(\forall y \in fld(R))(y < x)$;
ii) $(\forall x,y \in fld(R))(R(x,y) \leftrightarrow S(x,y))$.
iii) $S(x,y) \wedge y \in fld(R) \rightarrow x \in fld(R)$.
iv) fld(R) is a proper subset of fld(S).

LEMMA 5.8.20. Let R be an internal adequate binary relation. Then FODO(R) is an internal adequate binary relation. In addition, R extensional $\rightarrow$ FODO(R) extensional. FODO(R) sharply extends R. $(\forall x,y)(R(x,y) \rightarrow x < y) \rightarrow (\forall x,y)(FODO(R)(x,y) \rightarrow x < y)$.

Proof: See [Fr12], Lemma 5.8.18. QED

LEMMA 5.8.21. Let R be an internal adequate binary relation. Every set definable in (fld(R),R) is of the form $\{x: FODO(R)(x,y)\}$, where $y \in fld(FODO(R))$.

Proof: See [Fr12], Lemma 5.8.19. QED

Here we interpret Lemma 5.8.21 as a scheme of assertions about M**, where we take “definable” in the external sense. However, we also want to interpret Lemma 5.8.21 in a stronger, internal sense - using SAT_R from Lemma 5.8.17. This stronger form of Lemma 5.8.21 can also be proved with the help of internal inductions.

We now wish to transfinitely iterate the FODO operation. The base of the transfinite iteration will be the adequate relation

$$R_0(x,y) \leftrightarrow x = 0 \wedge y = 1.$$

In order to accomplish this, we must be a bit careful. Firstly, we must note that, conceptually, we are manipulating internal relations, and these internal relations are not points; they are elements of Y_2. Furthermore, these internal relations are not even coded as points. In contrast, recall that internal finite sequences f of points are coded as points using f#.

Secondly, note that the operation that sends appropriate R to FODO(R) is even further removed from being an object. It is merely a description of a relationship between objects (not even between points), given in a first order way, without parameters, over M**.

Our strategy is to properly define what we mean by a transfinite iteration of the operation up through a point, *as an object*. The objects for this purpose are the elements of the Y_k, $k \geq 1$. These are components of M** (going back to M*).

DEFINITION 5.8.20. Let T be a k+1-ary relation, $k \geq 1$. For $x \in C^{**}$, we write T_x for the cross section $\{(y_1,...,y_k): T(x,y_1,...,y_k)\}$.

Note that T_x is a k-ary relation.

LEMMA 5.8.22. Let $x \in C^{**}$. There is a unique internal ternary relation T such that
i) $T_0 = R_0$;
ii) For all $y < x$, $T_{y+1} = \text{FODO}(T_y)$;
iii) For all limits $y \leq x$, $T_y = \cup_{z<y}T_z$;
iv) For all $y \leq x$, T_y is adequate;
v) For all $y > x$, $T_y = \emptyset$.

Proof: See [Fr12], Lemma 5.8.20. As this is a particularly important juncture, we review the proof from there.

Define $\Gamma(T,x)$ if and only if $x \in C^{**} \wedge T$ is an internal ternary relation obeying i)-v).

First prove that for all x,T,T', $\Gamma(T,x) \wedge \Gamma(T',x) \rightarrow T = T'$. In fact, the T's such that $(\exists x)(\Gamma(T,x))$ are comparable in that any two agree on their common domain.

To prove existence, let $u > 0$, $u \in C^{**}$, and suppose $(\forall x < u)(\exists T)(\Gamma(T,x))$. We now show $(\exists T)(\Gamma(T,u))$.

The case u = 0 is obvious, by defining $T(a,b,c) \leftrightarrow a = 0 \wedge R_0(b,c)$.

Assume u is a successor, u = v+1. Let $\Gamma(T,v)$. Define

$$T'(a,b,c) \leftrightarrow$$
$$T(a,b,c) \vee (a = v+1 \wedge FODO(T_v)(b,c)).$$

To see that T′ is internal, it suffices to show that T′ is bounded. This follows from the boundedness of T and $FODO(T_v)$.

Assume u is a limit. Define

$$T^*(a,b,c) \leftrightarrow$$
$$a < u \wedge (\exists T)(\Gamma(T,a) \wedge T(a,b,c)).$$

To see that T* is internal, it suffices to show that T* is bounded above in M**. We have $(\forall a < u)(\exists! T)(\Gamma(T,a))$ in M**, by the first claim (uniqueness). Thus T* is a M** definable, defined without second order parameters. Hence T* is bounded above in M** by Lemma 5.8.3.

Now see the remainder of the proof of [Fr12], Lemma 5.8.20. QED

DEFINITION 5.8.21. For each $x \in C^{**}$, we let $L(x) = T_x$, where T is the ternary relation given by Lemma 5.8.22. Thus each $L(x) \in Y_2$.

DEFINITION 5.8.22. For each $x \in C^{**}$, we define $L[x] = fld(L(x))$. Note that $L[0] = \{0,1\}$, and that $L[x] \subseteq C^{**}$.

DEFINITION 5.8.23. We define $L[\infty]$ as the union of the L[x].

We caution the reader that $L[\infty] \subseteq C^{**}$ is not internal, because it is not bounded. It is, however, M** definable without any parameters.

DEFINITION 5.8.24. We define $L(\infty)$ to be the union of the L(x).

Thus $L(\infty)(x,y)$ if and only if there exists $z \in C$ such that $L(z)(x,y)$. Obviously $L(\infty) \subseteq C^2$.

The various L[x] correspond to the initial segments of the constructible hierarchy. The various L(x) correspond to the epsilon relations on the initial segments of the constructible hierarchy. L[∞] corresponds to the class of constructible sets. L(∞) corresponds to the epsilon relation on the class of constructible sets.

Clearly L(∞) is the version of the epsilon relation on the constructible sets in M**, and is a binary relation. Its field is L[∞].

We caution the reader that L[x] may not be an initial segment of points, and may not be a subset of [0,x). It may have elements that are greater than x.

LEMMA 5.8.23. $L(0) = R_0$. For all x ∈ C**, L(x+1) = FODO(L(x)). For all limits x ∈ C**, L(x) is the union of the L(y), y < x. For all x < y, L(x) is sharply extended by L(y). Each L(x) is extensional. Each L(x) has L(x)(y,z) → y < z.

Proof: See [Fr12], Lemma 5.8.21. QED

DEFINITION 5.8.25. Let x ∈ L[∞]. We write lrk(x) for the least y such that x ∈ L[y+1]. This is the L rank of x. Note that lrk is a function from C into C that is M** definable without parameters.

LEMMA 5.8.24. Let x,y ∈ C**. L(∞)(x,y) → (lrk(x) < lrk(y) ∧ x < y). L(∞)(x,y) ↔ L(lrk(y)+1)(x,y). L[∞] ∩ [0,x) ⊆ L[x].

Proof: See [Fr12], Lemma 5.8.22. QED

DEFINITION 2.8.26. A Δ_0 formula of L(∈,=) is a formula of L(∈,=) in which all quantifiers are ∈ bounded; i.e.,

$$(\exists x \in y)$$
$$(\forall x \in y)$$

where x,y are distinct variables.

LEMMA 5.8.25. Let $\varphi(x_1,\ldots,x_k)$ be a Δ_0 formula of L(∈,=). Let $y_1,\ldots,y_k$,z,w be such that $y_1,\ldots,y_k \in$ L[z],L[w]. Then $\varphi(y_1,\ldots,y_k)$ holds in (L[z],L(z)) if and only if $\varphi(y_1,\ldots,y_k)$ holds in (L[w],L(w)) if and only if $\varphi(y_1,\ldots,y_k)$ holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.23. QED

LEMMA 5.8.26. Extensionality, pairing, and union hold in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.24. QED

LEMMA 5.8.27. Infinity holds in (L[ω+1],L(ω+1)). Infinity holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.25. QED

LEMMA 5.8.28. Every L(x) is internally well founded. L(∞) is internally well founded. Foundation holds in every (L[x],L(x)). Foundation holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.26. QED

LEMMA 5.8.29. Let $n \geq 1$ and $\varphi_1,\ldots,\varphi_n$ be formulas of L(∈,=) that begin with, respectively, existential quantifiers $(\exists y_1),\ldots,(\exists y_n)$. For all z there exists w > z such that the following holds. Let $1 \leq i \leq n$. Let the free variables of φ_i be assigned elements of L[z]. If φ_i holds in (L[∞],L(∞)) then $(\exists y_i \in L[w])(\varphi_i(y_i))$ holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.27. QED

LEMMA 5.8.30. Let $\varphi(v_1,\ldots,v_k)$ be a formula of L(∈,=). For all z there exists w > z such that the following holds. Let $y_1,\ldots,y_k \in L[w]$. Then $\varphi(y_1,\ldots,y_k)$ holds in (L[∞],L(∞)) if and only if $\varphi(y_1,\ldots,y_k)$ holds in (L[w],L(w)).

Proof: See [Fr12], Lemma 5.8.28. Use Lemma 5.8.3. QED

DEFINITION 5.8.27. Collection is the scheme

$$(\forall x \in y)(\exists z)(\varphi) \rightarrow (\exists w)(\forall x \in y)(\exists z \in w)(\varphi)$$

where φ is a formula of L(∈,=), x,y,z,w are distinct variables, and w is not free in φ.

LEMMA 5.8.31. Every instance of Separation holds in (L[∞],L(∞)). Every instance of Collection holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.29. Use Lemma 5.8.3. QED

DEFINITION 5.8.28. Let ZF\P be all axioms of ZF less Power Set, using Collection.

LEMMA 5.8.32. Every axiom of ZF\P with Collection holds in (L[∞],L(∞)).

Proof: See [Fr12], Lemma 5.8.30. QED

???LEMMA 5.8.33. For all i ≤ t-5, L[i] ⊆ [0,i+1).

Proof: Use the last claim of Lemma 5.8.3. QED

DEFINITION 5.8.29. It is very convenient to define x ⊆* y if and only if

$$x \in L[\infty] \wedge (\forall z \in L[\infty])(L(\infty)(z,x) \rightarrow L(\infty)(z,y)).$$

Also, x ⊆** y if and only if

$$x \in L[\infty] \wedge (\forall z \in L[\infty])(L(\infty)(z,x) \rightarrow z \in L[y]).$$

LEMMA 5.8.34. Let x ⊆** t-5. Then x < t-4.

Proof: Assume this is false. Let x be least such that x ≥ t-4 ∧ x ⊆** t-5. By indiscernibility, let y be least such that y ≥ t-5 ∧ y ⊆** t-6. By indiscernibility, jet z be least such that z ≥ t-4 ∧ z ⊆** t-6. By the last claim of Lemma 5.8.3, y < t-4. Also by indiscernibility, y ⊆** z ∧ z ⊆** y. Hence y = z, which is a contradiction. QED

LEMMA 5.8.35. Power Set holds in (L[∞],L(∞)).

Proof: Suppose Power Set fails in (L[∞],L(∞)). Let u be least such that ¬(∃v)(∀x ⊆** u → x < v). By the last claim of Lemma 5.8.3 applied to the constantly u function, u < 1. This is a contradiction, since by Lemma 5.8.34, we can set v = 2. QED

LEMMA 5.8.36. ZF holds in (L[∞],L(∞)).

Proof: By Lemmas 5.8.32 and 5.8.35. QED

LEMMA 5.8.37. There exists a countable model M# of ZF, with constants c1,...,c_n, $n = \lfloor k_6/10^4 \rfloor - 7$, such that the following holds.
i) c1 < ... < ct-3 are ordinals.
ii) Let r ≥ 1 and $\varphi(v_1,...,v_{2r})$ be a formula in ∈,= with free variables among c1,..., c2r. Let $1 \leq i_1,...,i_{2r} \leq t-4$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type. Let $y_1,...,y_r \in M\#$, $y_1,...,y_r$ < ci1,...,cir. Then $\varphi(ci1,...,cir,y_1,...,y_r) \leftrightarrow \varphi(ci_{r+1},...,ci_{2r},y_1,...,y_r)$.

Proof: By Lemma 5.8.36 and indiscernibility. 1,...,t-4 are present in M**, and become 1,...,n, since $t = \lfloor k_6/10^4 \rfloor - 3$. QED

LEMMA 5.8.38. There exists a countable model M# of ZF, with constants $c1,\ldots,c_n$, $t = \lfloor k_6/10^4 \rfloor - 7$, such that the following holds.
i) $c_1 < \ldots < c_{t-3}$ are ordinals.
ii) Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula in $\in,=$ with free variables among $c_1,\ldots, c_{2r}$. Let $1 \leq i_1,\ldots,i_{2r} \leq t-4$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type. Let $y_1,\ldots,y_r \in M\#$, $y_1,\ldots,y_r < i_1,\ldots,i_r$. Then $\varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)$.

Proof: By Lemma 5.8.37 and the usual inner model construction applied to the M# of Lemma 5.8.37. QED

THEOREM 5.8.39. The following is provable in ZFC. If IMCT(2↑10,2↑10) then ZFC is consistent.

Proof: From Lemma 5.8.36. Recall that all during section 5, we have been operating under the assumption that IMCT(n,k) holds, where $n \geq k$ and $k_6 \geq 10^5$. If we set $n = k = 2\uparrow 10$, then $k_6 = 2\uparrow 4 = 2^{16} = 65{,}536 > 10^5$. Now apply Lemma 5.1. QED

5.9. EQUIVALENCE WITH CON(SRP).

We fix M# from Lemma 5.8.38.

LEMMA 5.9.1. In M#, c_n is satisfied to be a purely (n-1)-subtle ordinal, where $n = \lfloor k_6/10^4 \rfloor - 7$.

Proof: Assume that c_n is not satisfied to be a purely (n-1)-subtle ordinal.

Using ZFC + V = L in M#, let $f:S_{n-1}(c_n) \to c_n$ be a regressive function, M# definable with parameter c_n, which witnesses that c_n is not purely (n-1)-subtle. Let $A,B \in S_{t-4}(\{c_1,\ldots,c_n\})$. We can assume that $\min(A) \leq \min(B)$. Let $\alpha = f(A)$. Since f is regressive, $\alpha < \min(A \cup B)$. Therefore by indiscernibility in M#, we have $\alpha = f(A) \leftrightarrow \alpha = f(B)$, using the strictly increasing enumerations of A,B. Hence $f(A) = f(B)$.

We have thus verified that f does not violate that c_n is purely (n-1)-subtle. This is a contradiction. QED

THEOREM 5.9.2. The following is provable in ZFC. IMCT(k,k) → Con(SRP[$\lfloor k_6/10^4 \rfloor$-8)).

Proof: By Lemma 5.9.1. QED

Below we will show that ACA' suffices to carry out the reversal in section 5. The major feature of ACA' surrounds the satisfaction relation for countable structures. The full satisfaction relation itself is not available in ACA'. However, the local satisfaction relations are. I.e., ACA' proves that

#) for any countable structure M and any positive integer n,
the satisfaction relation for M for formulas with at most n quantifiers exists,
where n is treated as a variable.

We use #) to define the full satisfaction relation for M by: M satisfies φ at assignment α if and only if the satisfaction relation for M for formulas with at most the number of quantifiers in φ holds at φ,α. Note that this is only a description of the full satisfaction relation, and ACA' does not prove that the full satisfaction relation exists.

ACA' proves soundness in the form

##) if the countable structure M satisfies φ, then φ is formally consistent.

WKL_0 proves completeness in the form

###) if a sentence φ is formally consistent, then it has a model M for which the full satisfaction relation exists.

Note that if n is treated as a numeral (e.g., S...S0, where S is the successor function), then the satisfaction relation for formulas with at most n quantifiers for M exists, provably in ACA_0 (but not in RCA_0).

THEOREM 5.9.3. ACA' proves the equivalence of the Invariant Maximal Clique Theorem, the Invariant Maximal Clique Theorem (extended), the Maximal Clique Characterization Theorem, and Con(SRP). In Theorems 5.8.39 and 5.9.2, we can replace ZFC with ACA'.

Proof: By Theorem 4.3.20 and the argument before Theorem 4.1.11, ACA' proves the chain of implications

Con(SRP) → IMCT(extended) → IMCC → IMCT.

Also see Theorem 4.1.4. It now suffices to show that ACA' + IMCT proves Con(SRP), completing the circle.

For this, it suffices to verify that ACA' is sufficient to carry out the steps in section 5. But we need to carry out these steps where k is a variable, and not a specific integer. For this reason, satisfaction relations do enter.

section 5.1. RCA_0 is sufficient.

section 5.2. RCA_0 is sufficient.

section 5.2. The Tarskian semantics given before Lemma 5.2.5 is stated in full generality, and could raise issues. However, in section 5.2, it is not used in any substantive way. We give an induction argument on P(r), which is in the form of a bounded existential quantifier followed by an unbounded universal quantifier. This level of induction is available in RCA_0.

section 5.3 - 5.4. RCA_0 is sufficient.

section 5.5. Lemmas 5.5.2, 5.5.3 use #) for M(S).

sections 5.6 - 5.9. The development continues to be based on #) for M(S) and other structures built explicitly from M(S). In the construction of M* according to Definition 5.7.15, for use in Lemma 5.7.20, we form a structure whose domain is the set of equivalence relations, which is third order. What is just is good is to take the set of all points which are the numerically least (using the enumeration of the countable domain) elements of their equivalence class, and augment with the appropriate structure. The construction of the Y's is available in ACA_0 since it involves low level internal notions. Then #) is used. For Theorems 5.8.30 and 5.9.2, ##) is used.

The remaining claim is also established by this verification that ACA' suffices. QED

THEOREM 5.9.4. Let X be the family of statements IMCT(k,n), for k,n ≥ 1. ACA' ∪ X and ACA' + {Con(SRP[m]: m ≥ 1} are logically equivalent. No finite subset of X is sufficient to derive all elements of X, over ZFC (assuming SRP is consistent).

Proof: For the first claim, let note that Con(SRP[m]) is implied by some IMCT(k,k) + ACA', by Lemma 5.9.2. Also note that every IMCT(k,n) is implied by some Con(SRP[m]) + WKL_0 by Theorems 3.1 and 4.3.20. For the second claim, suppose some finite subset of X derives all elements of X, over ZFC. By the first claim, there exists n such that ZFC + Con(SRP[n]) proves Con(SRP[n+1]). This violates Gödel's Second Incompleteness Theorem, assuming SRP is consistent. QED

THEOREM 5.9.5. The Invariant Maximal Clique Theorem is not provable from any set of consequences of SRP consistent with ACA'. The preceding claim is provable in RCA_0. For finite sets of consequences, the first claim is provable in EFA. In particular, the Invariant Maximal Clique Theorem is not provable in ZFC, assuming ZFC is consistent.

Proof: Suppose T is a set of sentences such that

SRP proves T.
T + ACA' is consistent.
T proves IMCT.

By Theorem 5.9.3, T + ACA' proves Con(SRP). Arguing in ACA', since SRP proves T + ACA', we see that Con(SRP) $\rightarrow$ Con(T + ACA'). Hence T + ACA' proves Con(T + ACA'). By Gödel's second incompleteness theorem, T + ACA' is inconsistent. This is a contradiction. The argument is obviously formalizable in RCA_0. If T is finite, then there is no need for RCA_0, and we can instead use EFA = $I\Sigma_0(\exp)$. QED

6. FURTHER RESULTS AND OPEN QUESTIONS.

In section 6.1, we introduce the order theoretic equivalence relations, and the standard order theoretic equivalence relations. These conditions are based on fundamental conditions that are satisfied by our three main equivalence relations: order equivalence, $Z^+(\infty)$ order equivalence, and upper $Z^+(\infty)$ order equivalence.

A basic division is whether a standard order theoretic equivalence relation is contained in $Z^+(\infty)$ order equivalence. We show that if it is, then it must be among only five (Theorem 6.1.6). In section 6.2, we determine the truth value of IMCT for all five of these equivalence relations.

We show that IMCT fails badly for the remaining standard order theoretic equivalence relations - the ones not contained in $Z^+(\infty)$ order equivalence. See Theorem 6.2.7.

The situation is neatly summarized by Theorem 6.2.9. In particular, the only standard order theoretic equivalence relations for which the Invariant Maximal Clique Theorem holds are upper Z+(∞) order equivalence, and strong Z+(∞) order equivalence. The latter is a subset of the former.

In section 6.3, we list various open problems, most of which appear in the text.

6.1. ORDER THEORETIC EQUIVALENCE RELATIONS.

All notions of invariance discussed in this paper are for subsets of subspaces of $Q[-\infty,\infty]^k$, $k \geq 1$, and are induced by equivalence relations on the $Q[-\infty,\infty]^k$, $k \geq 1$ in the following way. A subset S of a subspace of $Q[-\infty,\infty]^k$ is "invariant" if and only if for all x,y in the subspace, if x,y are "equivalent" and $x \in S$, then $y \in S$.

For any notion of invariance induced by an equivalence relation, the equivalence relation can be recovered because the minimal nonempty invariant sets are exactly the equivalence classes under any inducing equivalence relation.

So we will always focus on the equivalence relations and not on the induced invariance notions.

Thus far, we have based this paper on three notions of invariance for subsets of the $Q[-\infty,\infty]^k$, $k \geq 1$: order invariance, and upper $Z^+(\infty)$ order invariance - with $Z^+(\infty)$ order invariance mentioned in passing.

DEFINITION 6.1.1. We use $Q[-\infty,\infty]^*$ = the set of all nonempty finite sequences from $Q[-\infty,\infty]$ as the master space, and view our equivalence relations as having field $Q[-\infty,\infty]^*$.

We define the equivalence relation ~ associated with the structure $(Q[-\infty,\infty],<,Z^+(\infty))$, given by

$$x \sim y \leftrightarrow x,y \text{ are locally isomorphic in } (Q[-\infty,\infty],<,Z^+(\infty));$$

i.e.,

$x \sim y \leftrightarrow$ x,y are order equivalent $\wedge$ $(\forall i)(x_i \in Z^+(\infty) \leftrightarrow y_i \in Z^+(\infty))$

where the order equivalence of x,y (obviously) requires having the same length, lth(x) = lth(y).

For R = order invariance, $Z^+(\infty)$ order invariance, and upper $Z^+(\infty)$ order invariance, on the field $Q[-\infty,\infty]^*$, the following two basic conditions hold.

i. $(Q[-\infty,\infty],<,Z^+(\infty))$ Invariant. $(x,y) \sim (z,w) \rightarrow (R(x,y) \rightarrow R(z,w))$.

ii. Essentially Binary. $R(x,y) \leftrightarrow \mathrm{lth}(x) = \mathrm{lth}(y) \wedge (\forall i,j \in \{1,\ldots,\mathrm{lth}(x)\})(R((x_i,x_j),(y_i,y_j)))$.

Another important condition that holds of the three main equivalence relations is

iii. $Z^+(\infty)$ Standard. If $x,y \in Z^+(\infty)^*$ have the same length, then $R(x,y) \leftrightarrow$ x,y are order equivalent.

An important consequence of i,ii is

iv. Symmetric. If lth(x) = lth(y) and π is a permutation of $\{1,\ldots,\mathrm{lth}(x)\}$, then $R(x,y) \leftrightarrow R(\pi x,\pi y)$. Here π acts on coordinate positions.

DEFINITION 6.1.1. We say that R is an order theoretic equivalence relation on $Q[-\infty,\infty]^*$ ($Q[-\infty,\infty]k$) if and only if R is an equivalence relation on $Q[-\infty,\infty]^*$ ($Q[-\infty,\infty]^k$) satisfying conditions i-ii. We say that R is a standard order theoretic equivalence relation on $Q[-\infty,\infty]^*$ ($Q[-\infty,\infty]^k$) if and only if R is an order theoretic equivalence relation on $Q[-\infty,\infty]^*$ ($Q[-\infty,\infty]k$) satisfying $Z+(\infty)$ Standard.

DEFINIITON 6.1.2. We say that R is an (standard) order theoretic equivalence relation if and only if R is an (standard) order theoretic equivalence relation on $Q[-\infty,\infty]^*$.

THEOREM 6.1.1. The restriction of any (standard) order theoretic equivalence relation to $Q[-\infty,\infty]^2$ is an (standard) order theoretic equivalence relation on $Q[-\infty,\infty]^2$. Every (standard) order theoretic equivalence relation on $Q[-\infty,\infty]^2$ extends uniquely to an (standard) order theoretic equivalence relation. Every order theoretic equivalence relation is Symmetric. The number of (standard) order theoretic equivalence relations is the same as the number of (standard) order theoretic equivalence relations on domain $Q[-\infty,\infty]^2$, which is finite.

Proof: For the first claim, let R be a (standard) order theoretic equivalence relation. We need only check that Essentially Binary holds for the restriction R* of R to Q[-∞,∞]2. We have

$$R^*(x,y) \leftrightarrow (\forall i,j \in \{1,2\})(R((x_i,x_j),(y_i,y_j))) \leftrightarrow (\forall i,j \in \{1,2\})(R^*((x_i,x_j),(y_i,y_j))).$$

For the second claim, let $R \subseteq Q[-\infty,\infty]^4$ be an order theoretic equivalence relation on $Q[-\infty,\infty]^2$. For $x,y \in Q[-\infty,\infty]^*$, define $R'(x,y) \leftrightarrow lth(x) = lth(y) \wedge (\forall i,j \in \{1,\ldots,lth(x)\})(R(x_i,x_j,y_i,y_j))$.

We first check that R is the restriction of R' to $Q[-\infty,\infty]^2$. We have for all $x,y \in Q[-\infty,\infty]^2$,

$$\begin{gathered} R'(x,y) \leftrightarrow (\forall i,j \in \{1,2\})(R((x_i,x_j),(y_i,y_j))) \leftrightarrow \\ R((x_1,x_1),(y_1,y_1)) \wedge R((x_2,x_2),(y_2,y_2)) \wedge R((x_1,x_2),(y_1,y_2)) \wedge \\ R((x_2,x_1),(y_2,y_1)) \leftrightarrow \\ R((x_1,x_2),(y_1,y_2)) \leftrightarrow \\ R(x,y). \end{gathered}$$

We now verify that R' is an order theoretic equivalence relation.

For $(Q[-\infty,\infty],<,Z^+(\infty))$ Invariant, let (x,y),(z,w) be $(Q[-\infty,\infty],<,Z^+(\infty))$ equivalent, where $x,y,z,w \in Q[-\infty,\infty]^*$. Then x,y,z,w have the same length, and

$$R'(x,y) \leftrightarrow (\forall i,j \in \{1,\ldots,lth(x)\})(R(x_i,x_j,y_i,y_j)).$$
$$R'(z,w) \leftrightarrow (\forall i,j \in \{1,\ldots,lth(x)\})(R(z_i,z_j,w_i,w_j)).$$

Hence for all $i,j \in \{1,\ldots,k\}$, $(x_i,x_j,y_i,y_j),(z_i,z_j,w_i,w_j)$ are order equivalent and $Z^+(\infty)$ equivalent. Therefore $R(x,y) \leftrightarrow R(z,w)$.

For Essential Binary, let $x,y \in Q[-\infty,\infty]^*$. Then

$$\begin{gathered} R'(x,y) \leftrightarrow (\forall i,j \in \{1,\ldots,k\})(R(x_i,x_j,y_i,y_j)) \leftrightarrow \\ (\forall i,j \in \{1,\ldots,k\})(R'(x_i,x_i,y_i,y_i) \wedge R'(x_j,x_j,y_j,y_j) \wedge \\ R'(x_i,x_j,y_i,y_j) \wedge R'(x_j,x_i,y_j,y_i)) \leftrightarrow \\ (\forall i,j \in \{1,\ldots,k\})(R'(x_i,x_j,y_i,y_j)). \end{gathered}$$

Suppose R is Z+(∞) standard. Then R' is $Z^+(\infty)$ Standard by the following argument. Let $x,y \in Z^+(\infty)^*$ have the same length. Then

$$R'(x,y) \leftrightarrow (\forall i,j \in \{1,...,lth(x)\})(R(x_i,x_j,y_i,y_j)) \leftrightarrow$$
$$(\forall i,j \in \{1,...,lth(x)\})((x_i,x_j),(y_i,y_j) \text{ are order equivalent}) \leftrightarrow$$
$$(\forall i,j \in \{1,...,lth(x)\})(x_i < x_j \leftrightarrow y_i < y_j) \leftrightarrow$$
$$x,y \text{ are order equivalent.}$$

This establishes existence. The uniqueness is immediate from Essentially Binary. This verifies the second claim.

For the third claim, let R be an order theoretic equivalence relation. Let lth(x) = lth(y), and π be a permutation of {1,...,2lth(x)}. Then

$$R'(x,y) \leftrightarrow (\forall i,j \in \{1,...,lth(x)))(R'(x_i,x_j,y_i,y_j)).$$
$$R'(\pi x,\pi y) \leftrightarrow (\forall i,j \in \{1,...,lth(x)))(R'(\pi x_i,\pi x_j,\pi y_i,\pi y_j)) \leftrightarrow$$
$$(\forall i,j \in \{1,...,lth(x)))(R'(x_{\pi i},x_{\pi j},y_{\pi i},y_{\pi j}))$$

The second and fifth of these assertions are equivalent since the aggregate of quadruples i,j,i,j are the same as the aggregate of quadruples π_i,π_j,π_i,π_j.

The first part of the last claim is immediate from the first and second claims. For finiteness, it suffices to remark that the number of equivalence classes of $Q[-\infty,\infty]^4$ under ~ is finite. In fact, there are representatives from each equivalence class in A^4, where A is the set of all nonnegative multiples of 1/4 that lie in [0,4]. QED

LEMMA 6.1.2. Let R be a standard order theoretic equivalence relation on $Q[-\infty,\infty]^*$, which is not contained in $Z^+(\infty)$ order equivalence. Then
i. R((0,0),(1/2,1/2));
ii. R((0,1/2),(1/3,1/2)); or
iii. R((0,1),(1/2,1)); or
iv. R((0,1/2),(0,2/3)); or
v. R((1,3/2),(1,4/3)); or
vi. R((1,3/2),(3/2,2)).
We can also use
vi'. R((0,1),(0,2)) $\wedge$ R((1,5/2),(2,5/2)).

Proof: Let R be as given. Let R((p,q),(p',q')), where (p,q),(p',q') are not $Z^+(\infty)$ order equivalent. Because of clause i, we can assume that $(\forall a,b,c,d)(R((a,b),(c,d)) \rightarrow$ (a,b),(c,d) are order equivalent).

case 1. $p = q \wedge p' = q'$. Then $R((p,p),(p',p')) \wedge p \neq p' \wedge (p \notin Z+ \vee p' \notin Z+)$.

case 3.1. $p \notin Z+$. By changing p a little, find $R((p,p),(p'',p''))$, $p'' \notin Z+$. Hence ii holds.

case 3.2. $p' \notin Z+$. Argue as in case 3.1. Hence ii holds.

case 2. $p < q \wedge p' < q'$. Hence $\neg((p \neq p' \rightarrow p,p' \in Z+) \wedge (q \neq q' \rightarrow q,q' \in Z+))$.

case 2.1. $p \neq p' \wedge (p \notin Z+ \vee p' \notin Z+)$.

case 2.1.1. $p < p' \wedge p \notin Z+$. By changing p a little, find $R((p'',q),(p',q'))$, $R((p'',q),(p,q))$, $p'' \notin Z+$. Then iii or iv holds.

case 2.1.2. $p' < p \wedge p \notin Z+$. If $p \neq q'$ then change p a little, so iii or iv holds. Assume $p = q'$, so that $R((p,q),(p',p))$, $p' < p < q$. If $p' \notin Z+ \vee q \notin Z+$, then by changing p' or q a little, we see that iii or v holds. So assume $p',q \in Z+$. Then vii holds.

case 2.1.3. $p < p' \wedge p' \notin Z+$. Argue as in case 2.1.2. Then iii, iv, v, or vii holds.

case 2.1.4. $p' < p \wedge p' \notin Z+$. Argue as in case 2.1.1. Then iii or iv holds.

case 2.2. $q \neq q' \wedge (q \notin Z+ \vee q' \notin Z+)$.

case 2.2.1. $q' < q \wedge q \notin Z+$. By changing q a little, find $R((p,q''),(p',q'))$, $R((p,q''),(p,q))$, $q'' \notin Z+$. Then v or vi holds.

case 2.2.2. $q < q' \wedge q \notin Z+$. If $q \neq p'$ then change q a little, so v or vi holds. Assume $q = p'$, so that $R((p,q),(q,q'))$, $p < q < q'$. If $p \notin Z+ \vee q' \notin Z+$, then by changing p or q' a little, we see that iii or v holds. So assume $p,q' \in Z+$. Then vii holds.

case 2.2.3. $q' < q \wedge q' \notin Z+$. Argue as in case 2.2.1. Then iii, v, vi, or vii holds.

case 2.2.4. $q < q' \wedge q' \notin Z+$. Argue as in case 2.2.1. Then v or vi holds.

case 3. $p > q \wedge p' > q'$. Note that R((q,p),(q',p')), by Essentially Binary. Hence we can apply case 2. Hence at least one of iii - vii holds.

For the second claim, suppose R((1,3/2),(3/2,2)). Then R((2,5/2),(5/2,3)), R((1,5/2),(5/2,3)), R((1,5/2),(2,5/2)), R((2,5/2),(5/2,4)), R((5/2,3),(5/2,4)). Hence R((0,1),(0,2)), R((1,5/2),(2,5/2)). QED

We now determine the five standard order theoretic equivalence relations that are contained in $Z^+(\infty)$ order equivalence.

DEFINITION 6.1.3. For $x,y \in Q[-\infty,\infty]^4$, we define $x \approx y \leftrightarrow x \sim y \vee x \sim (y_2,y_1,y_4,y_3) \vee x \sim (y_3,y_4,y_1,y_2) \vee x \sim (y_4,y_3,y_2,y_1)$.

LEMMA 6.1.3. $\approx$ is an equivalence relation. Any order theoretic equivalence relation on $Q[-\infty,\infty]^2$ is an $\approx$ invariant subset of $Q[-\infty,\infty]^4$.

Proof: The three permutations of {1,2,3,4} displayed together with the identity permutation form a subgroup under composition. The second claim follows from Symmetric. QED

LEMMA 6.1.4. Every quadruple from (the graph of) $Z^+(\infty)$ order equivalence is $\approx$ to at least one of the following quadruples.
1. (0,1,0,2), (3/2,1,3/2,2), (5/2,1,5/2,2).
2. The (p,q,p,q).
3. The (a,b,c,d), where (a,b),(c,d) are order equivalent and $a,b,c,d \in Z^+(\infty)$.

Proof: Let (p,q,r,s) be given, were (p,q),(r,s) are $Z^+(\infty)$ order equivalent.

case 1. There are no positive integers in (p,q,r,s). Then (p,q) = (r,s), which falls under 2 above.

case 2. Exactly two coordinates in (p,q,r,s) are positive integers.

case 2a. q,s are positive integers. Then p = r are not positive integers. If q = s then It is clear that (p,q,r,s) is $\approx$ with a quadruple in 2 above. If $q \neq s$ then it is clear that (p,q,r,s) is ≈ with a quadruple in 1 above, according to whether p < min(q,s), p > max(q.s), or otherwise.

case 2b. p,r are positive integers. Since (p,q,r,s) ≈ (q,p,s,r), we can apply case 2a.

case 3. All coordinates in (p,q,r,s) are positive integers. Then (p,q,r,s) is ≈ to an item in 3 above.

QED

We now define three additional equivalence relations on Q[-∞,∞]* contained in $Z^+(\infty)$ order equivalence.

DEFINITION 6.1.4. We say that x,y are lower $Z^+(\infty)$ order equivalent if and only if x,y are order equivalent, where for all i, if $x_i \neq y_i$ then every $x_j \leq x_i$ and every $y_j \leq y_i$ lies in $Z^+(\infty)$.

DEFINITION 6.1.5. We say that x,y are strongly $Z^+(\infty)$ order equivalent if and only if $x = y \vee x,y \in Z^+(\infty)$ and x,y are order equivalent.

DEFINITION 6.1.6. We say that x,y are upper/lower $Z^+(\infty)$ order equivalent if and only if x,y are upper $Z^+(\infty)$ order equivalent or lower $Z^+(\infty)$ order equivalent.

THEOREM 6.1.5. $Z^+(\infty)$ order equivalence, strong $Z^+(\infty)$ order equivalence, upper $Z^+(\infty)$ order equivalence, lower $Z^+(\infty)$ order equivalence, and upper/lower $Z^+(\infty)$ order equivalence are distinct standard order theoretic equivalence relations contained in $Z^+(\infty)$ order equivalence.

Proof: All of these five are obviously contained in $Z^+(\infty)$ order equivalence. It is easy to see that the first four are standard order theoretic equivalence relations. Lower $Z^+(\infty)$ order equivalence is handled with inequalities reversed.

We now come to upper/lower $Z^+(\infty)$ equivalence. We first verify that this is indeed an equivalence relation. Reflexivity and symmetry are obvious, and so transitivity remains.

Let x,y be lower or upper $Z^+(\infty)$ order equivalent, and y,z be lower or upper $Z^+(\infty)$ equivalent. Let $x,y,z \in Q[-\infty,\infty]^k$. Clearly x,y,z are $Z^+(\infty)$ order equivalent.

We need only consider the case where x,y are lower $Z^+(\infty)$ equivalent and y,z are upper $Z^+(\infty)$ equivalent. Let $x_i \neq z_i$. Let $x,y \in Q[-\infty,\infty]^k$.

case 1. $x_i \neq y_i$, $y_i \neq z_i$. Then every $y_j \leq y_i$ is in $Z^+(\infty)$ and every $y_j \geq y_i$ is in $Z^+(\infty)$. Hence $y \in Z^+(\infty)^k$. Therefore $x,z \in Z^+(\infty)^k$, and so x,y,z are both lower and upper $Z^+(\infty)$ order equivalent.

case 2. $x_i \neq y_i$, $y_i = z_i$. Then every $x_j \leq x_i$ is in $Z^+(\infty)$ and every $y_j \leq y_i$ is in $Z^+(\infty)$. Suppose $z_j \leq z_i$. If $z_j = y_j$ then $y_j \leq z_i = y_i$, and so $y_j = z_j \in Z^+(\infty)$. If $z_j \neq y_j$ then $z_j \in Z^+(\infty)$. Thus we have shown that every $z_j \leq z_i$ lies in $Z^+(\infty)$, and hence x,z are lower $Z^+(\infty)$ order equivalent.

case 3. $x_i = y_i$, $y_i \neq z_i$. Then every $z_j \geq z_i$ is in $Z^+(\infty)$ and every $y_j \geq y_i$ is in $Z^+(\infty)$. Suppose $x_j \geq x_i$. If $x_j = y_j$ then $y_j \geq x_i = y_i$, and so $y_j = x_j \in Z^+(\infty)$. If $x_j \neq y_j$ then $y_j \in Z^+(\infty)$. Thus we have shown that every $x_j \geq x_i$ lies in $Z^+(\infty)$. Hence x,z are upper $Z^+(\infty)$ order equivalent.

We now show that upper/lower $Z^+(\infty)$ order equivalence is an order theoretic equivalence relation. Length preserving, $Z^+(\infty)$ standard, Symmetric, and $(Q[-\infty,\infty],<,Z^+(\infty))$ are clear. For Essentially Binary, first let x,y be upper/lower $Z^+(\infty)$ order equivalent. We need to show that $(x_i,x_j),(y_i,y_j)$ are upper/lower $Z^+(\infty)$ order equivalent. Suppose x,y are upper $Z^+(\infty)$ order equivalent. Then obviously $(x_i,x_j),(y_i,y_j)$ are upper $Z^+(\infty)$ order equivalent. Suppose x,y are lower $Z^+(\infty)$ order equivalent. Then obviously $(x_i,x_j),(y_i,y_j)$ are lower $Z^+(\infty)$ order equivalent.

Now suppose that $(\forall i,j \in \{1,\ldots,k\})((x_i,x_j),(y_i,y_j)$ are upper/lower $Z^+(\infty)$ order equivalent). Then clearly x,y are order equivalent.

case 1. There exists i,j such that $(x_i,x_j),(y_i,y_j)$ is not upper $Z^+(\infty)$ order equivalent. It is clear that x,y are not upper $Z^+(\infty)$ order equivalent. Hence x,y are lower $Z^+(\infty)$ order equivalent.

case 2. For all i,j, $(x_i,x_j),(y_i,y_j)$ are upper $Z^+(\infty)$ order equivalent. Then x,y are upper $Z^+(\infty)$ equivalent.

For distinctness, note that (3/2,1),(3/2,2) are $Z^+(\infty)$ order equivalent, but not equivalent under the other five. (0,1),(0,2) are upper $Z^+(\infty)$ order equivalent, but not lower $Z^+(\infty)$ order equivalent. (5/2,1),(5/2,2) are lower $Z^+(\infty)$

order equivalent, but not upper $Z^+(\infty)$ order equivalent. QED

THEOREM 6.1.6. $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation containing (3/2,1,3/2,2). Upper $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation containing (0,1,0,2). Lower $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation containing (5/2,1,5/2,2). Upper/lower $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation containing {(0,1,0,2),(5/2,1,5/2,3)}. Strong $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation. In fact, it is the only standard order theoretic equivalence relation that contains none of these three quadruples. These five distinct standard order theoretic equivalence relations are the only standard order theoretic equivalence relations that are included in $Z^+(\infty)$ order equivalence.

Proof: For the first claim, suppose R is a standard order theoretic equivalence relation in which (3/2,1),(3/2,2) are related. We claim that R extends $Z^+(\infty)$ order equivalence on domain $[-\infty,\infty]^2$. To see this, we have only to check that R contains all quadruples listed in 1-3 of Lemma 6.1.3. This is immediate except for (0,1,0,2) and (5/2,1,5/2,2).

Now (3/2,1,3/2,2) ≈ (3/2,1,3/2,3), and so by transitivity, R(3/2,1,3/2,2), R(3/2,1,3/2,3), R(3/2,2,3/2,3). Since (3/2,2,3/2,3) ≈ (0,1,0,2), we have R(0,1,0,2).

Also (3/2,1,3/2,2) ≈ (5/2,1,5/2,3) ≈ (5/2,2,5/2,3), and so by transitivity, R(3/2,1,3/2,2), R(5/2,1,5/2,3), R(5/2,1,5/2,2).

For the second claim, let R be a standard order theoretic equivalence relation containing (0,1,0,2). It suffices to check that R contains upper $Z^+(\infty)$ order equivalence on domain $Q[-\infty,\infty]^2$. Let (p,q),(r,s) be upper $Z^+(\infty)$ order equivalent. If $p,q,r,s \in Z^+(\infty)$ then R(p,q,r,s). Suppose otherwise. we can assume that $(p,q) \neq (r,s)$.

case 1. $q \neq s$. Then $q,s \in Z^+(\infty)$. If $p,r \in Z^+(\infty)$, and since (p,q),(r,s) are order equivalent, R(p,q,r,s). Hence $p,r \notin Z^+(\infty)$, using $Z^+(\infty)$ order equivalence. Therefore $p = r < q,s$. Hence (p,q,r,s) ≈ (0,1,0,2). Therefore R(p,q,r,s).

case 2. $q = s$. Then $p \neq r$, and (q,p),(s,r) are order equivalent. By case 1, R(q,p,s,r), and so R(p,q,r,s).

For the third claim, let R be a standard order theoretic equivalence relation containing (5/2,1,5/2,2). It suffices to check that R contains lower $Z^+(\infty)$ order equivalence on domain $Q[-\infty,\infty]^2$. We argue as above with the inequalities reversed.

The fourth claim follows immediately from the second and third claims.

That strong $Z^+(\infty)$ order equivalence is the least standard order theoretic equivalence relation is immediate. Suppose R is a standard order theoretic equivalence relation not containing any of these three quadruples. By Lemma 6.1.4, every quadruple in R must be in 2,3 of Lemma 6.1.4, and hence must lie in strong $Z^+(\infty)$ order equivalence.

For the final claim, let R be a standard order theoretic equivalence relation included in $Z^+(\infty)$ order equivalence. Then R is completely determined by membership of the three quadruples in 1 of Lemma 6.1.3. This is clear because if R' is another standard order theoretic equivalence relation included in $Z^+(\infty)$ with the same membership of these three quadruples, then R and R' must agree on $Q[-\infty,\infty]^4$, and so we can apply Theorem 6.1.1.

So we now determine which subsets of the three quadruples are exactly the ones lying in some standard order theoretic equivalence relation included in Z+(∞) order equivalence. By the previous claims, the following are such.

$$\emptyset$$
$$\{(0,1,0,2)\}$$
$$\{(5/2,1,5/2,2)\}$$
$$\{(0,1,0,2),(5/2,1,5/2,2)\}$$
$$\{(0,1,0,2),(3/2,1,3/2,2),(5/2,1,5/2,2)\}.$$

By the first claim, the only subset that includes {3/2,1,3/2,2} has already been listed (the last). So only the subsets of {(0,1,0,2},{5/2,1,5/2,2)} remain. We have listed all of its subsets. The last claim is from Lemma 6.1.4. QED

6.2. ORDER THEORETICALLY INVARIANT MAXIMAL CLIQUES.

We first refute the Invariant Maximal Clique Theorem for any standard order theoretic equivalence relation on Q[-

$\infty,\infty]^*$ not included in $Z^+(\infty)$ order equivalence. Here we use Lemma 6.1.2.

We now define order invariant graphs $G_1(n),\ldots,G_6(n)$ on $[0,n]^2$, as follows.

$x,y \in Q[0,n]^2$ are not adjacent in G_1 if and only if

i. $x_1 = x_2 \wedge y_1 = y_2$; or
ii. $x_1 = y_1 \wedge x_2 = y_2$.

$x,y \in Q[0,n]^2$ are not adjacent in G_2 if and only if

i. $x_1 = y_1 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
ii. $x_1 < x_2 < y_1 = y_2$; or
iii. $y_1 < y_2 < x_1 = x_2$; or
v. $x_1 = y_1 \wedge x_2 = y_2$.

$x,y \in Q[0,n]^2$ are not adjacent in G_3 if and only if

i. $x_2 = y_2 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
ii. $x_1 < x_2 < y_1 = y_2$; or
iii. $y_1 < y_2 < x_1 = x_2$; or
v. $x_1 = y_1 \wedge x_2 = y_2$.

$x,y \in Q[0,n]^2$ are not adjacent in G_4 if and only if

i. $x_1 = y_1 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
ii. $x_1 = y_1 \wedge x_2 = y_2$.

$x,y \in Q[0,n]^2$ are not adjacent in G_5 if and only if

i. $x_2 = y_2 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
ii. $x_1 = y_1 \wedge x_2 = y_2$.

$x,y \in Q[0,n]^2$ are not adjacent in G_6 if and only if

i. $x_1 = y_1 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
ii. $x_2 = y_2 \wedge x_1 < x_2 \wedge y_1 < y_2$; or
iii. $x_1 = y_1$ A $x_2 = y_2$.

Note that each of these non adjacency relations are reflexive and symmetric. Therefore the corresponding adjacency relation is irreflexive and symmetric.

LEMMA 6.2.1. Let R be an order theoretic equivalence relation on [0,n]2 with R((0,0),(1/2,1/2)), $n \geq 2$. Then G_1 has no R invariant maximal clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. $(0,0) \in S$. Hence $(1/2,1/2) \in S$. Since $(0,0),(1/2,1/2)$ are not adjacent, this is impossible.

case 2. $(0,0) \notin S$. Let $(0,0),(p,q)$ be not adjacent, where $(p,q) \in S$. Hence $p = q$. These pairs are not equal, and so $p = q \neq 0$.

case 2a. $p \in Z^+$. then $(p-1,p-1) \in S \vee (p+1,p+1) \in S$, by $Z+(\infty)$ Standard. Since $(p,p),(p-1,p-1)$ are not adjacent, and $(p,p),(p+1,p+1)$ are not adjacent, this is impossible.

case 2b. $p \notin Z^+$. Since $(0,0),(p,p)$ are not adjacent, this is impossible.

QED

LEMMA 6.2.2. Let R be an order theoretic equivalence relation on [0,n]2 with $R((0,1/2),(0,2/3))$, $n \geq 3$. Then G_2 has no R invariant maximal clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. $(1,2) \in S$. By $Z^+(\infty)$ Standard, $(1,3) \in S$. Since $(1,2),(1,3)$ are not adjacent, this is impossible.

case 2. $(1,2) \notin S$. Let $(1,2),(p,q)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p = 1$. By $Z^+(\infty)$ standard, $(1,n) \notin S$. Since $q < n$, we have $(n,n) \notin S$. By $Z+(\infty)$ Standard, $(1,1) \notin S$. Let $(1,1),(p,q)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p < q < 1$. Hence $p,q \notin Z^+$, and therefore $(p,(q+1)/2) \in S$. Since $(p,q),(p,(q+1)/2)$ are adjacent, this is a contradiction.

QED

LEMMA 6.2.3. Let R be an order theoretic equivalence relation on $[0,n]^2$ with $R((0,1/2),(1/3,1/2))$, $n \geq 3$. Then G_3 has no R invariant clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. $(1,2) \in S$. By $Z^+(\infty)$ Standard, $(1,2),(0,2) \in S$. Since $(1,2),(0,2)$ are not adjacent, this is impossible.

case 2. $(1,2) \notin S$. By $Z^+(\infty)$ Standard, $(1,n) \notin S$. Let $(1,n),(p,q)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p < q = n$. Hence $(n,n) \notin S$. By $Z^+(\infty)$ Standard, $(1,1) \notin S$. Let $(1,1),(p,1)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p < q < 1$. Hence $p,q \notin Z^+$, and therefore $((p+q)/2,q) \in S$. Since $(p,q),((p+q)/2,q) \in S$, this is a contradiction.

QED

LEMMA 6.2.4. Let R be an order theoretic equivalence relation on $[0,n]^2$ with $R((1,3/2),(1,4/3))$, $n \geq 3$. Then G4 has no R invariant clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. $(1,2) \in S$. By $Z^+(\infty)$ Standard, $(1,3) \in S$. Since $(1,2),(1,3)$ are not adjacent, this is impossible.

case 2. $(1,2) \notin S$. Let $(1,2),(p,q)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p = 1 < q$. By $Z^+(\infty)$ Standard, $(1,q) \in S$ for $q \neq 2,\ldots,n$. Hence $(1,q-1) \in S \vee (1,q+1) \in S$. Since $(1,q),(1,q-1)$ and $(1,q),(1,q+1)$ are not adjacent, this is impossible.

QED

LEMMA 6.2.5. Let R be an order theoretic equivalence relation on $[0,n]^2$ with $R((0,1),(1/2,1))$, $n \geq 3$. Then G_5 has no R invariant clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. $(1,2) \in S$. By $Z^+(\infty)$ Standard, $(2,3),(1,3) \in S$. Since $(2,3),(1,3)$ are not adjacent, this is impossible.

case 2. $(1,2) \notin S$. Let $(1,2),(p,q)$ be not adjacent, where $(p,q) \in S$. These pairs are not equal, and so $p < q = 2$. Hence $p \neq 1$. Therefore $(p-1,2) \in S \vee (p+1,2) \in S$. Since $(p,2),(p-1,2)$ and $(p,2),(p+1,2)$ are not adjacent, this is impossible.

QED

LEMMA 6.2.6. Let R be an order theoretic equivalence relation on $[0,n]^2$ with R((0,1),(0,2)) ∧ R((1,5/2),(2,5/2)), n ≥ 3. Then G_6 has no R invariant clique.

Proof: Let R be as given. Let S be an R invariant maximal clique.

case 1. (1,2) ∈ S. By $Z^+(\infty)$ Standard, (1,3) ∈ S. Since (1,2),(1,3) are not adjacent, this is impossible.

case 2. (1,2) ∉ S. Let (1,2),(p,q) be not adjacent, where (p,q) ∈ S. These pairs are not equal, and so p < q and p = 1 ∨ q = 2.

case 2a. p = 1. If q < 2 then (1,q),(1,q+1) ∈ S, and since (1,q),(1,q+1) are not adjacent, this is impossible. If q > 2 then (1,q),(1,q-1) ∈ S, and since (1,q),(1,q-1) are not adjacent, this is impossible. Also q = 2,...,n are impossible by $Z^+(\infty)$ Standard.

case 2b. q = 2. If p < 1 then (p,q),(p+1,q) ∈ S, and since (p,q),(p+1,q) are not adjacent, this is impossible. If p > 1 then (p,q),(p-1,q) ∈ S, and since (p,q)(p-1,q) are not adjacent, this is impossible.

QED

THEOREM 6.2.7. If R is a standard order theoretic equivalence relation not contained in $Z^+(\infty)$ order equivalence, then the Invariant Maximal Clique Theorem fails using R invariance. If R is a standard order theoretic equivalence relation on $Q[0,3]^2$ not contained in $[0,3]^2$ order equivalence, then the Invariant Maximal Clique Theorem fails for $[0,3]^2$ using R invariance. This is also true if we replace 3 by any specific integer > 3.

Proof: By Lemmas 6.1.2 and 6.2.1 - 6.2.6. QED

LEMMA 6.2.8. The Invariant Maximal Clique Theorem fails for $[0,3]^2$ using $Z^+(\infty)$ order equivalence, lower $Z^+(\infty)$ order equivalence, and upper/lower order equivalence. The Invariant Maximal Clique Theorem holds using upper $Z^+(\infty)$ order equivalence and strong $Z^+(\infty)$ order equivalence. The former is provable in RCA_0, and the latter is provably equivalent to Con(SRP) over ACA'.

Proof: IMCT for $[0,3]^2$ using lower $Z^+(\infty)$ order equivalence is the same as IMCT for $[1,4]^2$ using upper $Z^+(\infty)$ order equivalence. This is clear by using the bijection 4-x from ([0,3],>) onto ([1,4],<), whereby lower $Z^+(\infty)$ order equivalence gets transformed to upper $Z^+(\infty)$ order equivalence. Now invoke Lemma 4.1.9. The second claim follows from Theorem 5.9.3. QED

THEOREM 6.2.9. Let R be a standard order theoretic equivalence relation. The following are equivalent.
i. The Invariant Maximal Clique Theorem holds for R.
ii. The Invariant Maximal Clique Theorem holds for $[0,3]^2$ using R.
iii. R is contained in upper $Z^+(\infty)$ order equivalence.
Only upper and strong $Z^+(\infty)$ order equivalence fall under iii.
The implications ii → i, iii → i are provably equivalent to Con(SRP) over ACA', The remaining implications are provable in RCA_0.

Proof: Suppose iii is false. Then R is not contained in $Z^+(\infty)$ order equivalence, or R is contained in $Z^+(\infty)$ order equivalence but not in upper $Z^+(\infty)$ order equivalence. In the former case, ii fails by Theorem 6.2.7. In the latter case, by Lemma 6.1.6, R is lower $Z^+(\infty)$ order equivalence, and hence ii also fails by Lemma 6.2.8. So we have established i → ii → iii in RCA_0. Obviously iii → i is provably equivalent to Con(SRP) over ACA', by Theorem 5.9.3. That only upper $Z^+(\infty)$ order equivalence and strong $Z^+(\infty)$ order equivalence fall under iii is by Lemma 6.1.6. QED

We close this section with some more detailed information about IMCT using the four standard order theoretic equivalence relations contained in Z+(∞) order equivalence.

INVARIANT MAXIMAL CLIQUE ALTERNATIVE (J). Every order invariant graph on J^k has a $Z^+(\infty)$ order invariant maximal clique.

We write this as IMCA(J). We also consider these other alternatives arising from Theorem 6.1.5, written IMCT(J)↓, IMCT(J)↑↓, and IMCT(J)!.

INVARIANT MAXIMAL CLIQUE THEOREM (J)↓. Every order invariant graph on J^k has a lower $Z^+(\infty)$ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (J)↑↓. Every order invariant graph on J^k has an upper/lower $Z^+(\infty)$ order invariant maximal clique.

INVARIANT MAXIMAL CLIQUE THEOREM (J)!. Every order invariant graph on J^k has a strong $Z^+(\infty)$ order invariant maximal clique.

LEMMA 6.2.10. (RCA_0). If $|J \cap Z^+| = 1$ then IMCA(J). IMCA([1,2]). IMCT(<0,2]) → IMCA(<0,2]). IMCT(<0,2]) → IMCA([1,3>).

Proof: Let $|J \cap Z^+| = 1$. It is clear that if $x,y \in J^k$ are Z^+ order equivalent then x = y. Hence all subsets of J^k are Z^+ order invariant. So we can use any maximal clique in any given order invariant graph on J^k.

We claim that $x,y \in [1,2]^k$ are {1,2} order equivalent if and only if $x = y \vee (x = (1,...,1) \wedge y = (2,...,2)) \vee (x = (2,...,2) \wedge y = (1,...,1))$. To see this, let x,y be {1,2} order equivalent, and $x_i \neq y_i$. Then $x_i,y_i \in \{1,2\}$, and we may assume $x_i = 1 \wedge y_i = 2$. Then every $y_j \leq y_i$, and so by order equivalence, every $x_j \leq 1$. Hence x = (1,...,1), and by order equivalence, y = (2,...,2). This establishes the claim. Hence every $S \subseteq [1,2]$ with $(1,...,1) \in S \leftrightarrow (2,...,2) \in S$ is {1,2} order equivalent.

Let G be an order invariant graph on $[1,2]^k$. If (1,...,1),(2,...,2) are adjacent in G, then any maximal clique in G containing (1,...,1),(2,...,2) is {1,2} order invariant. If (1,...,1),(2,...,2) are not adjacent in G, then any maximal clique in G containing (3/2,...,3/2) is {1,2} order invariant. Both claims use the order invariance of G.

Now assume IMCT(<0,2]). We claim that $x,y \in <0,2]^k$ are {1,2} order equivalent if and only if x,y are upper {1,2} order equivalent. To see this, let $x,y \in <0,2]^k$ be {1,2} order equivalent. Let $x_i \neq y_i$. Then $x_i,y_i \in Z^+$, and so we can assume $x_i = 1 \wedge y_i = 2$. Then every $y_j \geq y_i$ is 2. By order equivalence, every $x_j \geq x_i$ is 1. Hence x,y are upper {1,2} order equivalent. Therefore every upper {1,2} order invariant subset of $<0,2]^k$ is {1,2} order invariant. Let G be an order invariant graph on $<0,2]^k$. By IMCT(<0,2]), let S be an upper {1,2} order invariant maximal clique in G. Then S is a {1,2} order invariant maximal clique in G.

For the last claim, assume IMCT(<0,2]). By the third claim, IMCA(<0,2]). Let G be an order invariant graph on [1,3>. Note that h(x) = 3-x is an order reversing bijection from <0,2] onto [1,3>. Hence h^{-1}[G] is an order invariant graph on <0,1]. Let S be a {1,2} order invariant maximal clique in h^{-1}[G]. Then h[S] is a maximal clique in G, since the notion of maximal clique is invariant under order reversal. But the notion of {1,2} order invariance is invariant under order reversal as well, provided the order reversal preserves {1,2}. Since h preserves {1,2}, we see that h[S] is a {1,2} order invariant maximal clique in G. QED

LEMMA 6.2.11. (RCA_0). If J has at least four positive integers, then IMCA(J) fails in dimension 2.

Proof: Let J be as given. We define the order invariant graph G on J^2 as follows. (p,q),(p',q') are not adjacent if and only if

i. $p < q \wedge p' < q' \wedge p = p' \wedge q \neq q'$; or
ii. $p < q \wedge p' < q' \wedge q = q' \wedge p \neq p'$.

Obviously this defines an irreflexive symmetric relation. So G is well defined. Let $n,n+1,n+2,n+3 \in J \cap Z^+$.

Assume IMCA(J), and let S be an {n,n+1,n+2,n+3} order invariant maximal clique in G.

We claim that $(n+1,n+2) \notin S$. Otherwise, by invariance, $(n,n+2),(n+1,n+2) \in S$. By i, (n,n+2),(n+1,n+2) are not adjacent in G, contradicting that S is a clique. Hence $(n+1,n+2) \notin S$.

Let $(p,q) \in S$, (n+1,n+2),(p,q) not adjacent in G. Then $p < q$.

case 1. $p = n+1$, $q \neq n+2$. If $q > n+2$ then by invariance, $(n+2,q) \in S$, contradicting that S is a clique. If $n+1 < q < n+2$ then by invariance, $(n,q) \in S$, contradicting that S is a clique.

case 2. $p \neq n+1$, $q = n+2$. If $p < n+1$ then by invariance, $(p,n+1) \in S$, contradicting that S is a clique. If $n+1 < p < n+2$ then by invariance, $(p,n+3) \in S$, contradicting that S is a clique.

Since both cases lead to a contradiction, the proof is complete. QED

THEOREM 6.2.12. Let J be an interval in Q. IMCA(J) holds if J contains at most one positive integer, or if J contains exactly two positive integers, at least one of which is an endpoint. IMCA(J) fails if J contains at least four positive integers. The first result is provable in WKL_0 + Con(ZFC + "there exists a subtle ordinal"), and the second result is provable in RCA_0.

Proof: Immediate from Lemmas 6.2.1, 6.2.2, and Theorems 4.1.10, 4.3.20. QED

There is obviously a substantial gap between the necessary and the sufficient, in Theorem 6.1.3. Nevertheless, there is a sharp contrast between using $Z^+(\infty)$ invariance and upper $Z^+(\infty)$ invariance, as can be seen by comparing Theorem 6.1.3 with the Invariant Maximal Clique Characterization at the end of section 4.1.

We now discuss IMCT(J)↓, where ↓ indicates that we are using lower $Z^+(\infty)$ invariance instead of our usual upper $Z^+(\infty)$ invariance.

INVARIANT MAXIMAL CLIQUE CHARACTERIZATION (↓). IMCC(↓). (Assuming RCA_0 + IMCC). Let J be an interval in Q[-∞,∞). IMCT(J)↓ holds if and only if no positive integer is the greatest element of J, or J contains at most 2 positive integers. IMCT(J)↓ fails for J = <p,∞], p ∈ Q[-∞,∞), and for J = <0,n+3], n ∈ Z^+.

Proof: The first claim follows easily from the duality between lower $Z^+(\infty)$ order invariance and upper $Z^+(\infty)$ order invariance, using strong order reversing bijections, analogous to the proof of Theorem 4.1.1. For the second claim, note that for any p ∈ Q[-∞,∞), there is an order reversing bijection from Q[1,7/2] onto Q[p,∞] mapping 7/2,3,2,1 to ∞,n+1,n,p, and an order reversing bijection from Q(1,7/2] onto Q(p,∞] mapping 3,2,1 to ∞,n+1,n, where n is a positive integer > p. As in the proof of Theorem 4.1.1, IMCT(Q[p,∞])↓ → IMCT(Q[1,7/2]), and IMCT(Q(p,∞])↓ → IMCT([1,7/2)). By the Invariant Maximal Clique Characterization of section 4, IMCT([1,7/2>) fails. Also, IMCT(<0,n+3]) fails using the first claim. QED

INVARIANT MAXIMAL CLIQUE THEOREM (↓). IMCT(↓). Every order invariant graph on $[1,n+(1/2)]^k$ has a lower $Z^+(\infty)$ order invariant maximal clique.

We do not know if IMCT(Q)↓ holds.

INVARIANT MAXIMAL CLIQUE CHARACTERIZATION (↑↓). (Assuming RCA_0 + IMCC). Let J be an interval in Q[-∞,∞]. The Invariant Maximal Clique Theorem (J)↑↓ holds if and only if J is not a closed interval with both endpoints from $Z^+(\infty)$ or $|J \cap Z^+| \leq 2$.

Proof: Let J be an interval in Q[-∞,∞). According to IMCC and the above IMCC(↓), IMCT(J)↑↓ if and only if min(J) does not exist ∨ min(J) $\notin Z^+$ ∨ $|J \cap Z^+| \leq 2$ ∨ max(J) does not exist ∨ max(J) $\notin Z^+$ ∨ $|J \cap Z^+| \leq 2$. The present disjunction is equivalent to J not a closed interval with endpoints from Z+ ∨ $|J \cap Z^+\} \leq 2$. So this establishes the claim in case J ⊆ Q]-∞,∞). We now consider the case where J has endpoint ∞. The present condition is equivalent to min(J) does not exist ∨ min(J) $\notin Z^+(\infty)$ ∨ $|J \cap Z^+| \leq 2$, which is equivalent to min(J) does not exist ∨ min(J) $\notin Z^+$ ∨ $|J \cap Z^+| \leq 2$. By IMCC, this is in turn equivalent to IMCT(J). It remains to show that IMCT(J) is equivalent to IMCT(J)↑↓ provided J has right endpoint ∞. This is clear by the second claim in IMCC(↓). QED

Let ACA+ be ACA_0 + "for all x ⊆ ω, the ω-th Turing jump of x exists".

THEOREM 6.2.13. (ACA'). IMCT(J)! holds for all upper bounded intervals J ⊆ Q[-∞,∞). (ACA^+). IMCT(J)! holds for all intervals J ⊆ Q.

Proof: First suppose J ⊆ Q(-∞,n], n ∈ Z^+. The invariance condition now involves only moving around vectors consisting entirely of positive integers. We can therefore use a sufficiently large finite ordinals α instead of a large cardinal λ, since the Ramsey property to be used is the usual finite Ramsey theorem. So we need only use the linear ordering α × Q[0,1), build the appropriate maximal clique, apply the finite Ramsey theorem to obtain indiscernibles $\beta_1 < \ldots < \beta_n$, cutting off the construction at $(\beta_n,0)$, or earlier, depending on J, and then isomorphing the structure onto J, mapping the β's (or an initial segment thereof) to positive integers in J. Now suppose J ⊆ Q, where sup(J) = ∞. This time, let λ = ω, and use λ × Q[0,1). Argue the same way, this time using the infinite Ramsey theorem applied to λ × {0}. QED

We do not know of any proof of IMCT(Q[-∞,∞])! = IMCT(extended)! that doesn't prove IMCT(Q[-∞,∞]) =

IMCT(extended). I.e., we use SRP^+ or WKL_0 + SRP to prove IMCT(Q[-∞,∞])!. We do not know if this is needed.

6.3. FINITE INDEPENDENT DOMINATOR THEOREMS.

We now present some finite forms of the Invariant Maximal Clique Theorems.

We begin by presenting two additional notions that are dual to maximal cliques: maximal independent sets, and independent dominators.

Maximal independent sets are dual to maximal cliques, and thus can be used to make an obviously equivalent reformulation of the Invariant Maximal Clique Theorem.

DEFINITION 6.3.1. Let G be a graph. We say that S ⊆ V(G) is independent if and only if no two elements of S are adjacent. We say that S ⊆ V(G) is a maximal independent set if and only if it is an independent set that is not properly contained in any independent set.

Maximal independent sets are dual to maximal cliques in the following sense.

THEOREM 6.3.1. Let G = (V,E) be a graph. S is a maximal independent set in G if and only if S is a maximal clique in the graph (V,V\E\{(x,x): x ∈ V|). S is a maximal clique in G if and only if S is a maximal independent set in the graph (V,V\E\{(x,x): x ∈ V}. These claims are provable in RCA_0 for countable graphs, and EFA for finite graphs.

THEOREM 6.3.2. Theorems 2.1 - 2.4 hold with "clique" replaced by "independent set", and "maximal clique" replaced by "maximal independent set".

Proof: By Theorem 6.3.1. QED

It is now clear that we have the Invariant Independent Set Theorem and the Invariant Independent Set Theorem (extended), corresponding exactly to the Invariant Maximal Clique Theorem and the Invariant Maximal Clique Theorem (extended).

DEFINITION 6.3.2. A dominator in G is an S ⊆ V, where every v ∈ V\S is adjacent to some w ∈ S. Note how a dominator "communicates" with every vertex. I.e., a dominator exhibits "total communication".

Important references for work on domination in graphs include [HL90a], [HL90b], [HHS98a], and [HHS98b].

According to private communication, S.T. Hedetniemi is updating [3], which lists about 1200 publications. Hedetniemi expects over 3000 publications in the updated version - and many more that are closely related to the topic, that are under consideration for listing.

THEOREM 6.3.3. Every graph has a minimal dominator. This is provable in RCA_0 for countable graphs, and EFA for finite graphs.

Proof: For finite graphs, we simply start with the trivial dominator consisting of the set of all vertices. We can then (arbitrarily or in some definite order) successively delete any vertex whose absence still leaves us with a dominator. If we do this as long as we can, then we arrive at a minimal dominator.

We postpone the proof for graphs till after Theorem 6.3.5. QED

Can be prove that every graph has a minimal dominator in the same way? Well, that proof for finite graphs could just as easily start with any dominator, and leave us with a minimal dominator.

THEOREM 6.3.4. Every dominator in every finite graph contains a minimal dominator. There is a countable graph with no minimal dominator. The former is provable in EFA, and the latter is provable in RCA_0.

Proof: We have already proved the first. For the second, let G be the following graph on Z. x,y are adjacent if and only if

i. $x,y < 0 \wedge x \neq y$; or
ii. $x < 0 \wedge -x \geq y > 0$; or
iii. $y < 0 \wedge -y \geq x > 0$.

It is clear that $(-\infty,0]$ is a dominator. In fact, the dominators contained in $(-\infty,0]$ are exactly the infinite subsets of $(-\infty,0]$ that include 0. Hence $(-\infty,0]$ does not contain a minimal dominator. QED

THEOREM 6.3.5. In any graph, the maximal independent sets and the independent dominators are the same. Every independent dominator is a minimal dominator. There is a finite graph which has a minimal dominator that is not independent. The first two claims are provable in RCA_0 for countable graphs.

Proof: Let S be a maximal independent set in the graph G. Let v ∈ V\S. Then S ∪ {v} is not a maximal independent set. Hence v is adjacent to some S. This verifies that S is a dominator.

Let S be an independent dominator. Let v ∈ V\S. Let v be adjacent to x ∈ S. Then S ∪ {v} is not independent. This establishes that S is a maximal independent set.

Now let v ∈ S. We claim that S\{v} is not a dominator. This is because v is not adjacent to any element of S\{v} (since S is independent).

For the third claim, consider the graph on {1,2,3,4}, where E = ((1,3),(2,4),(1,2)}. Then {1,2} is a minimal dominator that is not independent. QED

Proof of Theorem 6.3.3: We have only to prove that every G has a minimal dominator. By Theorem 2.?, $(V,E\backslash V^2)$ has a maximal clique S. By Theorem 6.3.1, S is a maximal independent set in G. By Theorem 6.3.5, S is a minimal dominator. QED

THEOREM 6.3.6. In ZF, "every graph has a minimal dominator" is provably equivalent to the axiom of choice.

Proof: ZFC proves "every graph has a minimal dominator" by again going through the chain maximal clique → maximal independent set → minimal dominator.

For the converse, we argue in ZF. Let R be an equivalence relation on a set A. Let G be the graph on A where x,y ∈ A are adjacent if and only if x ≠ y are R equivalent. Let S be a minimal dominator in G. Then every x ∈ A is equivalent to some element of S. We claim that distinct elements of S are not R equivalent. To see this, let x ≠ y be from S. Then clearly S\{x} is also a dominator, contradicting the minimality of S. QED

INVARIANT MAXIMAL CLIQUE THEOREM. IMCT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant maximal clique.

INVARIANT MAXIMAL INDEPENDENCE THEOREM. IMIT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant maximal independent set.

INVARIANT INDEPENDENT DOMINATOR THEOREM. IIDT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant independent dominator.

INVARIANT MINIMAL DOMINATOR THEOREM. IMDT. Every order invariant graph on $Q[0,n]^k$ has an upper Z^+ order invariant minimal dominator.

THEOREM 6.3.7. RCA_0 proves IMCT ↔ IMIT ↔ IIDT → IMDT.

Proof: From Theorems 6.3.1 and 6.3.5. QED

We do not know whether IMDT is provable in ZFC, or even in RCA_0.

DEFINITION 6.3.3. Let G = (V,E) be a graph. We say that S is a dominator for A ⊆ V if and only if every element of A\V is adjacent to an element of S. We say that S is an independent dominator for A ⊆ V if and only if S is a dominator for A ⊆ V that is an independent set.

Obviously, an independent dominator in G is the same as an independent dominator for V.

Note that an independent dominator for A may not be a subset of A. The independent dominators for A that are subsets of A are just the independent dominators in G[A].

THEOREM 6.3.8. EFA + IIDT proves the following. In every order invariant graph on $Q[0,n]^k$, every finite set has a finite upper Z^+ order invariant independent dominator.

Proof: Let G be an order invariant graph on $Q[0,n]^k$. By IIDT, let S be an upper Z^+ order invariant independent dominator in G. Now let A be a finite set of vertices. For each x ∈ A\S, choose y ∈ S adjacent to x, and let T be the set of choices. Then obviously (A ∩ S) ∪ T is a finite dominator for A. Now (A ∩ S) ∪ T ⊆ $[0,n]^k$ may not be upper Z^+ order invariant. Let B be the least superset of (A ∩ S) ∪ T such that B ⊆ $[0,n]_k$ is upper Z^+ order invariant. Then B is a finite subset of S, and hence is independent. QED

We do not know if the statement in Theorem 6.3.8 can be proved in ZFC, or even in RCA_0.

FINITE INVARIANT INDEPENDENT DOMINATOR THEOREM. FIIDT. In every order invariant graph on $Q[0,n]^k$, every finite set A has a finite dominator B such that any finite set C has a finite upper Z^+ order invariant dominator $D \supseteq B$.

THEOREM 6.3.9. The following are provably equivalent in EFA. IMCT, IMIT, IIDT, IMDT, and FIIDT.

Proof: We prove IMDT → FIIDT, and invoke THEOREM 6.3.7. Of course, the substantial EFA + FIIDT → IIDT remains. The proof of this result will appear in [Fr?].

Let G be an order invariant graph on $Q[0,n]^k$. By IIDT, let S be an upper Z+ order invariant independent dominator in G. Now let A be a finite set of vertices. Construct B from G,S,A as in the proof of Theorem 6.3.7. Now let C be a finite set of vertices. Construct D from G,S,A ∪ C as in the proof of Theorem 6.3.7, where the choosing used for for $x \in (A \cup C)\backslash S$ extends the choosing used for $x \in A\backslash S$. QED

Note that FIIDT is explicitly Π^0_4. We can place easy a priori bounds on the size of A,B,C,D as follows.

FINITE INVARIANT INDEPENDENT DOMINATOR THEOREM(size). FIIDT(size). In every order invariant graph on $Q[0,n]^k$, every finite set A has a finite dominator B, $|B| \leq |A|$, such that any finite set C has a finite upper Z^+ order invariant dominator $D \supseteq B$, $|D| \leq n^k|A \cup C|$.

Evidently, RCA_0 proves FIIDT(size) ↔ FIIDT. In addition, it is also clear that FIIDT(size) is explicitly Π^0_1 relative to the very well known decision procedure for dense linear orderings with endpoints.

Specifically, let fix k,n,m, and an order invariant graph G be the given order invariant graph on Q[0,n]. Then FIIDT(size) takes the form of a sentence in (Q[0,n],0,...,n), with four alternating blocks of quantifiers of size depending on n,k,m.

We can go further and conveniently provide explicitly bounds for all of the existential quantifiers, thereby obtaining an explicitly Π^0_1 sentence.

DEFINITION 6.3.4. For $p \in Q[0,\infty)$, define #(x) as the sum of the numerator and denominator of the reduced form of p. For

$x \in Q[0,\infty)^k$, define #(x) to be the sum of the #'s of the k coordinates of x. For finite $A \subseteq Q^k$, define #(A) as the sum of the #'s of the elements of A.

Note that for given r, there are only finitely many $A \subseteq Q^k$ with #(A) ≤ r. We can put obvious a priori estimates on the quantifiers using #. Here we use a safe but extremely crude estimate. Better estimates will be given in [Fr?].

FINITE INVARIANT INDEPENDENT DOMINATOR THEOREM(#). FIIDT(#). In every order invariant graph on Q[0,n], every finite set A has a finite dominator B, #(B) ≤ (8#(A))!, such that any finite set C has a finite upper Z^+ order invariant independent dominator $D \supseteq B$, #(D) ≤ (8#(A ∪ C))!.

where f is give by a reasonable expression like f(r) = (8r)!, and much better. Obviously, RCA_0 proves FIIDT(#) ↔ FIID, and FIIDT(#) is explicitly Π^0_1.

We can also use #(A) to give another explicitly Π^0_1 form of FIIDT by strengthening the notion of dominator.

DEFINITION 6.3.5. Let G be an order invariant graph on $Q[0,k]^k$, and A be a finite subset of $Q[0,k]^k$. We say that B is a *-dominator for A if and only if every x ∈ A\V is adjacent to some y ∈ S with #(y) ≤ (8#(x))!.

INVARIANT INDEPENDENT *-DOMINATOR THEOREM. II*DT. In every order invariant graph on $Q[0,k]^k$, every finite set has an upper Z^+ order invariant independent *-dominator.

INVARIANT INDEPENDENT *-DOMINATOR THEOREM. II*DT. In every order invariant graph on $Q[0,k]^k$, every finite set A has an upper Z^+ order invariant independent *-dominator B, where #(B) ≤ (9#A)!.

THEOREM 6.4.10. The following are provably equivalent in EFA. FIIDT, FIIDT(size), FIIDT(#), II*DT. The first is explicitly Π^0_4. The second is explicitly Π^0_1 relative to the well known decision procedure for dense linear orderings with endpoints. The third is explicitly Π^0_2, and the fourth is explicitly Π^0_1.

Proof: These claims are straightforward, and will be discussed in [Fr?]. QED

6.4. OPEN QUESTIONS.

We do not know whether the Invariant Maximal Clique Theorem (Q) is provable in ZFC, or even in RCA_0. The same issue arises with regard to the IMCT(J), where J is an interval in Q whose right endpoint is not a positive integer. The latter are consequences of IMCT(Q) in RCA_0 by Theorem 4.1.1.

We do not know whether IMCT(Q) is provably equivalent to a Π^0_1 sentence over WKL_0, or even ZFC.

We do not know anything about the provability in ZFC of IMCT when either k or n is reasonably small. We conjecture that IMCT on $[0,2]^2$ is provable in RCA_0. We do not know if IMCT on $[0,4]^4$ is provable in ZFC, or even in RCA_0. From Theorem 5.8.39, we know that IMCT on $[0,k]^k$, where $k = 10\uparrow$, is not provable in ZFC (assuming ZFC is consistent).

We do not know how to fill the gap between Theorems 6.2.8 and 6.2.9.

We do not know if IMCT(Q)$\downarrow$ holds. See section 6.2.

We do no know if IMCT(extended)! is provable in ZFC, or even in RCA_0. See section 6.2.

We do not know if IMDT is provable in ZFC, or even in RCA_0. See section 6.3.

We do not know if the statement about finite dominators in Theorem 6.3.8 is provable in ZFC, or even in RCA_0.

We have given a complete characterization of the standard order theoretic equivalence relations which can be used for the Invariant Maximal Clique Theorem (Theorem 6.2.9). We conjecture that this can done for all of the order theoretic equivalence relations.

We can hope to go further.

TEMPLATE 1. Let R,R' be order theoretic equivalence relations, and $J \subseteq Q[-\infty,\infty]$ be an interval with endpoints from $Q[-\infty,\infty]$. Every R invariant graph on J^k has an R' invariant maximal clique.

We can even further increase the generality of the Template, by bringing in specific dimensions k, rather than merely quantifying over all of them.

TEMPLATE 2. Let R,R' be order theoretic equivalence relations, J ⊆ [-∞,∞] be an interval with endpoints from Q[-∞,∞], and k ≥ 1. Every R invariant graph on J^k has an R' invariant maximal clique.

CONJECTURE 1. Every instance of these Template 1, with J bounded above, is provable or refutable in SRP. Every instance of Template 1 is provable or refutable in SRP^+.

CONJECTURE 2. Every instance of Template 2 is provable or refutable in SRP.

CONJECTURE 3. For any two instances of these Templates, combined, one provably implies the other, over ACA'.

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