DOES MATHEMATICS NEED NEW AXIOMS?

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The point of view of the set theory community is well represented here. I want to concentrate on the perspective of mathematicians outside set theory.

1. MATHEMATICIANS' VIEWPOINT.

New axioms are needed in order to settle various mathematically natural questions. Yet no well known mathematician outside set theory is even considering adopting any new axioms for mathematics, even though they are aware of at least the existence of the independence results.

The difference in perspective of set theorists versus mathematicians who are not set theorists, is enormous. Recall that mathematics goes back, say, 2,500 years - whereas set theory in the relevant sense dates back to the turn of the 20th century.

For 2,500 years, mathematici-ans have been concerned with matters of counting and geometry and physical notions.

These main themes gave rise to arithmetic, algebra, geometry, and analysis.

The interest in and value of mathematics is judged by mathematicians in terms of its relevance and impact on the main themes of mathematics.

It is generally recognized by most mathematicians that set theory is the most convenient vehicle for achieving rigor in mathematics.

For this purpose, there has evolved a more or less standard set theoretic interpretation of mathematics, with ZFC generally accepted as the current gold standard for rigor.

It is simply false that a number theorist is interested in and respects set theory just as they are interested in and respect group theory, topology, differential geometry, real and complex analysis, operators on Hilbert space, etcetera.

The reason for this attitude is quite fundamental and extremely important. A number theorist is of course interested in complex analysis because he uses it so much. But not so with operators on Hilbert space. Yet there is still a distant respect for this because of a web of substantive and varied interconnections that chain back to number theory. Set theory does not have comparable interconnections.

For the skeptical, the degree of extreme isolation can be subjected to various tests including citation references - broken down even into their nature and quality. Using a critical notion from statistics, set theory is an extreme outlier.

Nor is set theory regarded as intrinsically interesting to mathematicians, independent of its lack of impressive interconnections. Why?

For the mathematician, set theory is regarded as a convenient way to provide an interpretation of mathematics that supports rigor. A natural number is obviously not a set, an ordered pair is obviously not a set, a function is obviously not a set of ordered pairs, and a real number is obviously not a set of rationals.

For the mathematician, mathematics is emphatically not a branch of set theory. The clean interpretation of mathematics into set theory does not commit the mathematician to viewing problems in set theory as problems in mathematics.

The mathematician therefore evaluates set theory in terms of how well it serves its purpose - providing a clean, simple, coherent, workable way to formalize mathematics.

This point of view hardened as many mathematicians experimented for several decades with what has come to be known as set theoretic problems which turned out to be independent of ZFC.

There was a growing realization that the cause of these difficulties was excessive generality in the formulations of the problems which allowed for pathological cases which were radically different in character than normal mathematical examples. That if the problems were formulated in more concrete ways that still covered all known interesting cases, then the difficulties completely disappeared.

Furthermore, distinctions between these set theoretic problems causing difficulties and the most celebrated theorems and open problems in mathematics can be given FORMALLY. This is in terms of quantifier complexity and the closely related matter of absoluteness. Thus set theory comes out as an extreme outli-er which can be documented FORMALLY.

2. THE MALIGNED AXIOM OF CONSTRUCTIBILITY - MORE IS LESS AND LESS IS MORE.

The set theorist is looking for deep set theoretic phenomena, and so V = L is anathema since it restricts the set theoretic universe so drastically that all sorts of phenomena are demonstrably not present. Furthermore, for the set theorist, any advantage that V = L has in terms of power can be obtained with more powerful axioms of the same rough type that accomo-date measurable cardinals and the like - e.g., V = L(□), or the universe is an inner model of a large cardinal.

However, for the normal mathematician, since set theory is merely a vehicle for interpre-ting mathematics so as to establish rigor, and not mathematically interesting in its own right,

the less set theoretic difficulties and phenomena the better.

I.e., less is more and more is less. So if the mathematician were concerned with the set theoretic independence results - and they generally are not - then V = L is by far the most attractive solution for them.

This is because it appears to solve all set theoretic problems (except for those asserting the existence of sets of unrestricted cardinality), and is also demonstrably relatively consistent.

Set theorists also say that V = L has implausible consequences - e.g., there is a Sigma-1-2 well ordering of the reals, or there are nonmeasurable PCA sets.

The set theorists claim to have a direct intuition which allows them to view these as so implausible that this provides “evidence” against V = L.

However, mathematicians dis-claim such direct intuition about complicated sets of reals. Many say they have no direct intuition about all multivariate functions from N into N!

3. QUESTION ANSWERED BY CLASSICAL DESCRIPTIVE SET THEORY?

The classical descriptive set theory coming from large cardinals is most often cited by set theorists as the reason why mathematics needs large cardinal axioms. I have several objections to this claim.

a. Part of the argument is that large cardinals are needed to establish these results. But large cardinals are not needed to establish an alternative series of such results. E.g., V = L provides another, entirely different, set of answers to these questions. The set theorists answer saying V = L gives the wrong answers and large cardinals give the right answers, citing their direct intuition about projective sets of reals. I am very dubious about this direct intuition. I don't have it, and mathematicians in general disclaim it.

b. Another part of the argument is that, in light of a,

set theory needs large cardinals

and therefore

mathematics needs large cardinals.

But this inference depends on a reading of our question that makes this tautological.

Reading the question this way simply avoids the really interesting questions, replacing them by much less interesting questions. For instance, it avoids questions of how and under what circumstances the general mathematical community or individual mathematicians will adopt new axioms, should adopt new axioms, and if so, how this will be manifested.

Here is the closest I can come to the set theorists' point of view on our question.

There is an interesting notion of "general set theory in its maximal conceivable form" and that V = L has no basis in this context. However, the notion is at present virtually completely unexplained, and no work that I have seen provides any serious insight into what this really means. We simply do not know how to explicate any relevant notion of maximality.

I agree that

“general set theory in its maximal conceivable form” needs large cardinals axioms

is very likely to be true. But I can’t conclude even that

set theory needs large cardinal axioms

let alone

mathematics needs large cardinal axioms.

4. GENERAL PREDICTIONS.

The picture is going to change radically with the new Boolean relation theory and related developments, joining the issue of new axioms and the relevance of large cardinals in a totally new and unexpect-edly convincing way.

Because of the thematic nature of these developments, and the interaction with nearly all areas of mathematics, large cardinal axioms will begin to be accepted as new axioms for mathematics - with controversy. Use of them will still be noted, at least in passing, for quite some time, before full acceptance.

5. CIRCUMSTANCES SURROUNDING ACTUAL ADOPTION OF NEW AXIOMS.

The cirumstances that I envis-ion are a coherent body of consequences of large cardinals of a new kind.

a. They should be entirely mathematically natural. This standard is very high for a logician trying to uncover such consequences, yet is routinely met in mathematics (set theory included) by professionals at all levels of achievment.

b. They should be concrete. At least within infinitary discrete mathematics. Most ideally, involving polynomials with integer coefficients, or even finite functions on finite sets of integers.

c. They should be thematic. If they are isolated, they will surely be stamped as curiosities, and the math community will find a way to attack them through an ad hoc raising of the standards for being entirely natural. However, if they are truly thematic, then the theme itself must be attacked, which may be difficult to do. For instance, the same theme may already be inherent in well known basic, familiar, and useful facts.

d. They should have points of contact with a great variety of mathematics.

e. They should be open ended. I.e., the pain will never end until the adoption of large cardinals.

f. They should be elementary. E.g., at the level of early undergraduate or gifted high school. That way, even scientists and engineers can relate to it, so it is harder for the math community to simply bury them.

g. Their derivations should be accessible, with identifiable general techniques. This way, the math community can readily immerse itself in hands on crystal clear uses of large cardinals that beg to be removed - but cannot.

We have omitted an additional curcumstance:

h. They should be used in nor-mal mathematics as pursued before such thematic results.

For some mathematicians, h will be required before they consider the issue really joined. I already know that for some well known core mathematicians, h is definitely not required - that the issue is already sufficiently joined for them by Boolean relation theory.

Implicit in criteria a-g is that the body of examples and the theme launch a new area, with an eventual AMS classification number. This new area will be accepted as part of the general unremovable furniture of contemporary mathematics whose intrinsic interest is comparable to other established areas in mathematics. In this way, the issue of large cardinal axioms will be joined for a critical number of important core mathematicians.

6. LARGE CARDINALS OR THEIR 1-CONSISTENCY?

The statements coming out of Boolean relation theory are provably equivalent to the 1-consistency of large cardin-als. So instead of adopting the large cardinal axioms themselves, one can instead adopt their 1-consistency.

When put into proper perspective, this is more of a criti-cism of form than over substance. Adopting large cardin-als amounts to asserting

"every consequence of large cardinals is true."

Adopting the 1-consistency of large cardinals amounts to asserting

"every □-0-2 consequence of large cardinals is true."

The obviously more natural choice is to accept large cardinals, since the latter is syntactic and not an attrac-tive axiom candidate.

However, for the purposes of proving □-0-2 sentences, these two choices are essentially equivalent.

Another consideration is more practical. When the working mathematician wants to develop Boolean relation theory, the proofs are incomparably more direct and mathematically elegant when done under the assumption of the large cardinal axioms themselves than under the 1-consistency.

When I publish that "□ needs large cardinals to prove" I explicitly formalized this as "any reasonable formal system that proves □ must interpret large cardinals in the sense of Tarski." This gives a precise sense to "needs."

There is an interesting point of some relevance here. State-ments in Boolean relation theory are also consequences of the existence of a real valued measurable cardinal - a related kind of large cardinal axiom.

Let me put it somewhat differently. There is a substantial and coherent list of non syn-tactic axiom candidates, in-cluding large cardinal axioms and other axioms. In this list, only certain axiom candidates settle questions in Boolean relation theory. The most appropriate ones from various points of view are in fact the small large cardinal axioms.

That is the obvious move to make from the point of view of a working scientist. If they later prove to be inconsistent, then we can undergo theory revision. The key advance is that the issue of new axioms finally promises to get joined in a serious way for the mathematics community.

APPENDIX

Two open questions in set theory.

The following are relevant to the panel discussion.

a. Prove that large cardinals provides a complete theory of the projective hierarchy.

Here a major challenge is to come up with an appropriate definition of "complete."

b. Prove that there are no "simple" axioms that settle the continuum hypothesis.

Here I mean "simple" in the same sense that the axioms of ZFC are "simple." For example, very short in primitive notation.