ORDER INVARIANT RELATIONS AND INCOMPLETENESS

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EXTENDED ABSTRACT

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Abstract. For all order invariant subsets of $Q[-n,n]^{2k}$, the sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part (trivial). Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal square whose sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part (nontrivial). We prove the latter in ZFC augmented with a standard large cardinal hypothesis, and show that ZFC does not suffice (assuming ZFC is consistent). We also establish this for a number of variants, including an explicitly finite form. We refer to these developments as "Π^0_1 mathematical incompleteness".

1. ORDER INVARIANCE AND SECTIONS

DEFINITION 1.1. Q is the set of all rationals. N is the set of all nonnegative integers. We use k,n,m,r,s,t,i,j for positive integers unless otherwise indicated. $Q[-n,n] = Q \cap [-n,n]$. For $A \subseteq Q^k$, $A^< = \{x \in A: x_1 < ... < x_k\}$, with $A^\leq$, $A^>$, $A^\geq$ defined analogously. $Q^{\leq k} = \cup_{i\leq k} Q^i$. $x,y \in Q^{\leq k}$ are order equivalent if and only if $lth(x) = lth(y)$, and for all $1 \leq i,j \leq lth(x)$, $x_i < x_j \leftrightarrow y_i < y_j$.

DEFINITION 1.2. $A \subseteq Q^k$ ($A \subseteq Q[-n,n]^k$) is order invariant if and only if for all order equivalent $x,y \in Q^k$ ($x,y \in Q[-n,n]^k$), $x \in A \rightarrow y \in A$. For $x,y \in Q^k$, $x*y$ is the concatenation of x,y, which lies in Q^{2k}. For $x \in Q^k$, max(x), min(x) are the largest, smallest coordinate of x, respectively.

Note that for fixed k,n, there are only finitely many order invariant subsets of Q^k and $Q[-n,n]^k$.

DEFINITION 1.3. Let A,B ⊆ Q^k, and p ∈ Q. A below p is {x ∈ A: max(x) < p}. A,B agree below p if and only if A below p equals B below p. The negative part of A is A below 0. The section of A at x ∈ Q^r is {y ∈ Q^{k-r}: x*y ∈ A}. The section of A at x ∉ $Q^{<k}$ is ∅, as we are not using tuples of length 0.

EXAMPLES. (1,-1,7,1/2) and (9/2,3,9,7/2) are order equivalent. {x ∈ $Q[-5,5]^4$: $x_1 \leq x_2 < x_3 < x_4 \vee x_3 \leq x_1 < x_2$} is an order invariant subset of $Q[-5,5]^4$.

We present the following straightforward result for expositional reasons. We use the exponent 2k merely for direct comparison with Proposition 2.2.

THEROEM 1.1. For all order invariant subsets of $Q[-n,n]^{2k}$, the sections at the x ∈ $\{0,...,n\}^{r<}$ have the same negative part.

Note how Theorems 1.1 and 1.2 compare with the Proposition 2.2. Actually, Theorems 1.1 has the following much sharper form.

THEOREM 1.2. For all order invariant subsets of $Q[-n,n]^{2k}$, the sections at order equivalent x,y ∈ $Q[-n,n]^{\leq 2k}$ agree below min(x*y).

The condition on sections in Theorem 1.2 is of course far stronger than the condition on sections in Proposition 2.2.

Of course Theorems 1.1, 1.2 hold for Q^{2k}, but we use $Q[-n,n]^{2k}$ for easy comparison with Proposition 2.2.

2. MAXIMAL SQUARES

DEFINITION 2.1. Let R ⊆ Q^{2k}. A square in R is a set S^2 ⊆ R. A maximal square in R is a square in R which is not a proper subset of any square in R.

THEOREM 2.1. Every R ⊆ Q^{2k} has a maximal square.

Proof: By Zorn's Lemma, or explicitly by using an enumeration $p_1,p_2,...$ of Q^k, as follows. We go through the enumeration, one by one, and delete or keep them according to whether the set A of kept p_i's has $A^2 \subseteq R$. After infinitely many (greedy) steps, we form an $S \subseteq Q^k$ such that S^2 is a maximal square in R. For if S^2 is not maximal, then we must have left out a p_i that we really didn't leave out. QED

We now combine Theorem 2.1 with (the idea of) Theorem 1.1.

PROPOSITION 2.2. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal square whose sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part.

Proposition 2.2 is provable in ZFC extended by a certain large cardinal hypothesis, but no in ZFC (assuming ZFC is consistent). We do not know if Proposition 2.2 for Q^{2k} is provable in ZFC.

We now present several alternative formulations, and state our results in full detail.

3. SQUARES, ROOTS, GRAPHS, CLIQUES

DEFINITION 3.1. Let $R \subseteq V^2$. A square in R is a set $S^2 \subseteq R$. A maximal square in R is a square in R which is not a proper subset of any square in R. A root of R is a set S such that $S^2 \subseteq R$. S is a maximal root of R if and only if S is a root of R which is not a proper subset of any root of R.

DEFINITION 3.2. A graph on V is a pair G = (V,E), where $E \subseteq V^2$ is irreflexive and symmetric. S is a clique in G if and only if $S \subseteq V$, and for all distinct $x,y \in S$, x E y. A maximal clique in G is a clique in G which is not a proper subset of any clique in G.

THEOREM 3.1. Every $R \subseteq V^2$ has a maximal square. Every $R \subseteq V^2$ has a maximal root. Every graph on V has a maximal clique.

Proof: If V is countable, then these are proved the same way we proved Theorem 2.1. For general V, use Zorn's Lemma, or a transfinite argument along a well ordering of V. The statements are provably equivalent to the axiom of choice over ZF. QED

4. FORMALITIES

DEFINITION 4.1. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP abbreviates "stationary Ramsey property".

DEFINITION 4.2. SRP is the formal system ZFC + $\{(\exists\lambda)(\lambda$ is k-SRP$)\}_k$. SRP^+ is ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-SRP$)$. SRP[k] is ZFC + $(\exists\lambda)(\lambda$ is k-SRP$)$.

DEFINITION 4.3. RCA_0 and WKL_0 are the first two of our five main systems of reverse mathematics. See [WIKIa]. EFA is exponential (elementary) function arithmetic. I originally introduced the system as "exponential function arithmetic". See [WIKIb].

DEFINITION 4.4. A Π^0_1 sentence is a sentence asserting that some given Turing machine never halts at the empty input tape. A Π^0_2 sentence is a sentence asserting that some given Turing machine halts at every input tape.

5. INFINITE INCOMPLETENESS

For the reader's convenience, Proposition 2.2 reappears here as Proposition 5.1.

PROPOSITION 5.1. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal square whose sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part.

PROPOSITION 5.2. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal root whose sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part.

PROPOSITION 5.3. Every order invariant graph on $Q[-n,n]^k$ has a maximal clique whose sections at the $x \in \{0,...,n\}^{r<}$ have the same negative part.

Here are our sharpest statements of this kind.

PROPOSITION 5.4. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal square whose sections at order equivalent $x,y \in \{0,...,n\}^{\leq 2k}$ agree below $\min(x*y)$.

PROPOSITION 5.5. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal root whose sections at order equivalent $x,y \in \{0,...,n\}^{\leq k}$ agree below $\min(x*y)$.

PROPOSITION 5.6. Every order invariant graph on $Q[-n,n]^k$ has a maximal clique whose sections at order equivalent $x,y \in \{0,...,n\}^{\leq k}$ agree below $\min(x*y)$.

Propositions 5.1 - 5.6 are clearly infinitary as stated since the maximal object must (normally) be infinite. However, it is a nice student exercise to put Propositions 5.1 - 5.6 into Π^0_1 form via Gödel's completeness theorem. Specifically, for each such R or G, there is a sentence in first order predicate calculus with equality whose countable models correspond to the required maximal object. For this reason, we will take the liberty of referring to Propositions 5.1 - 5.6 as Π^0_1 mathematical incompleteness.

THEROEM 5.7. Propositions 5.1 - 5.6 are provably equivalent over RCA_0, and provably equivalent to the consistency of SRP over WKL_0. It follows that Propositions 5.1 - 5.6 are
i. provable in SRP^+ but not in SRP (assuming SRP is consistent).
ii. unprovable in ZFC (assuming ZFC is consistent).
iii. neither provable nor refutable in SRP (assuming SRP is 1-consistent).
iv. neither provable nor refutable in ZFC (assuming SRP is 1-consistent).

THEROEM 5.8. For each fixed k (for each fixed n) Propositions 5.1 - 5.6 are provable in SRP. For each m there exists k,n,R such that Propositions 5.1, 5.2, 5.4, 5.5 are not provable in SRP[m] (assuming SRP is consistent). For each m there exists k,n,G such that Propositions 5.3, 5.6 are not provable in SRP[m] (assuming SRP is consistent). These results hold even if we require that the maximal clique be recursive in 0' (i.e., Δ^0_2) in the sense of recursion theory.

We target the following two statements for independence from ZFC:

PROPOSITION 5.9. Every order invariant graph on $Q[-4,4]^8$ has a maximal clique whose sections at $x \in \{0,1,2,3,4\}^{4<}$ have the same negative part.

PROPOSITION 5.10. Every order invariant graph on $Q[-4,4]^8$ has a maximal clique whose sections at order equivalent $x,y \in \{0,1,2,3,4\}^{\leq 4}$ agree below min(x*y).

We may have trouble showing these are independent of ZFC, but are confident that we can get away with single digit numbers for both.

Although we can also prove the analogous statements for Q^k in SRP, we do not know if they are provable in ZFC. E.g.,

PROPOSITION 5.11. Every order invariant graph on Q^k has a maximal clique whose sections at $x \in N^{r<}$ have the same negative part.

PROPOSITION 5.12. Every order invariant graph on Q^k has a maximal clique whose sections at order equivalent $x,y \in N^{\leq k}$ agree below min(x*y).

All of the Propositions in this section can be put in explicitly Π^0_1 via Gödel's completeness theorem.

6. INFINITE INCOMPLETENESS - BLOCKS

DEFINITION 6.1. A block of positive integers is a tuple i,i+1,...,j.

PROPOSITION 6.1. Every order invariant subset of $Q[-n,n]^{2k}$ has a maximal square whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

PROPOSITION 6.2. Every order invariant graph on $Q[-n,n]^k$ has a maximal root whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

PROPOSITION 6.3. Every order invariant graph on $Q[-n,n]^k$ has a maximal clique whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

THEOREM 6.4. Theorems 5.7 and 5.8 hold for Propositions 6.1 - 6.3.

We have not been able to prove the independence of block statements just using negative parts. A basic statement of this kind for graphs is

PROPOSITION 6.5. Every order invariant graph on $Q[-n,n]^k$ has a maximal clique whose sections at (1,...,n-1),(2,...,n) have the same negative part.

The strongest one of this kind that we have looked at is

PROPOSITION 6.6. Every order invariant graph on $Q[-n,n]^k$ has a maximal clique whose sections at equal length blocks in $\{0,...,n\}^{\leq k}$ have the same negative part.

All of the Propositions in this section can be put in explicitly Π^0_1 via Gödel's completeness theorem.

7. INFINITE INCOMPLETENESS - STEP MAXIMALITY

Step maximality is a natural condition to place on maximal cliques. It allows us to derive independence results with Q^k replacing $Q[0,n]^s$. It also allows us to use blocks as in Proposition 6.1, whose strength is unknown.

DEFINITION 7.1. For $A \subseteq Q^k$, $A|\leq p = A \cap (-\infty,p]^k$. For graphs $G = (Q^k,E)$ and $p \in Q$, $G|\leq p$ is the graph $(Q^k|\leq p, E|\leq p)$. We define $A|<p$ and $G|<p$ analogously.

DEFINITION 7.3. S is a step maximal square, root, clique in $R \subseteq Q^{2k}$, $R \subseteq Q^{2k}$, G on Q^k, if and only if for all $n \in N$, $S|\leq n$ is a maximal square, root, clique in $R|\leq n$, $R|\leq n$, $G|\leq n$, respectively.

It is easy to verify that every step maximal square, root, clique is a maximal square, root, clique, respectively. However, the converses fail.

THEOREM 7.1. Every subset of Q^k has a step maximal square and a step maximal root. Every graph on Q^k has a step maximal clique.

Proof: We consider the graph case, the others being handled in the same way. First obtain a maximal clique in $R|\leq 1$. Then extend it to a maximal clique in $R|\leq 2$. Continue in this way for all $R|\leq n$. Take the union. QED

PROPOSITION 7.2. Every order invariant subset of Q^{2k} has a step maximal square whose sections at (0,...,r),(1,...,r+1) have the same negative part.

PROPOSITION 7.3. Every order invariant subset of Q^{2k} has a step maximal root whose sections at (0,...,r),(1,...,r+1) have the same negative part.

PROPOSITION 7.4. Every order invariant graph on Q^k has a step maximal clique whose sections at (0,...,r),(1,...,r+1) have the same negative part.

Here are the strongest versions that we consider.

PROPOSITIOIN 7.5. Every order invariant subset of Q^{2k} has a step maximal square whose sections at order equivalent x,y $\in N^{\leq 2k}$ agree below min(x*y).

PROPOSITION 7.6. Every order invariant subset of Q^{2k} has a step maximal root whose sections at order equivalent x,y $\in$ $N^{\leq k}$ agree below min(x*y).

PROPOSITION 7.7. Every order invariant graph on Q^k has a step maximal clique whose sections at order equivalent x,y $\in N^{\leq k}$ agree below min(x*y).

THEOREM 7.8. Theorems 5.7 and 5.8 hold for Propositions 7.2 - 7.7.

Again we target single digit k,r, with k = 4 for Propositions 7.5, 7.6, and 7.7.

All of the Propositions in this section can be put in explicitly Π^0_1 via Gödel's completeness theorem.

8. INFINITE INCOMPLETENESS - A#, $R_<$[A]

We now give an alternative Π^0_1 incompleteness which has some advantages. It leads to our state of the art finite incompleteness in section 8.

DEFINITION 8.1. Let $A \subseteq Q^k$ and $R \subseteq Q^{2k}$. A# is the least $E^k \supseteq A \cup N^k$. $R_<[A] = \{y: (\exists x \in A)(x*y \in R \wedge \max(x) < \max(y))\}$.

DEIFNITION 8.2. The upper shift of $x \in Q^k$, ush(x), is the result of adding 1 to all nonnegative coordinates of x. For $A \subseteq Q^k$, ush(A) = {ush(x): x ∈ A}.

THEOREM 8.1. Every $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A]$.

PROPOSITION 8.2. Every order invariant $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A] \supseteq$ ush(A).

We give the following variant in the form of a fixed point statement like Theorem 8.1.

PROPOSITION 8.3. Every order invariant $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A] \cup$ ush(A).

THEROEM 8.4. Theorems 5.7 and 5.8 hold for Propositions 8.2, 8.3.

Propositions 8.2 and 8.3 can be put in explicitly Π^0_1 form using Gödel's completeness theorem. This is a little more involved than it is for the Propositions in sections 5-7 because of the use of +1 in ush.

9. INFINITE INCOMPLETENESS - HUGE CARDINALS

DEFINITION 9.1. Let $A,B \subseteq Q^k$. The positive shift of A is $\{x+1: x \in A \wedge \min(x) > 0\}$. A 1-contains B if and only if $A \supseteq B$, and every section of B at any $x \in Q^{k-1}$ is a section of A at some $y \in Q^{k-1}$.

PROPOSITION 9.1. Every order invariant $R \subseteq Q^{2k}$, $k \geq 3$, has an $A \subseteq Q^k$ that 1-contains the upper shift of $A^{\geq} = A\#^{\geq}\backslash R_<[A]$.

PROPOSITION 9.2. Every order invariant $R \subseteq Q^{2k}$, $k \geq 3$, has an $A \subseteq Q^k$ that 1-contains the positive shift of $A^{\geq} = A\#^{\geq}\backslash R_<[A]$.

THEOREM 9.3. Propositions 9.1 and 9.2 are provably equivalent to Con(HUGE) over WKL_0. This remains true even if we require that A be recursive in 0′ (i.e., Δ^0_2) in the sense of recursion theory. On the other hand, if we use the positive shift instead of the upper shift in Propositions 8.2 and 8.3, then the resulting statements are provable in ZFC.

Propositions 9.1 and 9.2 can be put into explicitly Π^0_1 form via Gödel's completeness theorem.

10. FINITE INCOMPLETENESS - SRP

DEFINITION 10.1. $F:Q^m \to Q^r$ is order theoretic if and only if graph(F) $\subseteq Q^{m+r}$ is order theoretic; i.e., a Boolean combination of inequalities in variables and constants. $R \subseteq Q^{2k}$ is a reduction if and only if for all $x,y \in Q^k$, $x R x$, and if $x R y$ then $\max(x) \geq \max(y)$. A sequence is R independent if and only if all terms lie in Q^k, and no two distinct terms are related by R.

THEOREM 10.1. Let $F:Q^{2k} \to Q^k$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,\ldots,x_n$ with each $F(x_i*x_{i+1}) R x_{i+2}$, $i \leq n-2$.

PROPOSITION 10.2. Let $F:Q^{2k} \to Q^k$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,\ldots,x_n,\text{ush}(x_n)$ with each $F(x_i*x_{i+1}) R x_{i+2}$, $i \leq n-2$.

PROPOSITION 10.3. Let $F:Q^{2k} \to Q^k$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,\ldots,x_n,\text{ush}(x_1),\ldots,\text{ush}(x_n)$ with each $F(x_i*x_{i+1}) R x_{i+2}$, $i \leq n-2$.

These propositions are written in explicitly Π^0_2 form, but we can apply the well known decision procedure for (Q,<) to put it in explicitly Π^0_1 form,. Or we can directly put an a priori bound on the numerators and denominators needed for the x's and y's in terms of k,n, and the numerators and denominators used in the parameters of F.

THEOREM 10.4. Propositions 10.2 and 10.3 are provably equivalent to Con(SRP) over EFA.

We can also look for such an infinitely long sequence, and the resulting infinite forms would be provably equivalent to Con(SRP) over WKL_0.

We are targeting k = 4 and arbitrary n. Also n = 4 and arbitrary k.

We do not know if Propositions 9.2 and 9.3 are provable if we replace 2k by k.

11. FINITE INCOMPLETENESS - HUGE

We use three very simple piecewise linear functions. In fact, they are what we call pure (Q,<,0,+1) Boolean. I.e., defined using <,0,+1,¬,∧,∨, and variables over Q. Under this terminology, pure (Q,<) Boolean is the same as order invariant.

DEFINITION 11.1. We define $F,G,H:Q^k \rightarrow Q^k$ as follows. F(x) = x+2 if min(x) ≥ 0; x otherwise. $G(x) = (x_1+2,\ldots,x_{k-1}+2,2k+1)$ if $0 \leq x_1,\ldots,x_{k-1} < 2k-1$; x otherwise. $H(x) = (x_1-2,\ldots,x_{k-1}-2,2k+1)$ if $x_k = 2k+1$; x otherwise.

PROPOSITION 11.1. Let $T:Q^{\leq nk} \rightarrow Q^{k\geq}$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,F(x_1),G(x_1),H(x_1),\ldots,x_n,F(x_n),G(x_n),H(x_n)$, with each $T(0,x_1,F(x_1),G(x_1),H(x_1),\ldots,x_{i-1},F(x_{i-1}),G(x_{i-1}),H(x_{i-1}))$ R x_i. Here we have left out *'s for readability.

This proposition is written in explicitly Π^0_2 form, but we can apply the well known decision procedure for (Q,<) to put it in explicitly Π^0_1 form,. Or we can directly put an a priori bound on the numerators and denominators needed for the x's and y's in terms of k,n.

THEOREM 11.2. Proposition 11.1 is provably equivalent to Con(HUGE) over EFA.

12. TEMPLATES

TEMPLATE A. Let φ be a purely universal sentence in a 2k-ary predicate symbol with < and constants from the domain Q[0,1]. Every order invariant subset of $Q[0,1]^{2k}$ has a maximal square satisfying φ.

TEMPLATE B. Let φ be a purely universal sentence in a k-ary predicate symbol with < and constants from the domain Q[0,1]. Every order invariant subset of $Q[0,1]^{2k}$ has a maximal root satisfying φ.

TEMPLATE C. Let φ be a purely universal sentence in a k-ary predicate symbol with < and constants from the domain Q[0,1]. Every order invariant graph on $Q[0,1]^k$ has a maximal clique satisfying φ.

Note that all of the Propositions in sections 5 and 6 are instances of one of these three Templates. Specifically, Q[-n,n] with 0,...,n corresponds to Q[0,1] with 1/n+1,...,1/1.

TEMPLATE D. Let φ be a purely universal sentence in a 2k-ary predicate symbol with < and constants from the domain Q. Every order invariant subset of Q^{2k} has a maximal square satisfying φ.

TEMPLATE E. Let φ be a purely universal sentence in a k-ary predicate symbol with < and constants from the domain Q. Every order invariant subset of Q^{2k} has a maximal root satisfying φ.

TEMPLATE F. Let φ be a purely universal sentence in a k-ary predicate symbol with < and constants from the domain Q. Every order invariant graph on Q^k has a maximal clique satisfying φ.

Note that all of the Propositions in section 7 are instances of one of these three Templates.

CONJECTURE. All instances of these 6 Templates are provable or refutable in SRP. In fact, all are provable in SRP or refutable in RCA_0. We know that these two conjectures fail if we replace SRP by any SRP[m] (assuming SRP is 1-consistent).

TEMPLATE G. Fix a piecewise linear $T:Q^k \rightarrow Q^k$. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,...,x_n,T(x_n)$ with each $F(x_i*x_{i+1})$ R x_{i+2}, $i \leq n-2$.

TEMPLATE H. Fix a piecewise linear $T:Q^k \rightarrow Q^k$. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,...,x_n,T(x_1),...,T(x_n)$ with each $F(x_i*x_{i+1})$ R x_{i+2}, $i \leq n-2$.

CONJECTURE. All instances of these 6 Templates are provable or refutable in SRP. In fact, all are provable in SRP or refutable in EFA. We know that these two conjectures fail if we replace SRP by any SRP[m] (assuming SRP is 1-consistent).

TEMPLATE I. Fix pure (Q,<,0,+1) Boolean $F:Q^k \to Q^{rk}$. Let $T:Q^{\leq nk} \to Q^{k\geq}$ be order theoretic. Every order invariant reduction $R \subseteq Q^{2k}$ has an independent $x_1,F(x_1),...,x_n,F_1(x_n)$, with each $T(0,x_1,F_1(x_1),... x_{i-1},F(x_{i-1})) R x_i$. Here we have left out *'s for readability.

CONJECTURE. All instances of Template I are provable or refutable in HUGE. In fact, all are provable in HUGE or refutable in EFA. We know that these two conjectures fail if we replace HUGE by any HUGE[m] (assuming HUGE is 1-consistent).

All instances of Templates A-I can be put into explicitly Π^0_1 form via Gödel's completeness theorem. All instances of Templates G-I are in explicitly Π^0_2 form. They can be put in explicitly Π^0_1 form using the well known decision procedure for (Q,<,+).

13. PROOFS

The provability of the Propositions in sections 5-7 from Con(SRP) is done almost exactly as in section 9 of [Fr14] (and earlier in section 4 of [Fr11]). The provability of Con(SRP) from the various Propositions builds on what was essentially done in section 5 of [Fr11].

13. APPENDIX - FORMAL SYSTEMS USED

EFA Exponential function arithmetic. Based on exponentiation and bounded induction. Same as $I\Sigma_0(\exp)$, [HP93], p. 37, 405.
RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].
WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].
ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si99,09].
ZF(C) Zermelo set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Je14].
SRP[k] ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), for fixed k. Section 9.1, [Fr14].
SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. Section 9.1, [Fr14].

SRP$^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). Section 9.1, [Fr14].

HUGE[k] ZFC + $(\exists\lambda)(\lambda$ is k-HUGE), for fixed k.

HUGE ZFC + $(\exists\lambda)(\lambda$ is k-huge), as a scheme in k.

HUGE$^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-huge).

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^{(k)}(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings). For more about huge cardinals, see [Ka94], p. 331.

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