METAMATHEMATICS OF ULM THEORY

by

Harvey M. Friedman
Department of Mathematics
Ohio State University
http://www.math.ohio-state.edu/~friedman/

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Abstract. The classical Ulm theory provides a complete set of invariants for countable abelian p-groups, and hence also for countable torsion abelian groups. These invariants involve countable ordinals. One can read off many simple structural properties of such groups directly from the Ulm theory. We carry out a reverse mathematics analysis of several such properties. In many cases, we reverse to ATR_0, thereby demonstrating a kind of necessary use of Ulm theory.

1. INTRODUCTION.

Ulm theory is crucial in the theory of torsion abelian groups (every element has finite order). It involves transfinite recursion. Kaplansky, Infinite Abelian Groups, discussed two test problems for Ulm theory:

a) Let G,H be countable torsion abelian groups, where G+G and H+H are isomorphic. Then G,H are isomorphic.

b) Let G,H be countable torsion abelian groups, where G is a direct summand of H and H is a direct summand of G. Then G,H are isomorphic.

Kaplansky emphasizes that his test problems a,b immediately follow from Ulm theory, and hence ATR_0.

CONJECTURES: Test problems a) and b) are equivalent to $\Pi^1_1\text{-}CA_0$ over RCA_0. Test problems a) and b) for countable reduced torsion abelian groups are equivalent to ATR_0 over RCA_0.

We formulate a number of other attractive test problems and analyze them from the reverse mathematics point of view. One of our equivalents with ATR_0 over ACA_0 was mentioned in [Si99], page 203.

We will mostly use ACA_0 as the base theory for the reversals. This is very convenient. We do expect that in each case, with additional work, one can replace ACA_0 with the weaker base theory RCA_0 by deriving ACA_0 from RCA_0 plus the statement in question. This should be worked out in the context of a complete systematic metamathematical analysis of infinite abelian group theory as represented, say, by [Fu70/73], [Gr70], and [Ka69].

The main purpose of this paper is to show that the fairly exotic logical nature of Ulm theory is required in the theory of infinite abelian groups, in the sense that basic structural facts require such logically exotic arguments.

However, the preponderance of results in countable infinite Abelian group theory can be proved in such systems as ACA_0 and RCA_0. It would be interesting to have a full understanding of countably infinite abelian group theory at these more typical lower levels of reverse mathematics.

Countably infinite abelian group theory is a beautiful context in which to do a systematic metamathematical analysis via reverse mathematics because

a) Theorems in countably infinite abelian group theory are naturally stated in the language of reverse mathematics, without coding issues.

b) The theorems in countably infinite abelian group theory represent all of the five principal levels of reverse mathematics.

Such a systematic analysis has begun in [FSS83], with exposition and further results in [Si99]. See [Si99], sections III.6, V.7, VI.4, as well as page 411 for documentation of a) and b).

We will carry out the equivalence of ATR_0 over the base theory RCA_0 for two variants of an attractive test problem for Ulm theory. In particular, we will show that the following are equivalent to ATR_0 over RCA_0:

I. Let p be prime and G,H be countable abelian p-groups. Either G is embeddable into H or H is embeddable into G .
II. There exists a prime p such that the following holds. Let G,H be countable reduced abelian p-groups. Either G is embeddable into H or H is embeddable into G .

Note that II is just a weak form of I. Here G is the direct sum of countably infinitely many copies of G.

Here are the statements about countable Abelian torsion groups that we consider in this paper.

1. Either G is embeddable into H or H is embeddable into G .
2. There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.
3. There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.
4. In every infinite sequence of groups, one group is embeddable in a later group.
5. In every infinite decreasing (⊇) chain of groups, one group is embeddable in a later group.
6. There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.
7. If G+G and H+H are isomorphic then G and H are isomorphic.
8. If G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

We prove in ATR_0 that for all primes p, 1-8 hold for countable reduced Abelian p-groups.

We prove in Π^1_1-CA_0 that for all primes p, 1-8 hold for countable abelian p-groups. For 1,4,5,6 we get away with ATR_0.

We prove in ATR_0 that 2,3,6,7,8 hold for countable reduced abelian torsion groups. This is refutable in ACA_0 for 1,4,5.

We prove in Π^1_1-CA_0 that 2,3,6,7,8 hold for countable abelian torsion groups. For 6 we get away with ATR_0.

We prove that for any statement φ among 1-5 and primes p, the following are provably equivalent in ACA_0:

i) ATR_0;
ii) φ holds for countable reduced Abelian p-groups;
iii) for all primes p, φ holds for countable reduced Abelian p-groups.

We prove that for statement 1 and primes p, the following are provably equivalent in RCA_0:

i) ATR_0;
ii) 1 holds for countable reduced Abelian p-groups;
iii) for all primes p, 1 holds for countable Abelian p-groups.

2. INFINITE ABELIAN GROUP THEORY BACKGROUND.

We now summarize results from infinite abelian group theory that we will use. We make use of the exposition in [BE70], as well as some new results there.

It is important to note that the entire standard Ulm theory of countable reduced abelian torsion groups is provable in ATR_0. [Si99], page 199 for some detailed background.

Some of the Ulm theory and much of infinite abelian group theory can actually be proved in ACA_0 or even in RCA_0. We will make limited use of this fact.

A divisible abelian group is an abelian group G such that for all $x \in G$ and $n \geq 1$, there exists $y \in G$ such that $x = ny$.

An abelian group is said to be nontrivial if and only if it has more than one element.

A reduced abelian group is an abelian group with no nontrivial divisible subgroups.

Let p be a prime. An abelian p-group is an abelian group where every element is of order a power of the prime p.

An abelian torsion group is an abelian group such that every element has finite order.

We will state a considerable amount of background information that can be gleaned from the literature cited. However, before doing this, we wish to clarify the reverse mathematics situation with regard to the decomposition of countable abelian groups into reduced and divisible groups. This situation is of the utmost importance for the results in this paper.

LEMMA A. (RCA_0) The following are equivalent.
i) Π^1_1-CA_0;
ii) every countable Abelian group is a direct sum of a divisible group and a reduced group.

Proof: See [Si99], p. 230. QED

We will need a strengthening of this equivalence.

LEMMA B. (RCA_0) The following are equivalent.
i) Π^1_1-CA_0;
ii) every countable Abelian group has a largest divisible subgroup;
iii) there exists a prime p such that every countable Abelian p-group has a largest divisible subgroup;
iv) every countable abelian group is a direct sum of a divisible group and a reduced group;
v) there exists a prime p such that every countable abelian p-group is a direct sum of a divisible group and a reduced group.

Proof: Implicit in [Si99], p. 230. Simpson correctly emphasizes ii) of Lemma A since it is more striking mathematically than, e.g., ii) of Lemma B. QED

Throughout 2.1 - 2.10, let p be any prime and G,H be any countable abelian p-groups.

2.1. Without regard to formalization, define pG = {px: x $\in$ G}. Define $p^0G = G$, $p^{\alpha+1}G = p(p^\alpha G)$, $p^\lambda G = \cap\{p^\beta G: \beta < \lambda\}$, where α is countable ordinal and λ is a countable limit ordinal. There is a least α such that the equation $p^\alpha G = p^{\alpha+1}G$ holds. α is called the length of G, written l(G).

2.2. (ATR0) The above construction can be performed along any well ordering of a subset of ω, although it may not achieve the equation.

In [Si99], p. 201, it is proved in ATR_0 that there are well orderings of ω such that this construction achieves the equation, under the crucial assumption that G is reduced. In fact, that any two such constructions yield the same groups at corresponding positions, and that the final group is {0}.

An Ulm resolution is defined to be such a construction along a well ordering that exactly stabilizes. I.e., there is a final stage where the group is p times itself. I.e., at the final stage, the group is divisible. Thus the domain of an Ulm resolution has a greatest element, which is l(G).

Thus in [Si99], it is proved that if G is reduced then G has an Ulm resolution.

ATR_0 is just adequate enough to provide a good theory of countable ordinals in terms of well orderings of subsets of $\mathbb{N}$. See [Si99] on ATR_0. It will therefore cause no confusion to use ordinal notation when discussing Ulm resolutions and lengths within ATR_0.

2.3. (ATR_0) It is true in general that G has an Ulm resolution. We do not need to know that G is reduced. However, this is not provable in ATR_0. This situation is clarified as follows.

Suppose G has an Ulm resolution on a well ordering of a subset of $\mathbb{N}$, and H is divisible. Then G+H has an obvious Ulm resolution on that same ordering. If G is reduced, then the Ulm resolution of G ends with {0} and the Ulm resolution of G+H ends with H.

2.4. (Π^1_1-CA_0) A basic decomposition theorem states that G is isomorphic to the direct sum of a reduced group and a divisible group. Using C, we now see that G must have an Ulm resolution. In fact, the Ulm resolution of G ends with the largest divisible subgroup of G. The decompositin theorem is not available in ATR_0.

2.5. (ATR_0) The following are equivalent:
i) G has an Ulm resolution;
ii) G has an Ulm resolution which ends with the largest divisible subgroup of G;
iii) G is the direct sum of a reduced group and a divisible group;
iv) G has a largest divisible subgroup.

2.6. (ACA_0) ACA_0 lacks a good theory of well orderings of subsets of $\mathbb{N}$ and countable ordinals. However, we can still discuss Ulm functions in ACA_0. We have already discussed Ulm resolutions on well orderings of subsets of $\mathbb{N}$ in a way that mades perfectly good sense in ACA_0. We cannot prove, however, in ACA_0 that if G is reduced then G has an Ulm resolution. And we cannot prove in ACA_0 that any two Ulm resolutions of G (if they exist) are in any sense equivalent.

2.7. (ACA_0) Suppose G has an Ulm resolution. The Ulm resolution ends with the largest divisible subgroup of G. G is the direct sum of a reduced group and the largest divisible subgroup.

2.8. (ATR_0) We define H[p] as the set of all elements of H of order p.

Assume that G has an Ulm resolution. We write $p^{\alpha}G[p]$ for $(p^{\alpha}G)[p]$. For any countable ordinal α,

$$p^{\alpha}G[p]/p^{\alpha+1}G[p]$$

is a group with every element of order p. It is therefore a vector space over the field {0,1,...,p-1}, and has a well defined dimension. For $\alpha < l(G)$, we let $U_G(\alpha)$ be this dimension. This is the α-th Ulm invariant of G.

Finally, we define the Ulm function of G (on the given well ordering) as the function $f:l(G) \rightarrow \omega \cup \{\infty\}$, where $f(\alpha) = U_G(\alpha)$. Note that the domain of an Ulm function is one less than the domain of the Ulm resolution.

2.9. (ATR_0) In ATR_0, we can speak of the Ulm function of G, provided G has an Ulm resolution, without being tied to a specific well ordering of a subset of ω. This is because of its adequate theory of countable ordinals in terms of well orderings of subsets of ω.

2.10. (ACA_0) Suppose G has an Ulm resolution on a given well ordering of a subset of ω. Then we can define the Ulm function of G on that ordering by paragraph G.

Let G,H have Ulm resolutions on a common well ordering of a subset of ω. Ulm's theorem tells us the following. G and H are isomorphic if and only if they have the same Ulm function on the ordering. See [Si99], p. 200.

2.11. (ATR_0) It is important to have a necessary and sufficient condition for a function f to be an Ulm function; i.e., the Ulm function of some countable abelian p-group:

i) f is a function of the form $f:\alpha \to \omega \cup \{\infty\}$, where α is a countable ordinal.
ii) if α is a sucessor ordinal then $f(\alpha-1)$ is nonzero.
iii) if $\lambda < \mu \leq \alpha$ are two limit ordinals then f takes on infinitely many nonzero values in the interval $[\lambda,\mu)$.

See [BE70], p. 48, [Ka69], [Fu70/73]. Let G,H have Ulm functions $f:\alpha \to \omega \cup \{\infty\}$ and $g:\beta \to \omega \cup \{\infty\}$. Then G+H is also a countable reduced abelian p-group and its Ulm function h has domain $\max(\alpha,\beta)$, and $h(x) = f(x)+g(x)$. NOTE: In this sum, interpret undefined values as 0.

If G is isomorphic to $G_r + G_d$, where G_r is reduced and G_d is divisible, then the Ulm function of G is the same as the Ulm function of G_r.

The characterization of the Ulm functions above is the same for countable abelian p-groups and for countable reduced abelian p-groups.

2.12. (ACA_0) A subset A of any abelian p-group G is said to be independent if and only if any sum $n_1a_1 + ... + n_ka_k$, where $a_1,...,a_k \in A$, $k \geq 1$, and $n_1,...,n_k \in \{1,...,p-1\}$, cannot vanish. We define r(G) = p-rank of G as the cardinality of any maximal independent set. All maximal independent sets have the same cardinality. r(G+H) = r(G) + r(H). r(G) = r(G[p]). See [BE70], p. 48.

If G is embeddable into H then $r(G) \leq r(H)$.

2.13. (ACA_0) Let G be divisible. Then there exists a unique $0 \leq n \leq \infty$ such that G is isomorphic to $Z(p^\infty)^n$. Here $Z(p^\infty)$ is the group of rational numbers whose denominator is a power of p, modulo 1. r(G) = n. No $Z(p^\infty)^n$, n finite, can be properly embedded into itself.

r(G) is the least $0 \leq n \leq \infty$ such that G is embeddable into $Z(p^\infty)^n$. G is embeddable into $Z(p^\infty)^\infty$. See [Fu70], p. 107.

2.14. (ATR_0) Suppose G has an Ulm resolution. We define the rank function of G as $r:l(G) \to \omega \cup \{\infty\}$, where for all $\alpha < l(G)$, $r(\alpha) = r(p^\alpha G)$.

If λ is a limit ordinal $\leq l(G)$, then for all $\alpha < \lambda$, $r(\alpha) = \infty$. See [BE70], p. 40.

2.15. (ATR_0) Let f,r be, respectively, the Ulm and rank function of G. Let □ be the greatest limit ordinal □ l(G). Write l(G) = □+n. If n > 0 then $p^{\square}G$ exists and is the direct sum of cyclic groups and the largest divisible subgroup of G.

For i < n, f(□+i) is the number of cyclic summands of $p^{\square}G$ of order p^{i+1}. If G is reduced, then r(□+i) is the number of cyclic summands of $p^{\square}G$ of order ≥ p^{i+1}. Hence for i < n, r(□+i) = f(□+i) + ... + f(□+n-1). In particular, r(□+n-1) = f(□+n-1). If n = 0 then r is identically .

For general G, let t be the rank of its largest divisible subgroup. Then r(□+i) is t + the number of cyclic summands of $p^{\square}G$ of order ≥ p^{i+1}. Hence for i < n, r(□+i) = f(□+i) + ... + f(□+n-1) + t. In particular, r(□+n-1) = f(□+n-1) + t. If n = 0 then r is identically .

Let G,H have rank functions r,s and Ulm functions f,g. Then the rank function t of G+H has domain max(□,□), and t(x) = r(x)+s(x).

2.16. (ATR_0) Suppose G and H have Ulm resolutions. Let G,H have rank functions r,s. Then G is embeddable into H if and only if l(G) □ l(H) and for all □ < l(G), r(□) □ s(□). This condition is automatic below the greatest limit ordinal □ min(l(G),l(H)). (This important result about embeddability is from [BE70]. They prove it for arbitrary cardinality.)

As a consequence, G is embeddable into H if and only if
i) l(G) + □ □ l(H); or
ii) l(G) □ l(H), l(H)-l(G) is finite, and r □ s at all ordinals finitely below l(G).

If G,H are reduced, then condition ii) follows from

ii') l(G) □ l(H), l(H)-l(G) is finite, and f □ g at all ordinals finitely below l(G).

2.17. (ACA_0) The treatment of M,N,O in ATR_0 can be carried out in ACA_0 in the following sense. We start with an Ulm resolution of G on a specific well ordering of a subset of □. We then define the rank function on that ordering just as we defined the Ulm function on that ordering. Then the claims in M,N,0 are provable.

2.18. (ACA$_0$). For finite n we write G^n for the group of all n-tuples from G via coordinatewise addition. G^n is still a countable reduced abelian p-group. Any Ulm resolution for G provides an Ulm resolution for G^n in the obvious way. The corresponding Ulm function of G^n is the corresponding Ulm function of G with all values multiplied by n. The corresponding rank function of G^n is the corresponding rank function of G multiplied by n.

2.19. (ACA$_0$). We write G for the direct sum of countably infinitely many copies of G. Formally, this is the set of all infinite sequences from G, where all but finitely many terms are 0, under coordinatewise addition. G is still a countable reduced abelian p-group. Any Ulm resolution for G provides an Ulm resolution for G in the obvious way. The corresponding Ulm function of G is the corresponding Ulm function of G with all nonzaero values replaced by . The corresponding rank function of G is the corresponding rank function of G with all values replaced by .

2.20. (ACA$_0$) Let G be a countable abelian torsion group. For each i, let G_i be the subgroup consisting of the elements of order a power of the i-th prime. Then G is isomorphic to the direct sum of the nontrivial G_i. Furthermore, if G is isomorphic to a direct sum of nontrivial countable abelian p-groups with different p, then these abelian p-groups must be the same as the nontrivial G_i.

3. PROOFS FOR COUNTABLE REDUCED ABELIAN p-GROUPS.

Throughout this section, p is a prime number and G,H are countable reduced abelian p-groups.

LEMMA 3.1. (ATR$_0$) Either G is embeddable into H or H is embeddable into G .

Proof: The rank functions of G and H are constantly . If l(G) □ l(H) then G is embeddable into H , and if l(H) □ l(G) then H is embeddable into G . QED

The following is quoted from an unpublished manuscript of ours dated May 4, 1986, as Exercise V.7.6 on page 203 of [Si99].

LEMMA 3.2. (ATR$_0$) There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: A direct summand of a group with an Ulm resolution must have an Ulm resolution. Therefore all the groups involved have Ulm resolutions, which allows us to use the Ulm theory freely.

Let f,g be the Ulm functions of G,H. Let l(G) = □ and l(H) = □, where □ □ □. Let □ be the greatest nonsuccessor ordinal □ □ such that between any two limit ordinals □ □, f and g are simultaneously nonzero at infinitely many arguments. Then min(f,g) restricted to □ is an Ulm function.

Now look at f,g on the interval [□,□+□). Only the part of this interval that is < alpha is relevant, and this part could be empty, or could be the whole interval. In any case there are only finitely many elements at which both f,g are nonzero. If there are none, then define h to be min(f,g) restricted to □. If there are some, then let n be greatest such that f,g are both nonzero at □+n, and define h to be min(f,g) restricted to □+ n+1. Thus dom(h) is an element of [□,□+□). We write dom(h) = □+m. Clearly h is an Ulm function.

We need to find Ulm functions h′ and h′′ such that

i) dom(h′) = dom(f) = □;
ii) dom(h′′) = dom(g) = □;
iii) h+h′ = f;
iv) h+h′′ = g.

There is no problem finding such h′ and h′′ by subtraction, but they might not be Ulm functions even if we use the formula   -   = . The "infinitely many nonzero values between limit points" condition may fail.

To fix this problem, we replace h by h*, where h* agrees with h at or above □. Below □, we take every other ordinal at which h = min(f,g) is nonzero and redefine it to be 0. Then h* is still an Ulm function. It is also clear that h′ = f-h* is an Ulm function, perhaps truncated so that the last value - if it exists - is nonzero. The construction of h′′ is analogous.

Thus h* is the Ulm function of a common direct summand K of G,H. We need to show that every common direct summand K′ is embeddable into K. But the rank function r for K is infinity below □, and the rank function r' for K′ is ≥ r at or above

□, to the extent that r′ is defined. Hence K′ is embeddable into K. QED

We remark that the following is false: Let G,H be countable reduced abelian p-groups. There is a common direct summand K of G and H such that every direct summand of G and H is a direct summand of K.

LEMMA 3.3. (ATR_0) There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.

Proof: Again, all of the groups involved are reduced, and therefore have Ulm resolutions. Therefore we can use the Ulm theory freely.

Let f,g be the Ulm functions of G ,H . Let l(G) = □ and l(H) = □, where □ □ □. Let □ be the greatest nonsuccessor ordinal □ □ such that between any two limit ordinals □ □, f and g are simultaneously   at infinitely many arguments. Then min(f,g) restricted to □ is an Ulm function.

Now look at f,g on the interval [□,□+□). Only the part of this interval that is < □ is relevant, and this part could be empty, or could be the whole interval. In any case there are only finitely many elements at which both f,g are . If there are none, then define h to be min(f,g) restricted to □. If there are some, then let n be greatest such that f,g are both infinity at □+n, and define h to be min(f,g) restricted to □+ n+1. Thus dom(h) is an element of [□,□+□). We write dom(h) = □+m. Clearly h is an Ulm function.

Obviously h+f = f and h+g = g. Hence the countable reduced abelian group J with Ulm function h is a direct summand of G ,H .

Now let J′ be a direct summand of G ,H , and let h′ be the Ulm function of J′. Then h′ is nonzero only at arguments where f,g are both infinity. Hence dom(h′) < □+□ since h′ is an Ulm function. In fact dom(h′) ≤ □+m = dom(h). It is now clear that h′+h = h. Therefore J' is a direct summand of J. QED

LEMMA 3.4. (ATR_0) In every infinite sequence of countable reduced abelian p-groups, one group is embeddable in a later group.

Proof: Let G[1],G[2],... be given. There are $i_1 < i_2 < ...$ such that the lengths of G[i_1},G[i_2],... are ordinals □$_1$ □ □$_2$ □ ... such that either the successive differences are all finite, or the successive differenes are all infinite. In the latter case, obviously each G[i_j] is embeddable into G[i_{j+1}], and we are done. So we assume that the successive differences are all finite, and that the □'s are distinct. Let □ be the greatest nonsuccessor ordinal □ □$_1$. We can write the □'s as □+m_1 < □+m_2 < ... . The rank functions of the G[i_1],G[i_2],... constitute an infinite sequence of decreasing finite sequences from □ □ { }, whose lengths are strictly increasing. By basic wqo theory, one of these finite sequences is dominated by a subsequence of a later finite sequence. But since the later finite sequence is decreasing, we can move this subsequence to the left to occupy an initial segment without lowering any of the terms. Hence that finite sequence is dominated by that later finite sequence; i.e., that rank function is dominated by that later rank function. Therefore the corresponding group is embeddable in the later corresponding group. QED

LKEMA 3.5. (ATR_0) There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.

Proof: Let r,s be the rank functions of G,H. Let l(G) = □ and l(H) = □, where □ □ □. Let □ be the greatest nonsuccessor ordinal □ □. Let t = min(r,s) defined on □. Since t is up to □, and nonzero and decreasing from □ up to but not including □, we see that t corresponds to an Ulm function with domain □, and so is the rank function of some group M. (This corresponding Ulm function is obtained by successive subtraction above □). If M' is any group embeddable in G and H then its rank function t' must have t' □ t and domain □ □. QED

LEMMA 3.6. (ATR_0) If G+G and H+H are isomorphic then G and H are isomorphic.

Proof: l(G+G) = l(G) and l(H+H) = l(H), and the Ulm functions of G+G and H+H must be double that of G and H. Hence G and H have the same Ulm functions. QED

LEMMA 3.7. (ATR_0) If G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

Proof: $l(G) \leq l(H) \leq l(G)$ and the Ulm function of G is $\leq$ the Ulm function of H. Therefore G and H have the same Ulm functions. QED

4. PROOFS FOR COUNTABLE ABELIAN p-GROUPS.

Throughout this section, p is a prime number and G,H are countable abelian groups. $\Pi^1_1\text{-}CA_0$ proves that G,H have largest divisible subgroups and Ulm resolutions. Sometimes by arguing more carefully, we can get away without having Ulm resolutions of G,H.

LEMMA 4.1. (ATR_0) G is embeddable into H or H is embeddable into G .

Proof: If G,H are reduced then apply Lemma 3.1. If, say, G is not reduced, then let J be a nontrivial divisible subgroup of G. Then H is embeddable into J , and hence into G . QED

For the remainder of this section, we write $G \cong G_r + G_d$ to mean that G is isomorphic to some reduced subgroup G_r of G plus the largest divisible subgroup G_d of G. Of course, G_d may not provably exist in ACA_0. But if it does, then G_r exists (although it is only unique up to isomorphism).

LEMMA 4.2. (ATR_0) Suppose $G \cong G_r + G_d$ and K be any countable abelian group. Then K is a direct summand of G if and only if $K \cong K_r + K_d$ with K_r a direct summand of G_r and K_d a direct summand of G_d.

Proof: Suppose K is a direct summand of G. Now G has an Ulm resolution, and this yields an Ulm resolution for K. So we can write $K \cong K_r + K_d$. The if part is trivial. So suppose K is a direct summand of G. Write $K_r + K_d + J \cong G_r + G_d$. By the same reasoning, J has an Ulm resolution. Hence we can write $K_r + K_d + J_r + J_d \cong G_r + G_d$, or $(K_r + J_r) + (K_d + J_d) \cong G_r + G_d$. By the uniqueness of the fundamental decomposition, we see that $K_r + J_r \cong G_r$ and $K_d + J_d \cong G_d$, as required. QED

LEMMA 4.3. (ATR_0) Suppose G,H have largest divisible subgroups. Then there is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: Write $G \cong G_r + G_d$ and $H \cong H_r + H_d$. By Lemam 3.2, let K be a minimal common direct summand of G_r,H_r with respect to embeddings. And let J be the minimum of G_d and H_d. We claim that K + J is as required. Suppose W is also a common direct summand of G and H. By Lemma 4.2, write $W \cong W_r + W_d$, where W_r is a direct summand of G_r,H_r, and W_d is a direct summand of G_d,H_d. Then W_r is embeddable into K and W_d is embeddable into J. QED

LEMMA 4.4. (Π^1_1-CA_0) There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: Immediate from Lemma 4.3. QED

LEMMA 4.5. (ATR_0) Suppose G,H have largest divisible subgroups. Then there is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.

Proof: Write $G \cong G_r + G_d$ and $H \cong H_r + H_d$. By Lemam 3.3, let J be a minimal common direct summand of G_r ,H_r with respect to direct summands. And let X be the minimum of G_d and H_d. We claim that J + X is as required. Suppose W is also a common direct summand of G and H . By Lemma 4.2, write $W \cong W_r + W_d$, where W_r is a direct summand of G_r ,H_r , and W_d is a direct summand of G_d ,H_d . Then W_r is a direct summand of J and W_d is a direct summand of X. Hence W is a direct summand of J + X. QED

LEMMA 4.6. (Π^1_1-CA_0) There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.

Proof: Immediate from Lemma 4.5. QED

LEMMA 4.7. (ATR_0) Either G has a largest divisible subgroup or $Z(p^{\infty})$ is embeddable into G.

Proof: We use the method of psudoresolutions as in the proof in [Si99], p. 201, within ATR_0 that every countable reduced Abelian p-group has an Ulm resolution.

If G has an Ulm resolution then the last group must be the largest divisible subgroup of G. So we can assume that G has no Ulm resolution.

We know that there is a partial Ulm resolution of G along any well ordering of □. It must not stabilize, since there is no Ulm resolution of G.

Therefore there must be a partial Ulm pseudoresolution of G on a non well founded ordering which does not stabilize. Now there are partial Ulm resolutions that do not stabilize, on arbitrarily large well orderings on □ which support ordinal addition and exponentiation, with the usual laws, and where the component groups decrease. Hence we must have a partial Ulm resolution that does not stabilize, on a non well founded ordering which also supports ordinal addition and exponentiation, with the usual laws, and where the component groups decrease.

Let $x_1 > x_2 > \ldots$ in the ordering, and consider $□^{x1} > □^{x2} > \ldots$. There are obviously decreasing chains strictly between each of the terms; i.e., $□^{xi} + x_i$, $□^{xi} + x_{i+1}$, ... is a decreasing chain. We view this as a decreasing sequence of left cuts, each of which have no least element.

Now just as in [Si99], p. 201, each of these right cuts determines a divisible subgroup of G; i.e., the union of the groups in the Ulm pseudoresolution whose position lies in the right cut. In this way, we obtain an infinite increasing chain of divisible subgroups of G. The union of these divisible subgroups of G must be a divisible subgroup of G that is isomorphic to Z(p) . QED

We need a refinement of Lemma 4.7.

LEMMA 4.8. (ATR_0) Suppose Let $G_1,G_2,\ldots$ be countable abelian p-groups. Suppose that Z(p) is not embeddable into any G_i. Then the sequence $G_{1d},G_{2d},\ldots$ of largest divisible subgroups of $G_1,G_2,\ldots$ exist. Furthermore, a sequence $G_{1r},G_{2r},\ldots$ of reduced subgroups of $G_1,G_2,\ldots$ exist, where each G_i □ $G_{ir} + G_{id}$. In addition, a sequence of isomorphisms exist.

Proof: Repeat the proof of Lemma 4.2 by constructing simultaneously Ulm peudoresolutions of the G's on a single non well founded ordering, without caring yet whether and when the peudoresolutions stabilize. Argue that none of them can continue strictly decreasing into the non well founded part, for otherwise an embedding of Z(p) can be constructed as in the proof of Lemma 4.2. Therefore, they all stabilize in the well founded part. We can then read off the largest divisible subgroups from the Ulm resolutions. We can uniformly construct the reduced parts and the associated isomorphisms by the “effectiveness” of the construction in the fundamental decomposition theorem. QED

LEMMA 4.9. (ATR_0) In every infinite sequence of countable abelian p-groups, one group is embeddable in a later group.

Proof: Let $G_1,G_2,\ldots$ be given. We are done if $Z(p^\infty)$ is embeddable into some group after the first, since every countable Abelian p-group is embeddable in $Z(p^\infty)$. Otherwise, by Lemma 4.8 we can write $G_i = G_{ir} + G_{id}$, $i \geq 2$. Since the divisible parts G_{id} are each of the form $Z(p^\infty)^n$, $0 \leq n \leq \omega$, and we can determine n “effectively,” we can pass to an infinite subsequence of the G’s whose divisible parts are increasing in rank (exponent). Then apply Lemma 3.4. QED

LEMMA 4.10. (ATR_0) There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.

Proof: Suppose $Z(p^\infty)$ is embeddable into G. Then the groups embeddable into G and H are exactly the groups embeddable into H. Therefore H is as required.

So we may assume that $Z(p^\infty)$ is not embeddable into G and not into H. By Lemma 4.7, G and H have largest divisible subgroups. Write $G \cong G_r + G_d$, $H \cong H_r + H_d$. It is clear that any function on any countable ordinal □ which is below the greatest limit ordinal □ □ and decreasing from □ on, is the rank function of a countable reduced abelian p-group. Hence we can find countable reduced abelian groups G’ and H’ whose rank function is the same as that of G and H. Hence G’ and G have the same subgroups up to isomorphism, as well as H’ and H. Now apply Lemma 3.5 to G’ and H’. QED

LEMMA 4.11. (Π^1_1-CA_0) If G+G and H+H are isomorphic then G and H are isomorphic.

Proof: Since G,H have Ulm resolutions, we can repeat the proof of Lemma 3.6. QED

LEMMA 4.12. (Π^1_1-CA0) If G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

Proof: Since G,H have Ulm resolutions, we can repeat the proof of Lemma 3.7. QED

5. PROOFS FOR COUNTABLE REDUCED ABELIAN TORSION GROUPS.

Throughout this section, G,H are countable reduced abelian torsion groups.

The $\square^{1}_{1}$-AC_0 appears in this section. This is provable in ATR_0. See [Si99], p. 205.

LEMMA 5.1. (RCA_0) Z_2 is not embeddable into Z_3 and Z_3 is not embeddable into Z_2 .

Proof: Obvious. QED

LEMMA 5.2. ($\square^{1}_{1}$-AC_0) Let $G_1,G_2,\ldots$ and $H_1,H_2,\ldots$ be countable abelian p-groups with primes $p_1,p_2,\ldots$, where p_i is the i-th prime. Let G be the direct sum of the G's and H be the direct sum of the H's. Then G is embeddable into H if and only if for all i, G_i is embeddable into H_i. Also G is a direct summand of H if and only if for all i, G_i is a direct summand of the H_i.

Proof: For each prime p_i, the elements of G of order a power of p_i are those all of whose coordinates other than the i-th are 0.

Let h be an embedding from G into H. By considering orders of elements, h must map each G_i into . On the other hand, suppose each G_i is embeddable into H_i. Using $\square^{1}_{1}$-AC_0, we can put such embeddings together in order to embed G into H.

Now let h be an isomorphism from G + J onto H. The elements of G + J of order a power of p_i are exactly G_i + J', where J' is the subgroup of J consisting of the elements of order a power of p_i. And the elements of H of order a power of p_i are exactly H_{p_i}. Hence each G_i is a direct summand of H_i.

Finally suppose that for each i there is a J_i such that G_i + J_i is isomorphic to H_i. Again by $\square^{1}_{1}$-AC_0 we can choose these J's and the relevant isomorphisms, and let J be their direct sum. Then G + J is isomorphic to the direct sum of certain of the H's. But then we can set J* to be the direct sum of J with the direct sum of the remaining H's, so that G + J* is isomorphic to H. QED

In Lemmas 5.3 - 5.?, G,H are countable reduced abelian torsion groups. Define Gi to be the subgroup of elements whose order is a power of the i-th prime.

LEMMA 5.3. (ATR_0) There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: For each i there is a direct summand K_i of G_i and H_i such that every direct summand of G_i and H_i is embeddable into K_i. Using Σ^1_1-AC_0, let K be the direct sum of the K's. Then K is a direct summand of G and H. Let J be any direct summand of G and H. Then J is a countable abelian torsion group. So for each i, J_i is a direct summand of G_i and H_i. Hence for each i, J_i is embeddable into K_i. So by Σ^1_1-AC_0, J is embeddable into K. QED

LEMMA 5.4. (ATR_0) There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.

Proof: For each i there is a direct summand K_i of G_i and H_i such that every direct summand of G_i and H_i is a direct summand of K_i. Using Σ^1_1-AC_0, let K be the direct sum of the K's. Then K is a direct summand of G and H . Let M be any direct summand of G and H . Then M is a countable abelian torsion group. So for each i, M_i is a direct summand of G_i and H_i . Hence for each i, M_i is a direct summand of K_i. So by Σ^1_1-AC_0, M is a direct summand of K. QED

LEMMA 5.5. (RCA_0) There is an infinite sequence of finite abelian torsion groups, no one of which is embeddable in any other one.

Proof: Take $Z_2, Z_3, Z_5, \ldots$. QED

LEMMA 5.6. (RCA_0) There is an infinite decreasing sequence of countable abelian torsion groups, no one of which is embeddable in any later one.

Proof: Take G_i to be the direct sum of Z_j, where j ranges over all but the first i primes. QED

LKEMA 5.7. (ATR_0) There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.

Proof: By Lemma 3.5, for each i there is a group M_i embeddable into G_i and H_i such that every group embeddable into G and H is embeddable into M_i. Using Σ^1_1-AC_0, let M be the direct sum of the M's. Again using Σ^1_1-AC_0, M is embeddable into G and H. Let N be any group embeddable into G and H. Then N is a countable abelian torsion group. So for each i, N_i is embeddable into G_i and H_i. Hence for each i, N_i is embeddable into M_i. Using Σ^1_1-AC_0, N is embeddable into M. QED

LEMMA 5.8. (ATR_0) If G+G and H+H are isomorphic then G and H are isomorphic.

Proof: Suppose G+G and H+H are isomorphic. Note that for each i, $(G+G)_i \cong G_i + G_i$, and $(H+H)_i \cong H_i + H_i$. Hence for each i, $G_i + G_i \cong H_i + H_i$. By Lemma 3.6, for each i, $G_i \cong H_i$. Using Σ^1_1-AC_0, $G \cong H$. QED

LEMMA 5.9. (ATR_0) If G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

Proof: Suppose $G + J \cong H$ and $H + M \cong G$. Then for each i, $G_i + J_i \cong H_i$ and $H_i + M_i \cong G_i$. By Lemma 3.7, for each i, $G_i \cong H_i$. By Σ^1_1-AC_0, $G \cong H$. QED

6. PROOFS FOR COUNTABLE ABELIAN TORSION GROUPS.

Throughout this section, G,H are countable abelian torsion groups.

LEMMA 6.1. (Π^1_1-CA_0) There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: We prove this Lemma from Lemma 5.3 by repeating the proof of Lemma 4.4 from Lemma 3.2. QED

LEMMA 6.2. (Π^1_1-CA_0) There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of J.

Proof: We prove this Lemma from Lemma 5.4 by repeating the proof of Lemma 4.6 from Lemma 3.3. QED

LEMMA 6.3. (ATR_0) There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.

Proof: We prove this Lemma from Lemma 4.10 by repeating the proof of Lemma 5.7 from Lemma 3.5. QED

LEMMA 6.4. (Π^1_1-CA_0) If G+G and H+H are isomorphic then G and H are isomorphic.

Proof: This Lemma is proved from Lemma 4.11 by repeating the proof of Lemma 5.8 from Lemma 3.6. QED

LEMMA 6.5. (Π^1_1-CA_0) if G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

Proof: This lemma is proved from Lemma 4.12 by repeating the proof of Lemma 5.9 from Lemma 3.7. QED

7. REVERSALS TO ATR_0 OVER ACA_0.

The statements 1-8 from the Introduction have all been proved in ATR_0, proved in Π^1_1-CA_0, or refuted in RCA_0. We have obtained reversals only to ATR_0. We have obtained no reversals to Π^1_1-CA_0 among 1-8. But recall the equivalence of the existence of a largest divisible subgroup of any countable abelian group (or even countable abelian 2-group) with Π^1_1-CA_0. See [Si99], p. 230.

Let T be a finite sequence tree; i.e., a set of finite sequences including the empty sequence, closed under initial segments. We define the important countable abelian p-group G(T,p) as follows.

The elements consist of formal sums of the form

*) $n_1x_1 + \ldots + n_kx_k$

where $k \geq 0$, $x_1,\ldots,x_k$ are distinct nonempty elements of T, and $1 \leq n_1,\ldots,n_k \leq p-1$. In case k = 0 we write 0. The order in which the terms are listed is irrelevant, and so if we were to be very formal, we would write

$\{\langle n_1,x_1\rangle,\ldots,\langle n_k,x_k\rangle\}$

where the empty set is the zero.

Addition of two elements is defined first by taking the usual sum where coefficients are added for the same generator. This results in an expression of the form *), where there may be coefficients ≥ p. We then successively replace each $n_i x_i$ for which $n_i \geq p$ by

$(n_i-p)(x_i)+(x_i')$

where y_i' is the result of chopping off the last term of y_i. If y_i is of length 1 then we just use

$(n_i-p)(x_i)$.

We again normalize the result by putting it in the form *), where there still may be coefficients ≥ p. We then continue this process of replacements and normalizations until all of the coefficients lie in [1,p-1). Simple inductive arguments prove that any sequence of replacements and normalizations terminates with the same final result, with a good a priori bound on how many steps are needed. And also that this addition is commutative and associative.

We define the additive inverse of the group element *) to be

$(p-n_1)(x_1) + ... + (p-n_k)(x_k)$.

Of course, the additive inverse of 0 is 0.

All of this can obviously be formalized in RCA_0.

LEMMA 7.1. (RCA_0) A finite sequence tree T is well founded if and only if G(T,p) is reduced.

Proof: Suppose T is not well founded, and let f be a path through T. Thus we have nonempty elements of T, $y_1,y_2,...$, each an immediate successor of the preceding, $lth(y_1) = 1$, which form both a sequence and a set. It is easy to see that the subgroup of G(T,2) generated by these generators exists and is divisible. On the other hand, let H be a divisible subgroup of G(T,2). Let x be an element of H. Suppose 2y = x. Then every maximal element of x has an immediate successor in y. From this we easily obtain an infinite path through T. QED

Let □ be a linear ordering of a subset of □. We define G(□,p) = G(T,p), where T = T(□) is the tree of finite sequences from the field of □ that are decreasing under □.

LEMMA 7.2. (ACA_0) Let □ be a well ordering of a subset of □. G(□,p) is reduced. G(□,p) has a unique Ulm resolution along ≤. The Ulm invariants exist and are all 1.

Proof: Clearly T(□) is well founded, and so we can apply Lemma 7.1.

We now define a map from the nonzero elements of G(□,p) into points in □. Let x ≠ 0 be in G(□,p). If x is nonzero, let x* be the least point in □ that appears as a term of any generator appearing in x. For points u in □ define G_u to be the set of all x □ G(□,p) such that x* ≥ u, or x is 0. Note that G_u is a subgroup of G. If u is the least element of □ then G_u = G. If u is the successor of v then $G_u = pG_v$. If u is a limit point then G_u is the intersection of all G_v, v < u. Also if u □ v then G_u contains G_v. And the intersection of all of the G_u is {0}. The x in G_u with px = 0 are just the x □ G(<,p) all of whose generators are of length 1, where the term is ≥ u. Hence the u-th Ulm invariant is 1. The Ulm resolution is unique by considering the set of places at which two of them differ. QED

LEMMA 7.3. (ACA_0) Let □ and □′ be well orderings of subsets of □. Then G(□,p) is embeddable into G(□′,p) if and only if □ is order embeddable into □′.

Proof: Suppose there is an order embedding from □ into □′. Then there is an embedding of T(□) into T(□′) which preserves < > and immediate successors. Such a tree embedding obviously induces a group embedding.

Now let h be an embedding from G(≥,p) into G(□′,p). Let $G(□,p)_u$ be the Ulm resolution of G(□,p), indexed by points u in □, and $G(□′,p)_v$ be the Ulm resolution of G(□′,p), indexed by points v in □′.

Using ACA_0, let f(u) be the greatest v such that h maps $G(□,2)_u$ into $G(□′,2)_v$. We know that v must exist since we can use the least element of □′.

Let u < v in □. Then h maps $G(□,p)_u$ into $G(□′,p)_{f(u)}$. Since there exists x □ $G(□,p)_u$ such that px ≠ 0, there exists y □

$G(□',p)_{f(u)}$ such that py ≠ 0. Hence f(u)+1 exists. Also h maps $G(□,p)_u$ into $G(□',p)_{f(u)+1}$. Hence f(v) >' f(u). This shows that f is an order embedding of □ into □'. QED

LEMMA 7.4. (ACA_0) Let □ and □' be well orderings of subsets of □, and 1 □ n □ □. $G(□,2)^n$ has a unique Ulm resolution along □. The Ulm invariants exist and are all n. $G(□,p)^n$ is embeddable into $G(□',p)^n$ if and only if □ is order embeddable into □'.

Proof: Adapt the proofs of Lemmas 7.2 and 7.3. QED

LERMMA 7.5. The following implies ATR_0. Either G is embeddable into H or H is embeddable into G .

Proof: By Lemma 7.4, we can derive the weak comparability of well orderings. We showed that this implies ATR_0. See [FH90]. QED

LEMMA 7.6. (ACA_0) Let K,M be any abelian groups, and G = K+M; i.e., G is the set of all ordered pairs from K amd M under coordinatewise addition. Let (G_u), u in □, be an Ulm resolution of G. Then $(G_u$ □ K+{0}) is an Ulm resolution of K+{0} possibly extended by a tail of copies of {0}. In particular, K has an Ulm resolution along an initial segment of □.

Proof: If u is least in □ then G_0 = G, and so G_0 □ K+{0} = K+{0}. Let u be a limit in ≤. Then □{G_v □ K+{0}: v < u} = □{G_v: v < u} □ K+{0} = G_u □ K+{0} as required.

Now let u be a point in □. We must verify that p(G_u □ K+{0}) = pG_u □ K+{0}. Let (x,y) □ p(G_u □ K+{0}). Write (x,y) = (pz,pw), where (z,w) □ G_u □ K+{0}. Then w = 0, and so y = 0. Thus we must check that (pz,0) □ pG_u □ K+{0}. Now (z,0) □ G_u. Hence (pz,0) □ pG_u. Also z □ K. Hence (pz,0) □ pG_u □ K+{0}.

Finally let (x,y) □ pG_u □ K+{0}. Then x □ K and y = 0. Write (x,0) = (pz,0), where (z,0) □ G_u. Since G = K+M, z □ K. Hence (pz,0) in p(G_u □ K+{0}). However, note that we may get to {0} at some point, and so we may have a tail of copies of {0}. QED

We remark that RCA_0 is sufficient to show that (G_u □ K+{0}) is an extended Ulm resolution of K+{0}.

LEMMA 7.7. (ACA_0) Let □ and □′ be well orderings of subsets of □. Suppose G,H are countable reduced abelian p-groups with Ulm resolutions along □ and □′ respectively. Suppose G is a direct summand of H. Then □ is order isomorphic to an initial segment of □′.

Proof: Let □, □′, G, H be as given. Without loss of generality we can assume that H = G+K. By Lemma 7.6, G has an Ulm resolution along an initial segment of □′. But G has an Ulm resolution along □. We can easily prove by transfinite induction that for every point x in □ there is a unique point y in □' such that the x-th point in the Ulm resolution along □ is identical to the y-th point in the Ulm resolution along the initial segment of □', and that this defines an isomorphism from □ onto the initial segment of □′ used for the Ulm resolution along □′. QED

LEMMA 7.8. (ACA_0) Let □ and □′ be well orderings of subsets of □. Suppose G,H are countable reduced abelian p-groups with Ulm resolutions along □ and □′ respectively. Assume the corresponding Ulm invariants for H are all nonzero. Suppose □ is order isomorphic to an initial segment of □′. Then G is a direct summand of H .

Proof: Let □, □′, G, H be as given. Let ≤ be order isormophic to the initial segment □′′ of □′. Then G has an Ulm resolution along □′′. Hence H has an Ulm resolution along □′ with Ulm invariants . Therefore G + H has an Ulm resolution along □′ with Ulm invariants . By Ulm's theorem, H □ G + H . QED

LEMMA 7.9. (ACA_0) The following implies ATR_0. There is a direct summand J of G and H such that every direct summand of G and H is emeddable into J.

Proof: Let □ and □′ be well orderings of subsets of □. Set G = G(□,p) and H = G(□′,p). Then G has an Ulm resolution along □ with Ulm invariants , and H has an Ulm resolution along □' with Ulm invariants . Let J be as given.

By Lemma 7.6, J has an Ulm resolution on a well ordering □′′ of a subset of □. By Lemma 7.7, □′′ is order isomorphic to initial segments of □ and □′. We claim that one of these initial segments must not be proper. This will show that □ and □′ are order comparable. We will have then derived the comparability of well orderings, which implies ATR_0 according to [FH90].

Suppose they are both proper. Then we have a points x,y in □,□′ such that □′′ is isomorphic to the initial segment of □ below x and the initial segment of □′ below y. Let □* be □′′ with a single point tacked on the end. Then □* is isomorphic to initial segments of □ and □′. By Lemma 7.8, G(□*,p) is a direct summand of G and H . Therefore G(□*,p) is embeddable into J. By Lemma 7.3, □* is order embeddable into □′′, which is impossible. QED

LEMMA 7.10. (ACA_0) The following implies ATR_0. In any infinite sequence of countable reduced abelian p-groups, one group is embeddable in another group.

Proof: Using the G(□,p) construction and Lemma 7.3, this implies that in any infinite sequence of well orderings of subsets of □, one ordering is embeddable into another one. In [Sh93], it is proved that this implies ATR_0 over RCA_0. QED

LEMMA 7.11. (ACA_0) The following implies ATR_0. Let $G_1,G_2,\ldots$ be an infinite descending chain of countable reduced abelian 2-groups under containment. Then one group is embeddable into an earlier group.

Proof: This implies that if $□_1,□_2,\ldots$ is an infinite descending chain of well orderings of subsets of □ under containment, then one ordering is embeddable into an earlier one. This is a slight weakening of the following statement that [Sh93] proved implies ATR_0 over RCA_0: if $□_1,□_2,\ldots$ is an infinite descending chain of well orderings of subsets of □ such that each ordering is embeddable into the previous ordering, then some ordering is embeddable into a later ordering. However, Shore's proof can be adapted to show that this weakening also implies ATR_0 over RCA_0. QED

8. REVERSALS TO ACA_0 OVER RCA_0.

either G is embed into H or H is embed into G .

start with a 1-1 function from □ into □, and get its range.

height is the largest n such that it is divisible by p^n. If there is no largest, then height is . h:G into H. Then the hgt(x) ≤ hgt(h(x)). if x is divisible by pn then h(x) is divisible by pn.

The leaves of T consist of {<0,0>, <0,1,0>, <0,2,1,0>, <0,3,2,1,0>, ...}.

The leaves of T′ are {<n,f^{-1}(n),f^{-1}(n)-1,...,1,0>: n □ rng(f)} □ {<n>: n □ rng(f)}.

In G(T,p), for all n ≥ 1 there exists y with pny = <0>. I.e., <0> has infinite height.

In G(T′,p), there is no nonzero element of infinite height.

So G(T′,p) is embeddable into G(T,p), say by h. Then the height of x is at most the height of h(x). But for x = <n>, n □ rng(f), hgt(x) = f^{-1}(n)+1, in G(T′,p). But in G(T,p), height is at most one more than the largest coordinate, with at most p exceptions. So we get the existence of the range of f.

now for the infinite decreasing sequence of abelian p-groups, one is embeddable in a later one.

We can enumerate the values of the one-one f in numerical order, as n_i = the i-th value of f. We need to prove {n_1,n_2,...} exists. We will assume, without loss of generality, that all values of f are powers of 2 greater than 1.

We build a crucial tree T. We say that x □ T is strong if and only if there are paths of arbitrarily long finite length starting at x. We define #(x) as the greatest integer n such that there is a strong y which is n steps above y in T. If x is not strong then #(x) is taken to be 0. We will arrange that for all x of length 1, #(x) is finite.

In fact, we will arrange

#(n_1) = 1
#(n_2) = 2, #(n_2 + 1) = 3
#(n_3) = 4, #(n_3 + 1) = 5, #(n_3 + 2) = 6
#(n_4) = 7, #(n_4 + 1) = 8, #(n_4 + 2) = 9, #(n_4 + 3) = 10
etcetera.

We have blocks B[1],B[2],..., where B[1] = {n_1}, B[2] = {n_2,n_2 + 1}, ... i.e., B[i] = [n_i,n_i + i). And # on B[i+1] continues from # on B[i] by successively adding 1. # off of blocks is 0.

T_1 = T. The T_i successively drop all tuples whose first term is the largest in its block. i.e., define B[i,j] to be $[n_i, n_i + i-j)$ or so, so that they become empty.

Look at $G(T_j,p)$. The nodes emanating from B[i,j] have #'s raning from blah to blee. make it more in tune with powers of 2. so it has nice formulas. Look at embeddings from $G(T_i)$ into $G(T_{i+1})$. Must map length one nodes emanating from one block into nodes emanating from a later block. From such a map, we can obviously produce the range of f.

□-0-1 function on □ bounded by an exponential can be arranged to be the sharp function, with sharp function 0 off of the domain.

how about: sequence of groups, one embeddable into a different one?

same idea, except you can take mutually disjoint sets, no blocks. then must jump from one place to later place.

LEMMA a. (RCA_0) Let T be a tree whose leaves are of different lengths, and where the tree above any nonempty element is finite and has no branching, and where the set of all leaves exists. Let G = G(T,p) . Then the sequence of groups p^nG exists. Every nonzero element of G has finite height.

Proof: We need a good criterion for whether a nonzero element x of G is divisible by p^n. This holds if and only if all of the generators appearing in x have at least n successors. This forms a set since the set of all leaves exists. Every element of G has finite height since the tree above any nonempty element is finite. QED

LEMMA b. (RCA_0) Let K be a direct summand of G = G(T,p) , where T is as in Lemma a, and K has elements of arbitrarily high order. Then the sequence of groups p^nK, $n \geq 0$, exists, all of which are nontrivial. Every element of K has finite height. There are arbitrarily large n such that $(p^nK)[p] \neq (p^{n+1}K)[p]$.

Proof: By the proof of Lemma 7.6, the sequence of groups p^nK, $n \geq 0$, exists. Since K has elements of arbitrarily high order, these groups are nontrivial. Also T is infinite. Now let $m \geq 0$. Let x be an element of K of order p^{m+1}. Then $p^m x \in (p^m K)[p]$. Since $p^m x$ is of finite height, let $r > m$ be least such that $p^m x \notin (p^r K)$. Then $(p^{r-1}K)[p] \neq (p^r K)[p]$. QED

LEMMA c. (RCA_0) Let T′ be an infinite tree whose leaves are of different lengths, and where the tree above any nonempty element is finite and has no branching. We do not assume that the set of all leaves exists. Let $H = G(T',p)$. For all $n \geq 0$, $(p^n H)[p] \neq (p^{n+1}H)[p]$ if and only if T′ has a leaf of length n+1. For all $n \geq 1$, if there is a leaf of T′ of length n then $Z(p^n)$ is a direct summand of H.

Proof: Suppose T′ has a leaf, x, of length n+1. We write x for (x,0,0,...). Then $p^{n+1}x = 0$ and so $p^n x \in p^n H[p]$. However, $p^n x \notin p^{n+1}H$, since there is only one way to successively divide $p^n x$ by p, and that terminates in n steps with x. On the other hand, suppose T′ has no leaf of length n+1. Let $y \in (p^n H)$, $py = 0$. All generators appearing in y are elements of T of length 1. They all must have at least n successors in T, the last of which are elements of T of length n+1. But none of these are leaves, and so x is divisible by p^{n+1}. I.e., $x \in (p^{n+1}H)[p]$.

Suppose there is a leaf, x, of T′ of length $n \geq 1$. Let T′′ be the subtree of T′ resulting from removing x and all of its nonempty predecessors from T′. Then $Z(p^n) + G(T'',p) \cong G(T',p)$. QED

LEMMA d. (RCA_0) Let T′,H be as in Lemma c. Let K be a direct summand of H. For all $n \geq 0$, if $(p^n K)[p] \neq (p^{n+1}K)[p]$ then T′ has a leaf of length n+1.

Proof: Suppose $(p^n K)[p] \neq (p^{n+1}K)[p]$. Write $K + J \cong H$. Let $x = p^n y$, $px = 0$, $x \notin (p^{n+1}K)[p]$, $y \in K$. Then $(x,0) \in (p^n H)[p]$. Now $x \notin (p^{n+1}K)$. Hence $(x,0) \notin p^{n+1}H$. Therefore $(p^n H)[p] \neq (p^{n+1}H)[p]$. By Lemma c, T′ has a leaf of length n+1. QED

LEMMA e. (RCA_0) The following implies ACA_0. Let $f:\mathbb{N} \to \mathbb{N}$ be one-one, where there are arbitrarily large n that are not values of f. Then there exists a one-one $g:\mathbb{N} \to \mathbb{N}$ such that no value of g is a value of f.

Proof: Let $h:\mathbb{N} \to \mathbb{N}$. We wish to prove that rng(h) exists. We can assume that there are arbitrarily large numbers that are not values of h. Let P(n) if and only if n is of the form $<<a_1,b_1>,...,<a_m,b_m>>$, $m \geq 0$, where

i) for all i, b_i is the least number such that $h(b_i) = a_i$;
ii) $a_1 < ... < a_m$;
iii) there is no element of rng(h) below a_1;
iv) there is no element of rng(h) between the a's.

Obviously P is in Π^0_1 form and so its negation is in Σ^0_1 form, and so must be the range of a one-one function $f:\mathbb{N} \to \mathbb{N}$ (we are using P fails of arbitrarily large numbers). See [Si99], p. 71.

We claim that P holds of arbitrarily large numbers. To see this, note that for each m there exists b such that there are at least m values of g below b. Then we can apply finite Σ^0_1 comprehension below b to obtain a solution to P of length m. See [Si99], p. 71.

Now let g be as given. I.e., g is a one-one enumeration of solutions to P. Note that n is a value of h if and only if "n is a term of a value of g" if and only if "n is a term of every value of g of length > n." Therefore rng(h) exists. QED

LEMMA f. (RCA_0) The following implies ACA_0. For all countable abelian p-groups G,H, there is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.

Proof: It suffices to apply Lemma e by deriving the statement in Lemma e. Let $f:\mathbb{N} \to \mathbb{N}$, where there are arbitrarily large numbers that are not values of f.

Let T be the tree whose leaves are the $<n,...,3n-1>$, $n \geq 1$. Let $T' = T \cup \{<n,...,3n-1,f^{-1}(n)>: n$ is a value of $f\}$. Let G = G(T,p) and H = G(T',p) . Note that T obeys the hypotheses of Lemmas a-d, and T' obeys the hypothesis of Lemma c,d. Let K be as given.

By Lemma c, $Z(p^{2n})$ is a direct summand of H if n is not a value of f. Also for all n, $Z(p^{2n})$ is a direct summand of G. Hence there are arbitrarily large n such that $Z(p^n)$ is a common summand of G and H. Therefore there are arbitrarily large n such that $Z(p^n)$ is embeddable into K. Hence K has elements of arbitrarily large order.

By Lemma b, the sequence of nontrivial groups p^nK exist, with intersection {0}. And there are arbitrarily large n such that $(p^nK)[p] \neq (p^{n+1}K)[p]$. By Lemma d, for all $n \geq 0$, if $(p^nK)[p] \neq (p^{n+1}K)[p]$ then T has a leaf of length n.

Note that the n for which

*) $(p^{2n}K)[p] \neq (p^{2n+1}K)[p]$

is existentially defined. Therefore it is the range of values of a function (although we do not know that the range exists as a set). And *) implies T' has a leaf of length 2n, and so leaf of length 2n+1, and therefore n is not a value of f. By [Si99], p. 71, the solutions to *) are the values of a one-one function. We have verified the statement in Lemma e as required. QED

9. RESULTS AND OPEN QUESTIONS.

We consider the following statements about groups.

1. Either G is embeddable into H or H is embeddable into G .
2. Either G is embeddable into H or H is embeddable into G .
3. Let G+G be isomorphic to (embeddable into) G and H+H be isomorphic to (embeddable into) H. Then G is embeddable into H or H is embeddable into G.
4. There is a direct summand K of G and H such that every direct summand of G and H is embeddable into K.
5. There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of (embeddable into) J.
6. Let G+G be isomorphic to G and H+H be isomorphic to H. There is a direct summand J of G and H such that every direct summand of G and H is a direct summand of (embeddable into) J.
7. In every infinite sequence of groups, one group is embeddable in a later (different) group.
8. In every infinite decreasing (⊇) chain of groups, one group is embeddable in an earlier group.
9. There is a group M embeddable into G and H such that every group embeddable into G and H is embeddable into M.
10. If G+G and H+H are isomorphic then G and H are isomorphic.
11. If G is a direct summand of H and H is a direct summand of G then G and H are isomorphic.

The following two theorems follow from the results proved thus far. (Replacing 2 by any prime p requires a straightforward adaptation).

THEOREM 5.1. 1-11 are provable in ATR_0 under the hypotheses that the groups are countable reduced abelian p-groups with the same p. 4-6,9-11 are provable in ATR_0 under the hypotheses that the groups are countable reduced abelian torsion groups.

THEOREM 5.2. Any of 1-8 are provably equivalent to ATR_0 over ACA_0 under the hypotheses that the groups are countable reduced abelian 2-groups. This is true if 2 is replaced by any prime p. Any of 4-6 are provably equivalent to ATR_0 under the hypotheses that the groups are countable reduced abelian torsion groups.

OPEN QUESTIONS: We do not know how to reverse any of 9-11. We conjecture that they are equivalent to ATR_0 over ACA_0. We do not know if they are provable in ACA_0. Also, we leave open the question of improving the base theory to RCA_0 in Theorem 4.2.

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