ADJACENT RAMSEY THEORY

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Abstract. We introduce Adjacent Ramsey Theory, which investigates solutions to the shift equation $f(x_1,...,x_k) = f(x_2,....,x_{k+1})$ and the shift inequality $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1})$, in the context of functions $f:N^k \to N^r$. Existence theorems are proved, and shown to have strong metamathematical properties such as unprovability within Peano Arithmetic. Adjacent Ramsey Theorems represent a new level of naturalness and simplicity for independence results at the level of Peano Arithmetic.

1. $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

TO BE COMPLETED.

We use N for the set of all nonnegative integers, and [t] for $\{0,...,t-1\}$, $t \in N$.

THEOREM 1.1. For all $f:N^k \to [2]$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

THEOREM 1.2. For all $f:N^k \to [2]$, there exist $x_1 < ... < x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

THEOREM 1.3. For all $f:N^k \to [3]$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

THEOREM 1.4. For all $f:N^k \to [t]$, there exist $x_1 < ... < x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$.

We have proved that theorems 1.2, 1.3 correspond to roughly k fold iterated exponentiation. Theorem 1.1 however has a small upper bound.

After we obtained our results, we became aware of

Shift graphs and lower bounds on Ramsey numbers rk(l;r), by D. Duffus, H. Lefmann, and V. Rodl, Discrete Mathematics, volume 137, Issues 1-3 (January 1995), 177-187

which has considerable overlap with Theorems 1.1 - 1.4, but stated in very different language.

THEOREM 1.5. For all even $k \geq 1$ and $f:[k+1]^k \rightarrow [2]$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$. For all odd $k \geq 1$ and $f:[k+2]^k \rightarrow [2]$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$.

Proof: Fix $k \geq 1$, k even. Consider the sequence (0,...,k-1), (1,...,k),(2,...,k,0),(3,...,k,0,1),...,(k,0,...,k-1),(0,...,k). There are k+2 terms, which is an even number of terms. Hence the values of f at some adjacent pair must be equal.

Fix $k \geq 1$, k odd. Consider the sequence (0,...,k-1), (1,...,k),(2,...,k,k+1),(3,...,k,k+1,0),...,(k+1,0,...,k-2),(0,...,k-1). There are k+3 terms, which is an even number of terms. Hence the values of f at some adjacent pair must be equal. QED

THEOREM 1.6. Theorem 1.5 is best possible in the sense that k+1 cannot be replaced by k, and k+2 cannot be replaced by k+1.

Proof: The first claim is obvious. For the second claim, let $k \geq 1$ be odd. Define the automorphism $\alpha:[k+1]^k \rightarrow [k+1]^k$ given by

$$\alpha(x_1,\ldots,x_k) = (x_2,\ldots,x_k,y), \text{ where } \{x_1,\ldots,x_k,y\} = [k+1].$$

It is easy to see that all orbits of α have k+1 elements, which is even. So we can obviously define f on each orbit of α so that the value of f as successive terms in the orbit alternate between 0 and 1. QED

THEOREM 1.7. For all $f:[2k+1]^k \to [2]$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$.

Proof: By induction on $k \geq 1$. For the basis case, let $f:[3] \to [2]$. Then $f(0) = f(1) \vee f(0) = f(2) \vee f(1) = f(2)$.

Suppose the statement is true for fixed $k \geq 1$. Let $f:[2k+3]^{k+1} \to [2]$. Suppose there exists $1 \leq x_1 < \ldots < x_{k+2}$ such that $f(x_1,\ldots,x_{k+1}) \neq f(x_1,\ldots,x_k,x_{k+2})$. Then

$$f(0,x_1,\ldots,x_k) = f(x_1,\ldots,x_{k+1}) \vee f(0,x_1,\ldots,x_k) = f(x_1,\ldots,x_k,x_{k+2})$$

and we are done. So we assume that for all $1 \leq x_1 < \ldots < x_{k+2}$,

$$f(x_1,\ldots.,x_{k+1}) = f(x_1,\ldots,x_k,x_{k+2}).$$

Now define $g:\{1,\ldots,2k+1\}^k \to [2]$ by $g(x_1,\ldots,x_k) = f(x_1,\ldots,x_k,x_k+1)$. We can obviously view $\{1,\ldots,2k+1\}$ with $\{0,\ldots,2k\} = [2k+1]$. By the induction hypothesis, fix $1 \leq x_1 < \ldots < x_{k+1}$ such that $g(x_1,\ldots,x_k) = g(x_2,\ldots,x_{k+1})$. Then $f(x_1,\ldots,x_k,x_{k+1}) = f(x_2,\ldots,x_{k+1},x_{k+1}+1)$. QED

LEMMA 1.8. For all $k \geq 1$, there exists $f:[2k]^k \to [2]$ such that the following holds. For all $x_1 < \ldots < x_{k+1} < 2k$, $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$.

Proof: THERE SEEMS TO BE A PROBLEM WITH THIS. NEED TO LOOK FURTHER. I STILL BELIEVE THIS TO BE TRUE.

LEMMA 1.9. Let $k,r,d \geq 1$. Suppose there exists $f:N_k \to [t]$ such that for all $x_1 < \ldots < x_{k+1}$ if $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$ then $x_1 \geq d$. There exists $g:N^{k+1} \to [32t]$ such that for all $x_1 < \ldots < x_{k+2}$, if $g(x_1,\ldots,x_{k+1}) = g(x_2,\ldots,x_{k+2})$ then $x_1 \geq 2^d$.

Proof: Let k,r,d be as given. We now define $h_1,\ldots,h_7$.

$h1(x_1,\ldots,x_{k+1}) = 0$ if $x_1 = 0$; 1 otherwise.

$h_2(x_1,\ldots,x_{k+1})$ is the parity of the greatest i such that $\log(x_1) = \ldots = \log(x_i)$; 0 if $x_1 = 0$.

$h_3(x_1,\ldots,x_{k+1})$ is the parity of the greatest i such that $\log(x_1) < \ldots < \log(x_i)$; 0 if $x_1 = 0$.

Let $\alpha(x,y)$ is the first digit from the left that x,y differ at in base 2, assuming that they have the same leading digit.

$h_4(x_1,...,x_{k+1})$ is the parity of the greatest i such that $\alpha(x_1,x_2) = ... = \alpha(x_i,x_{i+1})$, assuming $\log(x_1) = ... = \log(x_{k+1})$; 0 otherwise.

$h_5(x_1,...,x_{k+1})$ is the parity of the greatest i such that $\alpha(x_1,x_2) < ... < \alpha(x_i,x_{i+1})$, assuming $\log(x_1) = ... = \log(x_{k+1})$; 0 otherwise.

$h_6(x_1,...,x_{k+1})$ is the parity of the greatest i such that $\alpha(x_1,x_2) > ... > \alpha(x_i,x_{i+1})$, assuming $\log(x_1) = ... = \log(x_{k+1})$; 0 otherwise.

$h_7(x_1,...,x_{k+1}) = f(\log(x_1),...,\log(x_k))$ if $\log(x_1) < ... < \log(x_k)$; $f(\alpha(x_1,x_2),...,\alpha(x_k,x_{k+1}))$ if $\alpha(x_1,x_2) < ... < \alpha(x_k,x_{k+1})$; $f(\alpha(x_k,x_{k+1}),...,\alpha(x_1,x_2))$ if $\alpha(x_1,x_2) > ... > \alpha(x_k,x_{k+1})$; 0 otherwise.

Note that $h_1,...,h_6$ are into [2], and h_7 is into [8]. We can combine these 7 functions into a single function g:Nk+1 → [96].

Let $x_1 < ... < x_{k+2}$ be such that $h_i(x_1,...,x_{k+1}) = h_i(x_2,...,x_{k+2})$, for all $1 \leq i \leq 8$. It suffices to show that $x_1 \geq 2^d$.

Clearly $h_1(x_1,...,x_{k+1}) = 1$. Hence $x_1 > 0$.

Suppose the greatest i such that $\log(x_1) = ... = \log(x_i)$ lies in [2,k]. Then i-1 is greatest such that $\log(x_2) = ... = \log(x_{i+1})$. Using h_2, this is impossible. Hence $i \in \{1,k+1]$.

Suppose the greatest j such that $\log(x_1) < ... < \log(x_j)$ lies in [2,k]. Then j-1 is greatest such that $\log(x_2) < ... < \log(x_{j+1})$. Using h_3, this is impossible. Hence $j \in \{1,k+1\}$.

Suppose the greatest p such that $\alpha(x_1,x_2) = ... = \alpha(x_p,x_{p+1})$ lies in [2,k-1]. Then p-1 is the greatest such that $\alpha(x_2,x_3) = ... = \alpha(x_{p+1},x_{p+2})$. Using h_4, this is impossible. Hence $p \in \{1,k\}$.

Suppose the greatest q such that $\alpha(x_1,x_2) < ... < \alpha(x_q,x_{q+1})$ lies in [2,k-1]. Then q-1 is the greatest such that $\alpha(x_2,x_3)$

$< \ldots < \alpha(x_{q+1},x_{q+2})$. Using h_5, this is impossible. Hence $q \in \{1,k\}$.

Suppose the greatest s such that $\alpha(x_1,x_2) > \ldots > \alpha(x_s,x_{s+1})$ lies in [2,k-1]. Then s-1 is the greatest such that $\alpha(x_2,x_3) > \ldots > \alpha(x_{s+1},x_{s+2})$. Using h6, this is impossible. Hence $s \in \{1,k\}$.

Since $\log(x_1) \leq \log(x_2)$, we see that

$\log(x_1) = \ldots = \log(x_{k+1})$ or
$\log(x_1) < \ldots < \log(x_{k+1})$.

case 1. $\log(x_1) = \ldots = \log(x_{k+1})$. Using h_4,h_5,h_6, $\alpha(x_1,x_2) = \ldots = \alpha(x_k,x_{k+1}) \vee \alpha(x_1,x_2) < \ldots < \alpha(x_k,x_{k+1}) \vee \alpha(x_1,x_2) = \ldots = \alpha(x_k,x_{k+1})$. The first alternative is impossible. Hence $\alpha(x_1,x_2) < \ldots < \alpha(x_k,x_{k+1}) \vee \alpha(x_1,x_2) > \ldots > \alpha(x_k,x_{k+1})$. We now use h_7. In the first case, $f(\alpha(x_1,x_2),\ldots,\alpha(x_k,x_{k+1})) = f(\alpha(x_2,x_3),\ldots,\alpha(x_{k+1},x_{k+2}))$. In the second case, $f(\alpha(x_k,x_{k+1}),\ldots,\alpha(x_1,x_2)) = f(\alpha(x_{k+1},x_{k+2}),\ldots,\alpha(x_2,x_3))$. By the choice of f, we have $\alpha(x_1,x_2) \geq d$ or $\alpha(x_k,x_{k+1}) \geq d$. Hence $\log(x_1) \geq d$, and so $x_1 \geq 2^d$.

case 2. $\log(x_1) < \ldots < \log(x_{k+1})$. Using h_7, $f(\log(x_1),\ldots,\log(x_{k+1})) = f(\log(x_2),\ldots,\log(x_{k+2}))$. By the choice of f, we have $\log(x_1) \geq d$. Hence $x_1 \geq 2^d$.

TO BE CONTINUED...

THEOREM 1.9 is enough to show that for each fixed $k \geq 1$, we have roughly k fold iterated exponentiation in the number colors; i.e., the range of the function.

But we are going to be proving additional lower bounds. In particular, for range [3].

2. $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

For $x,y \in N^k$, we write $x \leq y$ if and only if for all $1 \leq i \leq k$, $x_i \leq y_i$.

We start with three Adjacent Ramsey Theorems in weak form.

THEOREM A. For all $f:N^k \to N^2$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

THEOREM B. For all $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+3}$ such that $f(x_1,\ldots,x_k) \le f(x_2,\ldots,x_{k+1}) \le f(x_3,\ldots,x_{k+3})$.

THEOREM C. For all $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_2,\ldots,x_{k+1})-f(x_1,\ldots,x_k) \in 2N$.

We now present these three Adjacent Ramsey Theorems in strong form.

THEOREM D. For all $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_1,\ldots,x_k) \le f(x_2,\ldots,x_{k+1})$.

THEOREM E. For all $t \ge 1$ and $f:N^k \to N^r$, there exist x1 $< \ldots < x_{k+t-1}$ such that $f(x_1,\ldots,x_k) \le \ldots \le f(x_t,\ldots,x_{k+t-1})$.

THEOREM F. For all $t \ge 1$ and $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_2,\ldots,x_{k+1})-f(x_1,\ldots,x_k) \in tN^r$.

In the development, it will be useful to have these two hybrids.

THEOREM G. For all $f:N^k \to \{1,2\}$, $g:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$ and $g(x_1,\ldots,x_k) \le g(x_2,\ldots,x_{k+1})$.

THEOREM H. For all $f:N^k \to [t]$, $g:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$ and $g(x_1,\ldots,x_k) \le g(x_2,\ldots,x_{k+1})$.

We will show that Theorems A-H are provable in ACA but not in ACA_0. The actual result that we prove is considerably sharper.

THEOREM 2.1. Theorems A-H are true.

Proof: We first show that Theorem E implies the rest.

fill this in later.

We now prove Theorem E. We will use the following well known fact. Every infinite sequence from N^r has an infinite subsequence such that each term is $\le$ the next term.

Now let $f:N^k \to N^r$. By the Infinite Ramsey Theorem for k+1 tuples and two colors, we can find an infinite $A \subseteq N$ such that

i. For all $x_1 < ... < x_{k+1} \in A$, $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1})$; or
ii. For all $x_1 < ... < x_{k+1} \in A$, not $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1})$.

However, ii) is impossible by the well known fact cited above. Therefore we have i). QED

It is very convenient to use the following Lemma from proof Theory.

LEMMA 2.2. The following are provably equivalent in RCA_0.
i. $\in_0$ is well ordered.
ii. Every Π^1_1 sentence provable in ACA_0 is true.

Proof: Reference?? QED

THEOREM 2.3. Theorems A-H are provable in RCA_0 + $\in_0$ is well ordered.

Proof: The proof that Theorem E implies the rest goes through in RCA_0.

Now fix $k \geq 1$. Theorem E for k is a Π^1_1 sentence, and the proof we gave goes through in ACA_0. since the Infinite Ramsey Theorem for any fixed dimension is provable in ACA_0.

Also observe that the previous paragraph goes through in RCA_0. Now apply Lemma 2.2. QED

We now show that Theorems A-H are provably equivalent to "$\in_0$ is well ordered" in RCA_0.

The plan is to first derive $\in_0$ is well ordered from RCA_0 + Theorem D. We then prove Theorems A-H in RCA_0 + Theorem D by direct combinatorial arguments.

Define $\omega[0] = 1$, $\omega[k+1] = \omega^{\omega[k]}$.

We use base ω Cantor normal form for ordinals $< \in_0$. For each $k \geq 0$, define the set of expressions $W(k)$ and a strict linear ordering $<_k$ on $W(k)$, as follows.

$W(0)$ consists of 0 only. $<_0$ is empty. Suppose $W(k)$, $<_k$ have been defined, where $<_k$ is a strict linear ordering on $W(k)$. $W(k+1)$ consists of 0 and the expressions

$$c_1\omega^{x1} + \ldots + c_r\omega^{xr}$$

where r ≥ 1, c1,...,cr are positive integers, and $x_1 >_k \ldots >_k x_r$.

$<_{k+1}$ on W(k+1) is defined as follows. 0 is put at the bottom. Let x,y ∈ W(k+1) be nonzero and distinct. Write

$$x = c_1\omega^{x1} + \ldots + c_r\omega^{xr}$$
$$y = d_1\omega^{y1} + \ldots + d_s\omega^{ys}$$

case 1. There exists 1 ≤ i ≤ min(r,s) such that for all 1 ≤ j < i, $c_j\omega^{xi} = d_j\omega^{yi}$, and $c_i\omega^{xi} \neq d_i\omega^{yi}$. If $x_i <_k y_i$ then $x <_{k+1} y$. If $y_i <_k x_i$ then $y <_{k+1} x$. If $x_i = y_i$ and $c_i < c_j$ then $x <_{k+1} y$. If $x_i = y_i$ and $c_j < c_i$ then $y <_{k+1} x$.

case 2. Otherwise. Then x is a proper initial segment of y or y is a proper initial segment of x. In the former case, $x <_{k+1} y$. In the latter case, $y <_{k+1} x$.

We can easily prove by induction that W(k) ⊆ W(k+1) and $<_k \subseteq <_{k+1}$ and each $<_k$ is a well ordering that agrees with the standard ordinal interpretation of W(k). This is all well known and very standard.

Fix k ≥ 0. For x ∈ W(k) and i ≥ 0, we define x[i] as follows. For x = 0, take 0[i] = 0. For x > 0, take x[i] to be the i-th term of x if 1 ≤ i ≤ lth(x); 0 otherwise. Note that each nonzero x[i] has a coefficient and an exponent.

Let x,y ∈ W(k). Write CP(x,y) for the comparison position of x,y, which is the least i such that x[i] ≠ y[i] if x ≠ y; 0 otherwise.

Write CT(x,y) for the comparison term of x,y, which is x[CP(x,y)].

LEMMA 2.4. The following is provable in EFA. x ≠ y → CP(x,y) exists and is nonzero. 0 < i ≤ j ≤ lth(x) → $x[j] \leq_k x[i]$. $x \leq_k y \leftrightarrow x[CP(x,y)] \leq_k y[CP(x,y)]$.

Proof: Suppose x ≠ y. If x,y ≠ 0 then obviously CP(x,y) exists. Since x[0] = y[0], CP(x,y) > 0.

Suppose 0 < i ≤ j ≤ lth(x). Then since x is made of strictly decreasing terms, $x[j] \leq_k x[i]$.

Suppose $x \leq y$. If $x = y$ then obviously $x[CP(x,y)] \leq_k y[CP(x,y)]$. If $0 = x <_k y$ then $x[CP(x,y)] = 0 \leq_k y[CP(x,y)]$. If $0 <_k x <_k y$ then $CP(x,y) > 0$, and $x[CP(x,y)] <_k y[CP(x,y)]$.

Suppose $y < x$. Then $CP(x,y) \neq 0$. If $y = 0$ then $y[CP(x,y)] = 0 <_k x[CP(x,y)]$. If $y \neq 0$ then $y[CP(x,y)] <_k x[CP(x,y)]$. QED

LEMMA 2.5. The following is provable in EFA. Suppose $x,y,z \in W(k)$, $CP(x,y) \leq CP(y,z)$, $CT(x,y) \leq_k CT(y,z)$. Then $x \leq_k y$.

Proof: Let x,y,z be as given. By Lemma 1, if $CP(x,y) = 0$ then $x = y$. So we assume that $0 < CP(x,y) \leq CP(y,z)$. Using Lemma 2.s1,

$$x[CP(x,y)] = CT(x,y) \leq_k CT(y,z) = y[CP(y,z)] \leq_k y[CP(x,y)].$$

By Lemma 2.4, $x \leq_k y$. QED

Write $CC(x,y)$ for the comparison coefficient of x,y, which is the coefficient of $CT(x,y)$ if $CT(x,y) > 0$; 0 otherwise.

Write $CE(x,y)$ for the comparison exponent of x,y, which is the exponent of $CT(x,y)$ if $CT(x,y) > 0$; 0 otherwise.

LEMMA 2.6. The following is provable in EFA. Suppose $x,y,z \in W(k)$, $CP(x,y) \leq CP(y,z)$, $CC(x,y) \leq CC(y,z)$, $CE(x,y) \leq_k CE(y,z)$. Then $x \leq y$.

Proof: Let x,y,z be as given. By Lemma 2.5, it suffices to show that $CT(x,y) \leq_k CT(y,z)$.

Clearly, if $CT(x,y), CT(y,z) \neq 0$, then $CT(x,y) \leq_k CT(y,z)$.

Suppose $CT(y,z) = 0$. Then $CC(x,y) \leq CC(y,z) = 0$. Hence $CT(x,y) = 0$. QED

LEMMA 2.7. The following is provable in EFA. For all $k \geq 1$ there exists $f:W(k)^k \rightarrow N^{2k-1}$ such that the following holds. $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1}) \rightarrow x_1 \leq_k x_2$.

Proof: By induction on $k \geq 1$. We first let $k = 1$. Note that $W(1) = \{c\omega^0: c \in N\backslash\{0\}\} \cup \{0\}$. Set $f(x)$ to be the coefficient of x.

Now let $f:W(k)^k \rightarrow N^{2k-1}$ be as required. We define $g:W(k+1)^{k+1} \rightarrow N^{2k+1}$ as follows.

Let $x_1,\ldots,x_{k+1} \in W(k+1)$. Define $g(x_1,\ldots,x_{k+1}) =$

$$(CP(x_1,x_2),CC(x_1,x_2),f(CE(x_1,x_2),CE(x_2,x_3),\ldots,CE(x_k,x_{k+1}))).$$

Now let $x_1,\ldots,x_{k+2} \in W[k+1]$, $g(x_1,\ldots,x_{k+1}) \leq g(x_2,\ldots,x_{k+2})$. Clearly $g(x_2,\ldots,x_{k+2}) =$

$$(CP(x_2,x_3),CC(x_2,x_3),f(CE(x_2,x_3),CE(x_3,x_4),\ldots,CE(x_{k+1},x_{k+2})).$$

Hence

$$CP(x_1,x_2) \leq CP(x_2,x_3).$$
$$CC(x_1,x_2) \leq CC(x_2,x_3).$$
$$f(CE(x_1,x_2),CE(x_2,x_3),\ldots,CE(x_k,x_{k+1})) \leq f(CE(x_2,x_3),CE(x_3,x_4),\ldots,CE(x_{k+1},x_{k+2})).$$

By the induction hypothesis,

$$CE(x_1,x_2) \leq_k CE(x_2,x_3).$$

Hence by Lemma 2.6, $x_1 \leq_c x_2$. QED

LEMMA 2.8. The following is provable in RCA_0. Suppose Theorem D holds for $r = 2k-1$. Then $W(k)$ is well ordered by $<_k$.

Proof: Let k be as given. Let $g:N \to W(k)$. Let $f:W(k)^k \to N2^{k-1}$ be given by Lemma 2.7. It suffices to find $i < j$ such that $g(i) <_k g(j)$.

Define $h:N^k \to N^{2k-1}$ by

$$h(x_1,\ldots,x_k) = f(g(x_1),\ldots,g(x_k)).$$

By Theorem D for h, let $x_1 < \ldots < x_{k+1}$, where

$$h(x_1,\ldots,x_k) \leq h(x_2,\ldots,x_{k+1}).$$
$$f(g(x_1),\ldots,g(x_k)) \leq f(g(x_2),\ldots,g(x_{k+1})).$$

By the choice of f, $g(x_1) \leq g(x_2)$. Since g is arbitrary, $W(k)$ is well ordered by $<_k$. QED

Note that from Theorem 2.3 and Lemma 2.8, Proposition D is equivalent to "$\in_0$ is well ordered" in RCA_0. We now handle the rest of Theorems A-H.

LEMMA 2.9. RCA_0 + Theorem A proves Theorem H.

Proof: Let k,t,f,g be as given. Define $h:N^{k+1} \to N^2$ as follows. Let $x_1,\ldots,x_{k+1} \in N$.

case 1. $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1})$. Define $h(x_1,\ldots,x_{k+1}) = (f(x_1,\ldots,x_k),t+1-f(x_1,\ldots,x_k))$.

case 2. $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$. Define $h(x_1,\ldots,x_{k+1}) = (0,g(x_1,\ldots,x_k)+t+1)$.

Let $x_1,\ldots,x_{k+2}$ be distinct, $h(x_1,\ldots,x_{k+1}) \leq h(x_2,\ldots,x_{k+2})$.

case i. $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1}) \neq f(x_3,\ldots,x_{k+2})$. Then
$h(x_1,\ldots,x_{k+1}) = (f(x_1,\ldots,x_k),t+1-f(x_1,\ldots,x_k))$.
$h(x_2,\ldots,x_{k+2}) = (f(x_2,\ldots,x_{k+1}),t+1-f(x_2,\ldots,x_{k+1}))$.

$f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.
$t+1-f(x_1,\ldots,x_k) \leq t+1-f(x_2,\ldots,x_{k+1})$.
$f(x_2,\ldots,x_{k+1}) \leq f(x_1,\ldots,x_k)$.
$f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$.

This is a contradiction.

case ii. $f(x_1,\ldots,x_k) \neq f(x_2,\ldots,x_{k+1}) = f(x_3,\ldots,x_{k+2})$. Then
$h(x_1,\ldots,x_{k+1}) = (f(x_1,\ldots,x_k),t+1-f(x_1,\ldots,x_k))$,
$h(x_2,\ldots,x_{k+2}) = (0,g(x_2,\ldots,x_{k+1})+t+1)$.

This is a contradiction.

case iii. $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1}) = f(x_3,\ldots,x_{k+2})$. Then
$h(x_1,\ldots,x_{k+1}) = (0,g(x_1,\ldots,x_k)+t+1)$.
$h(x_2,\ldots,x_{k+2}) = (0,g(x_2,\ldots,x_{k+1})+t+1)$.

$g(x_1,\ldots,x_k)+t+1 \leq g(x_2,\ldots,x_{k+1})+t+1$.
$g(x_1,\ldots,x_k) \leq g(x_2,\ldots,x_{k+1})$.

case iv. $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1}) \neq f(x_3,\ldots,x_{k+2})$. Then
$h(x_1,\ldots,x_{k+1}) = (0,g(x_1,\ldots,x_k)+t+1)$.
$h(x_2,\ldots,x_{k+2}) = (f(x_2,\ldots,x_{k+1}),t+1-f(x_2,\ldots,x_{k+1}))$.
$g(x_1,\ldots,x_k)+t+1 \leq t+1-f(x_2,\ldots,x_{k+1})$.

This is a contradiction. QED

LEMMA 2.10. RCA_0 + Theorem H proves Theorem D. RCA_0 + Theorem G proves Theorem A.

Proof: Assume Theorem H. We first derive Theorem D with "distinct x1,...,xk+1".

Let k,r,f be as given in Theorem D. Define $g:N^k \to [1,r]$ and $h:N^k \to N$ as follows.

case 1. $(\exists i)(f(x_1,...,x_k)_i > f(x_2,...,x_{k+1})_i)$. Let i be least. Define $g(x_1,...,x_k) = i$, $h(x_1,...,x_k) = f(x_1,...,x_k)_i$.

case 2. Otherwise. $H(x_1,...,x_k) = 0$.

Let $x_1,...,x_{k+1}$ be distinct, $g(x_1,...,x_k) = g(x_2,...,x_{k+1})$, $h(x_1,...,x_k) \leq h(x_2,...,x_{k+1})$. We can assume case 1 applies to $x_1,...,x_{k+1}$, and to $x_2,...,x_{k+2}$. Let $g(x_1,...,x_k) = g(x_2,...,x_{k+1}) = i$. Then $f(x_1,...,x_k)_i \leq f(x_2,...,x_{k+1})_i$. This is a contradiction.

The above derivation can be repeated using Theorem G, obtaining Theorem A. This establishes the second claim.

We now obtain Proposition D with x1 < ... < xk+1 by raising the codimension as follows. FILL THIS IN. QED

LEMMA 2.11. RCA_0 + Theorem B proves Theorem A.

Proof: Let $k \geq 1$, $f:N^k \to N^2$. Define $g:N^{k+3} \to N$ as follows. Let $x_1,...,x_{k+1}$. Let $\alpha(x_1,...,x_{k+1})$ be the least $i \in \{1,2\}$ such that $f(x_1,...,x_k)_i > f(x_2,...,x_{k+1})_i$. We can assume that i always exists, if $x_1,...,x_{k+1}$ are distinct.

Let $x_1,...,x_{k+3} \in N$. Look at the triple of numbers, $\alpha(x_1,...,x_{k+1})$, $\alpha(x_2,...,x_{k+2})$, $\alpha(x_3,...,x_{k+3})$. We define $g(x_1,...,x_{k+3})$ by cases according to this triple.

case 1. 11v. Define $g(x1,...,x_{k+3}) = f(x_1,...,x_k)1 + 3$.

case 2. 22v. Define $g(x_1,...,x_{k+3}) = f(x_1,...,x_k)2 + 3$.

case 3. 122. Define $g(x_1,...,x_{k+3}) = 0$.

case 4. 211. Define $g(x_1,...,x_{k+3})) = 0$.

case 5. 121. Define $g(x_1,...,x_{k+3}) = 1$.

case 6. 212. Define $g(x_1,...,x_{k+3}) = 2$.

Let $x_1,\ldots,x_{k+5}$ be distinct, where $g(x_1,\ldots,x_{k+3}) \leq g(x_2,\ldots,x_{k+4}) \leq g(x_3,\ldots,x_{k+5})$. We argue according to the sequence $\alpha(x_1,\ldots,x_{k+1})$, $\alpha(x_2,\ldots,x_{k+2})$, $\alpha(x_3,\ldots,x_{k+3})$, $\alpha(x_4,\ldots,x_{k+4})$, $\alpha(x_5,\ldots,x_{k+5})$.

case i. 11abc. If a = 1 then $g(x_1,\ldots,x_{k+3}) = f(x_1,\ldots,x_k)_1 + 3$, $g(x_2,\ldots,x_{k+4}) = f(x_2,\ldots,x_{k+1})_1 + 3$. Hence $f(x_1,\ldots,x_k)_1 \leq f(x_2,\ldots,x_{k+1})_1$, contradicting that $\alpha(x_1,\ldots,x_{k+1}) = 1$. If a = 2 then $g(x_1,\ldots,x_{k+3}) > g(x_2,\ldots,x_{k+4})$, which is a contradiction.

case ii. 121ab. If a = 1 then $g(x_1,\ldots,x_{k+3}) = 1$, $g(x_2,\ldots,x_{k+4}) = 0$. If a = 2 then $g(x_2,\ldots,x_{k+4}) = 3$, $g(x_3,\ldots,x_{k+5}) = \leq 1$. Contradiction.

case iii. 122ab. If a = 2 then $g(x_2,\ldots,x_{k+4}) = f(x_2,\ldots,x_{k+1})_2 + 3$, $g(x_3,\ldots,x_{k+5}) = f(x_3,\ldots,x_{k+2})_2 + 3$. Hence $f(x_2,\ldots,x_{k+1})_2 \leq f(x_3,\ldots,x_{k+2})_2$, contradicting $\alpha(x_2,\ldots,x_{k+2}) = 2$. If a = 1 then $g(x_2,\ldots,x_{k+4}) > g(x_3,\ldots,x_{k+5})$, which is a contradiction.

case iv. 211ab. If a = 1 then $g(x_2,\ldots,x_{k+4}) = f(x_2,\ldots,x_{k+1})_1 + 3$, $g(x_3,\ldots,x_{k+5}) = f(x_3,\ldots,x_{k+2})_1 + 3$. Hence $f(x_2,\ldots,x_{k+1})_1 \leq f(x_3,\ldots,x_{k+2})_1$, contradicting $\alpha(x_2,\ldots,x_{k+2}) = 1$. If a = 2 then $g(x_2,\ldots,x_{k+4}) > g(x_3,\ldots,x_{k+5})$, which is a contradiction.

case v. 212ab. $g(x_1,\ldots,x_{k+3}) = 2$, $g(x_2,\ldots,x_{k+4}) \leq 1$. Contradiction.

case vi. 221ab. $g(x_1,\ldots,x_{k+3}) = f(x_1,\ldots,x_k)_1 + 3$, $g(x_2,\ldots,x_{k+4}) \leq 2$. Contradiction.

case vii. 222ab. $g(x_1,\ldots,x_{k+3}) = f(x_1,\ldots,x_k)_2 + 3$, $g(x_2,\ldots,x_{k+4}) = f(x_2,\ldots,x_{k+1})_2 + 3$. Hence $f(x_1,\ldots,x_k)_2 \leq f(x_2,\ldots,x_{k+1})_2$, contradicting $\alpha(x_1,\ldots,x_{k+1}) = 2$.

QED

LEMMA 2.12. RCA_0 + Theorem C implies Theorem G.

Proof: Use f for the parity and g for the values. I.e., $h(x_1,\ldots,x_k) = 2g(x_1,\ldots,x_k)+f(x_1,\ldots,x_k)-1$. QED

LEMMA 2.13. Over RCA_0, F → C → G → A → H → D → ϵ_0 is well ordered → A,B,C,D,E,F,G,H. B → A. E → D.

Proof: From above-- fill in more details.

THEOREM 2.14. Each of Propositions A-H is provably equivalent to "ϵ_0 is well ordered" in RCA_0.

Proof: Cleaning up from Lemma 2.13 - fill in.

3. f COMPUTABLE.

TO BE FILLED IN. Reasonable categories are recursive, primitive recursive, elementary recursive, polynomial time computable, and linear time log space computable. The latter two need to be defined carefully using base 2 representations.

THEOREM A (rec,primrec,elem,poly,lin/log). For all (rec,pimrec,elem,poly,lin/log) $f:N^k \to N^2$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

THEOREM B (rec,primrec,elem,poly,lin/log). For all (rec,primrec,elem,poly,lin/log) $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+3}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1}) \leq f(x_3,\ldots,x_{k+3})$.

THEOREM C (rec,primrec,elem,poly,lin/log). For all (rec,primrec,elem,poly,lin/log) $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_2,\ldots,x_{k+1})-f(x_1,\ldots,x_k) \in 2N$.

THEOREM D (rec,primrec,elem,poly,lin/log). For all (rec,primrec,elem,poly,lin/log) $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

THEOREM E (rec,primrec,elem,poly,lin/log). For all $t \geq 1$ and (rec,primrec,elem,poly,lin/log) $f:N^k \to N^r$, there exist $x1 < \ldots < x_{k+t-1}$ such that $f(x_1,\ldots,x_k) \leq \ldots \leq f(x_t,\ldots,x_{k+t-1})$.

THEOREM F (rec,primrec,elem,poly,lin/log). For all $t \geq 1$ and (rec,primrec,elem,poly,lin/log) $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_2,\ldots,x_{k+1})-f(x_1,\ldots,x_k) \in tN^r$.

THEOREM G (rec,primrec,elem,poly,lin/log). For all (rec,primrec,elem,poly,lin/log) $f:N^k \to \{1,2\}$, $g:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$ and $g(x_1,\ldots,x_k) \leq g(x_2,\ldots,x_{k+1})$.

THEOREM H (rec,primrec,elem,poly,lin/log). For all (rec,primrec,elem,poly,lin/log) $f:N^k \to [t]$, $g:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) = f(x_2,\ldots,x_{k+1})$ and $g(x_1,\ldots,x_k) \leq g(x_2,\ldots,x_{k+1})$.

Show that the last four get provable equivalence with 1-Con(PA).

Done by showing intermediate equivalence of 1-Con(PA) with ∈0 has no

recursive
prim rec
elementary
poly time
lin/log

infinite descending sequence. Done by adapting the previous section FILL IN DETAILS HERE.

Recursive is different, and using it gets equivalence with 2-Con(PA). This is what "no recursive infinite descending sequence through ∈0" is equivalent to.

It suffices to deal with Propositions D and E, as the rest imply D, and are implied by E, in rather direct ways.

4. f LIMITED.

TO BE FILLED IN.

We say that $f:N^k \to N^r$ is limited if and only if for all $x \in N^k$, $\max(f(x)) \leq \max(x)$.

THEOREM A (limited). For all limited $f:N^k \to N^2$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

THEOREM B (limited). For all limited $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+3}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1}) \leq f(x_3,\ldots,x_{k+3})$.

THEOREM C (limited). For all limited $f:N^k \to N$, there exist distinct $x_1,\ldots,x_{k+1}$ such that $f(x_2,\ldots,x_{k+1})-f(x_1,\ldots,x_k) \in 2N$.

THEOREM D (limited). For all limited $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

THEOREM E (limited). For all $t \geq 1$ and limited $f:N^k \rightarrow N^r$, there exist x1 < ... < x_{k+t-1} such that $f(x_1,...,x_k) \leq ... \leq f(x_t,...,x_{k+t-1})$.

THEOREM F (limited). For all $t \geq 1$ and limited $f:N^k \rightarrow N^r$, there exist $x_1 < ... < x_{k+1}$ such that $f(x_2,...,x_{k+1})-f(x_1,...,x_k) \in tN^r$.

THEOREM G (limited). For all limited $f:N^k \rightarrow \{1,2\}$, $g:N^k \rightarrow N$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$ and $g(x_1,...,x_k) \leq g(x_2,...,x_{k+1})$.

THEOREM H (limited). For all limited $f:N^k \rightarrow [t]$, $g:N^k \rightarrow N$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$ and $g(x_1,...,x_k) \leq g(x_2,...,x_{k+1})$.

These are all provably equivalent to 1-Con(PA) in RCA_0. The idea for reversal is derive "combinatorial well foundedness of ∈0" as in my paper Internal Tree Embeddings, in the Feferfest volume. FILL IN DETAILS HERE.

5. f FINITE.

TO BE FILLED IN.

THEOREM A (finite). For all k there exists p such that the following holds. For all limited $f:[p]^k \rightarrow N^2$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1})$.

THEOREM B (finite). For all k there exists p such that the following holds. For all limited $f:[p]^k \rightarrow N$, there exist distinct $x_1,...,x_{k+3}$ such that $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1}) \leq f(x_3,...,x_{k+3})$.

THEOREM C (finite). For all k there exists p such that the following holds. For all limited $f:[p]^k \rightarrow N$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_2,...,x_{k+1})-f(x_1,...,x_k) \in 2N$.

We now present these three Adjacent Ramsey Theorems in strong form.

THEOREM D (finite). For all k there exists p such that the following holds. For all limited $f:[p]^k \rightarrow N^r$, there exist $x_1 < ... < x_{k+1}$ such that $f(x_1,...,x_k) \leq f(x_2,...,x_{k+1})$.

THEOREM E (finite). For all k,t there exists p such that the following holds. For all limited $f:[p]^k \to N^r$, there exist x1 < ... < x_{k+t-1} such that $f(x_1,...,x_k) \leq ... \leq f(x_t,...,x_{k+t-1})$.

THEOREM F (finite). For all k,t there exists p such that the following holds. For all limited $f:[p]^k \to N^r$, there exist $x_1 < ... < x_{k+1}$ such that $f(x_2,...,x_{k+1})-f(x_1,...,x_k) \in tN^r$.

In the development, it will be useful to have these two hybrids.

THEOREM G (finite). For all k there exists p such that the following holds. For all limited $f:[p]^k \to \{1,2\}$, $g:[p]^k \to N$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$ and $g(x_1,...,x_k) \leq g(x_2,...,x_{k+1})$.

THEOREM H (finite). For all k,t there exists p such that the following holds. For all limited $f:[p]^k \to [t]$, $g:[p]^k \to N$, there exist distinct $x_1,...,x_{k+1}$ such that $f(x_1,...,x_k) = f(x_2,...,x_{k+1})$ and $g(x_1,...,x_k) \leq g(x_2,...,x_{k+1})$.

All of these are provably equivalent to 1-Con(PA) in EFA. These immediately imply the (limited) forms, so they imply 1-Con(PA). The other way around passes through Π02 reflection. FILL IN DETAILS HERE.

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