FLAT MENTAL PICTURES

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EXTENDED ABSTRACT

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Abstract. We present some conditions on equivalence relations, linear orderings, and a domain, and demonstrate a correspondence between such equivalence relations and linear orderings and models of set theories such as ZC, ZFC, and ZFC with various large cardinal hypotheses. The conditions are so basic as to arguably correspond to mental pictures. These mental pictures are flat, as opposed to the so called mental pictures of the iterative set theoretic universe V. The results can be formulated in terms of first order axiomatizations of higher order logic with the conditions acting as nonlogical axioms. In each case, the conditions that we present are not intended to characterize the underlying "mental picture", but rather just those aspects that are sufficient for the high interpretation power. The results can be construed as developing a foundation of iterative set theory based on arguably more fundamental flat notions.

1. EQUIVALENCE RELATIONS

We consider an equivalence relation E on the world W. We have a mental picture of an infinite sequence of E equivalence classes of strictly increasing size.

We write [x] for the E equivalence relation of $x \in W$. We write $A \equiv B$ for "there is a bijection from A onto B", $A \leq B$ for "A is injectable into B", and $A < B$ for "A is injectable into B but not vice versa".

EQ1. (W,E) is an equivalence relation.
EQ2. If [x] ≡ [y] then [x] = [y].
EQ3. (W,E) is first order equivalent to the equivalence relation obtained by removing the <-least equivalence class.
EQ4. (W,E) is first order equivalent to the equivalence relation obtained by removing any single equivalence class.
EQ5. (W,E) is first order equivalent to the equivalence relation obtained by removing any equivalence classes, as long as what remains obeys EQ2.
EQ6. (W,E) is second order equivalent to the equivalence relation obtained by removing the <-least equivalence class.
EQ7. (W,E) is second order equivalent to the equivalence relation obtained by removing any single equivalence class.
EQ8. (W,E) is second order equivalent to the equivalence relation obtained by removing any equivalence classes, as long as what remains obeys EQ2.

Next, we consider the mental picture with an ω+ω sequence of equivalence classes of strictly increasing size. It is convenient to use a predicate symbol P for the "omega-th" equivalence class.

EQ9. The extension of P is an equivalence class, where (∃x)([x] < P), (∀[x] < P)(∃[y] < P)([x] < [y]).
EQ10. (W,E) is second order equivalent to the result of removing any single equivalence class.

We use the simple theory of types, STC(E,P), with choice at every level, over the ground type W,)type 0) augmented by predicate symbols E,P, as the metatheory for the following results. In section 4, we fine tune the metatheory.

THEOREM 1.1. The following are provable in STC(E,P).
i. EQ1 + EQ2 + any of EQ3 - EQ5 has a model if and only if there is a model of second order ZC of height ω+ω with domain ⊆ W.
ii. If EQ1 + EQ2 + any of EQ6 - EQ8 has a model then there is a transitive model of ZFC + "there exists a stationary Ramsey cardinal" with domain W.

THEOREM 1.2. STC(E,P) + EQ1 - EQ8 is interpretable in ZFC + "there exists a measurable cardinal" and interprets ZFC + "there exists a stationary Ramsey cardinal".

2. COMPLETE DENSE LINEAR ORDERINGS

We consider a linear ordering $<$ on the world W. For the picture, we assume density, no endpoints, and order completeness.

LIN1. $<$ is a linear ordering with no endpoints, dense, and order complete.
LIN2. Every function on a proper initial segment is bounded above.
LIN3. There are arbitrarily large proper initial segments with property LIN2.
LIN4. $(\exists x)((W,<)$ and $((-\infty,x),<)$ are second order equivalent.
LIN5. $(\exists x)(\forall y < x)((W,<,y)$ and $((-\infty,x),<,y)$ are second order equivalent.
LIN6. $A \subseteq W \rightarrow (\exists x)((W,<,A)$ and $((-\infty,x),<,A \cap (-\infty,x))$ are second order equivalent).

LIN2 corresponds to an horizon at ∞. LIN3 corresponds to ever higher horizons. LIN4 - LIN6 are forms of reflection.

STC(<) is the simple theory of types with choice at every level, using a binary relation symbol $<$ on the ground type W (type 0).

THEOREM 2.1. The following are provable in STC(<).
i. LIN1 + LIN2 has a model.
ii. If LIN1 + LIN3 has a model then there is a transitive model of ZC with domain W.
iii. If LIN1 - LIN3 has a model then there is a transitive model of ZFC + "there exists infinitely many strongly inaccessible cardinals" with domain W.
iv. If LIN1 - LIN5 has a model then for all n, there is a transitive model of ZFC + "there exists a Π^1_n indescribable cardinal" with domain W.

THEOREM 2.2. STC(<) + LIN1 + LIN3 is interpretable in ZC. STC(<) + LIN1 - LIN3 is interpretable in ZFC + "there exists uncountably many strongly inaccessible cardinals". LIN1 - LIN6 is interpretable in ZFC + "there exists a second order indescribable cardinal".

THEOREM 2.3. All of the above results hold even if we weaken LIN1 to "$<$ is a linear ordering", where we use "strictly bounded above".

3. EXOTIC DOMAINS

Here we do not put any structure on the world, W.

W1. Every purely second order definable $(\wp(W),R)$, $R \subseteq \wp(W)^2$, is nontrivially self embeddable.

Here "purely" means without parameters, and "nontrivial" means not the identity function.

THEOREM 3.1. The following are provable in STC. If W1 has a model then I2 has a second order model. If I1 is true then W1 has a model.

THEOREM 3.2. STC + W1 interprets I2 and is interpretable in I1.

Here I1 and I2 are standard notation from, e.g., Aki Kanamori, The Higher Infinite.

4. LOGICAL METATHEORIES

So far, we have only used a rather strong metatheory, STC. STC has types 0,1,..., with = at each type, extensionality at each type, full comprehension at each type, and full choice at each type. We have the axioms and rules of inference for many sorted predicate calculus with equality at each sort. Here choice is stated in the most obvious way - a set of pairwise disjoint nonempty sets has a set with exactly one element in common with each of the nonempty sets. There are many variants of this system, and they are all easily seen to be mutually interpretable in various strong senses.

For some results, it is enough to have a bare bones system for second order logic with no choice. Here are two natural alternatives.

GROUP 1
= only on type 0

1. UTP(0,1). Types 0,1, where type 1 has the usual sets, and = only on type 0. Add the binary function symbol P on type 0 with $P(x,y) = P(z,w) \rightarrow x = z \wedge y = w$. Use full comprehension. Unary type theory with pairing in types 0,1.

2. BTP(0,1). Types 0,1, where type 1 has binary relations only. We use full comprehension. Binary type theory in types 0,1.

For other results, we need a serious dose of choice. We can often use only types 0,1. Here are some natural alternatives.

GROUP 2
= only on type 0

3. BTWO(0,1). BT(0,1) plus "there is a well ordering on type 0".
4. BTCH(0,1). BT(0,1) augmented with two unary function symbols CH_1,CH_2 (choice) from type 1 to type 0. Expand full comprehension to the expanded language, and add R(x,y) → R(CH_1(R),CH_2(R)).
5. UTWO(0,1). UTP(0,1) plus "there is a well ordering on type 0", appropriately formulated using P.
6. UTPCH(0,1). UTP(0,1) augments with CH1,CH2 as in BTCH(0,1).

For still other results, we need types 0,1,2. We will just present three versions with choice.

GROUP 3
= only on types 0,1, with extensionality

7. UTPCH(0,1,2). Types 0,1,2 where types 1,2 have the usual sets, pairing on type 0, and we have CH from type 2 to type 1. Full comprehension, P(x,y) = P(z,w) → x = z ∧ y = w, and: x in A implies CH(A) in A.
8. BTPCH(0,1,2). Analogous to UTP(0,1,2), except with binary relations on type 0 and binary relations on type 1, and CH1,CH2 from type 2 to type 1.
9. BTWO(0,1,2). Same as BTPCH except CH is replace by "there exists a well ordering of type 1 objects".

A fully systematic treatment of fragments of STC and variants with relations will have to wait. There are a number of interesting technical issues.

We can replace STC, STC(E), STC(<), STC(E), STC(E,P) by X(0,1,2,E), X(0,1,2,<), X(0,1,2,E) in sections 1-3 - where X is in Group 3 above. For various parts of results in sections 1-3 we can use Group 1 and Group 2 systems.