# $\Pi^0_1$ INCOMPLETENESS: finite set equations

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'Beautiful' is a word used by mathematicians with a semi rigorous meaning.

We give an 'arguably beautiful' explicitly $\Pi^0_1$ sentence independent of ZFC. See Proposition A from section 1.

## 1. $\Pi^0_1$ INDEPENDENT STATEMENT.

We use $[1,n]$ for the discrete interval $\{1,...,n\}$.

Let $A \subseteq [1,n]^k$. We write $A' = [1,n]^k \backslash A$. This treats $[1,n]^k$ as the ambient space.

Let $R \subseteq [1,n]^{3k} \times [1,n]^k$. We define

$$R<A> = \{y \in [1,n]^k: (\exists x \in A^3)(R(x,y))\}.$$

We say that R is strictly dominating if and only if for all $x,y \in [1,n]^k$, if $R(x,y)$ then $\max(x) < \max(y)$.

We start with a basic finite fixed point theorem.

THEOREM 1.1. For all $k,n \geq 1$ and strictly dominating $R \subseteq [1,n]^{3k} \times [1,n]^k$, there exists $A \subseteq [1,n]^k$ such that $R<A'> = A$. Furthermore, $A \subseteq [1,n]^k$ is unique.

We can obviously take complements, obtaining what we call the 'complementation theorem' for $R<A>$.

THEOREM 1.2. For all $k,n \geq 1$ and strictly dominating $R \subseteq [1,n]^{3k} \times [1,n]^k$, there exists $A \subseteq [1,n]^k$ such that $R<A> = A'$. Furthermore, $A \subseteq [1,n]^k$ is unique.

Here is a trivial modification of Theorem 1.1 with the same proof. We call this the 'complementation theorem' for $R<A\backslash\{t\}^k>$.

THEOREM 1.3. For all $k,n,t \geq 1$ and strictly dominating $R \subseteq [1,n]^{3k} \times [1,n]^k$, there exists $A \subseteq [1,n]^k$ such that $R<A\backslash\{t\}^k> = A'$. Furthermore, $A \subseteq [1,n]^k$ is unique.

We now incorporate the free set concept.

Let $R \subseteq [1,n]^{3k} \times [1,n]^k$. We say that A is R free if and only if A and $R<A>$ are disjoint.

The reader will recall the following familiar 'maximal free set' theorem.

THEOREM 1.4. For all $k,n \geq 1$, every $R \subseteq [1,n]^{3k} \times [1,n]^k$ has a maximal free set.

Note that the equation in Theorems 1.2, $R<A> = A'$, asserts that A is a very strong kind of free set.

THEOREM 1.5. For all $k,n \geq 1$, every strictly dominating $R \subseteq [1,n]^{3k} \times [1,n]^k$ has a free set A such that $R<A> = A'$. Furthermore, A is unique.

Note that we could equally well write $R<A> \supseteq A'$ instead of $R<A> = A'$, since we are also asserting A is R free. So

A is R free.
$R<A> = A'$.

work together, with the latter strictly implying the former.

What about

A is R free.
$R<A\backslash\{t\}^k> = A'$?

There are no implications between these two conditions, and they don't work together, as we now see.

THEOREM 1.6. The following is false. For all $k,n,t \geq 1$, every strictly dominating $R \subseteq [1,n]^{3k} \times [1,n]^k$ has a free set A such that $R<A\backslash\{t\}^k> = A'$.

We can sharpen Theorem 1.6 as follows.

THEOREM 1.7. Let $k,t \geq 1$. The following is false. Every strictly dominating $R \subseteq [1,t+1]^{3k} \times [1,n]^k$ has a free set A such that $R<A\backslash\{t\}^k> = A'$.

We will now consider various weakenings of the equation $R[A\backslash\{t\}^k] = A'$.

Our development depends heavily on a very strong regularity condition on R.

Two vectors $x,y$ □ $[1,n]^p$ have the same order type if and only if their coordinates have the same relative order. I.e., for all $1$ □ $i,j$ □ $p$,

$$x_i < x_j \;\square\; y_i < y_j.$$

We say that R □ $[1,n]^p$ is order invariant if and only if for all $x,y$ in $[1,n]^p$ of the same order type, $R(x)$ □ $R(y)$. Note that the number of such R is bounded by an exponential expression in p that does not depend on n.

We say that R □ $[1,n]^{3k}$ □ $[1,n]^k$ is order invariant if and only if R is order invariant as a subset of $[1,n]^{4k}$.

The imposition of order invariance is still not sufficient:

THEOREM 1.8. The following is false. For all $k,n,t \geq 1$, every strictly dominating R □ $[1,n]^{3k}$ □ $[1,n]^k$ has a free set A such that $R<A\backslash\{t\}^k> = A'$.

THEOREM 1.9. Let $k,t \geq 1$, the following is false. Every strictly dominating order invariant R □ $[1,t+1]^{3k}$ □ $[1,n]^k$ has a free set A such that $R<A\backslash\{t\}^k> = A'$.

However, we can weaken the equation $R<A\backslash\{t\}^k> = A'$ by asserting that a ‘squashed’ form of $R<A\backslash\{t\}^k>$ is the same as a ‘squashed’ from of A’, in the following sense.

We use $\log_t$ for the usual logarithm to the base $t \geq 2$, and $\text{flog}_t$ for the floor of $\log_t$. We extend $\text{flog}_t$ coordinatewise to elements of $[1,n]^k$. For A □ $[1,n]^k$, we define

$$\text{flog}_t\ A = \text{flog}_t[A].$$

Thus $\text{flog}_t$ ‘squashes’ A □ $[1,n]^k$ to $\text{flog}_t[A]$ □ $[0,\text{flog}_t(n)]^k$.

THEOREM 1.10. The following is false. For all $k,n,t \geq 1$, every strictly dominating order invariant R □ $[1,n]^{3k}$ □ $[1,n]^k$ has a free set A such that $\text{flog}_{t+1}\ R<A\backslash\{t\}^k> = \text{flog}_{t+1}\ A'$.

Counterexamples naturally arise to the statement in Theorem 1.10 with t fairly small relative to, say, $k^k$. This is in contrast to the situation with Theorems 1.6 and 1.8.

What happens if t is quite large relative to k? Below, we use (8k)! as a convenient number larger than k - we have not tried to be anywhere near optimal (we will in due course).

THEOREM 1.11. For all $k,n \geq 1$ and $t \geq (8k)!$, every strictly dominating order invariant $R \subseteq [1,n]^{3k} \times [1,n]^k$ has a free set A such that $\mathrm{flog}_{t+1}\ R<A\backslash\{t\}^k> = \mathrm{flog}_{t+1}\ A'$.

In fact, Theorem 1.11 can be proved in EFA.

We can follow the same development starting with the equations

$$R<R<A>> = R<A'>$$
$$R<R<[A\backslash\{t\}^k]>> = R<A'>$$
$$\mathrm{flog}_{t+1}\ R<R<A\backslash\{t\}^k>> = \mathrm{flog}_{t+1}\ R<A'>$$

instead of

$$R<A> = A'$$
$$R<A\backslash\{t\}^k> = A'$$
$$\mathrm{flog}_{t+1}\ R<A\backslash\{t\}^k> = \mathrm{flog}_{t+1}\ A'.$$

Note that each of the first two equations follow from the last three equations, respectively. However, there is no implication between the two third equations.

In this modified development, Theorems 1.1 - 1.10 remain unchanged, except that we have to drop uniqueness in Theorems 1.1, 1.2, 1.3, 1.5.

We now come to Theorem 1.11. Under this modified development, we arrive at

PROPOSITION A. For all $k,n \geq 1$ and $t \geq (8k)!$, every strictly dominating order invariant $R \subseteq [1,n]^{3k} \times [1,n]^k$ has a free set A such that $\mathrm{flog}_{t+1}\ R<R<A\backslash\{t\}^k>> = \mathrm{flog}_{t+1}\ R<A'>$.

Every statement that we have considered in this section is clearly explicitly $\Pi^0_1$.

Proposition A is independent of ZFC.

Here is much more precise information.

Let MAH = ZFC + {there exists a strongly n-Mahlo cardinal}$_n$.

Let MAH+ = ZFC + ‘for all n there exists a strongly n-Mahlo cardinal’.

THEOREM 1.12. MAH+ proves Proposition A. However, Proposition A is not provable in any consistent fragment of MAH that derives Z = Zermelo set theory. In particular, Proposition A is not provable in ZFC, provided ZFC is consistent. These facts are provable in $RCA_0$.

THEOREM 1.13. EFA + Con(MAH) proves Proposition A.

THEOREM 1.14. It is provable in ACA that Proposition A is equivalent to Con(MAH).

In $A\backslash\{t\}^k$, we are eliminating the single diagonal vector (t,...,t) from A. We can strengthen the statements by using A\t, which if defined to be A with all vectors removed in which t appears as a coordinate. Theorems 1.12 - 1.14 will still hold.

## 2. CONTROLLING PROPOSITION A.

We wish to control the strength of Proposition A by weakening it in simple ways. This seems to be a large subject, with much to work out, and we merely scratch the surface of it here.

PROPOSITION B. For all t ≥ (8k)!, every strictly dominating order invariant $R \subseteq [1,t^k]^{3k} \times [1,t^k]$ has a free set A such that $\mathrm{flog}_{t+1} R<R<A\backslash\{t\}^k>> = \mathrm{flog}_{t+1} R<A'>$.

PROPOSITION C. For all t ≥ (8k)!, every strictly dominating order invariant $R \subseteq [1,t!!]^{3k} \times [1,t!!]^k$ has a free set A such that $\mathrm{flog}_{t+1} R<R<A\backslash\{t\}^k>> = \mathrm{flog}_{t+1} R<A'>$.

PROPOSITION D. For all k,r ≥ 1 and t ≥ (8k)!, every strictly dominating order invariant $R \subseteq [1,t^r]^{3k} \times [1,t^r]^k$ has a free set A such that $\mathrm{flog}_{t+1} R<R<A\backslash\{t\}^k>> = \mathrm{flog}_{t+1} R<A'>$.

THEOREM 2.1. Proposition B is provable in ACA but not in PA. It is provable in EFA that Proposition B is equivalent to Con(PA).

THEOREM 2.2. Proposition C is provable in Z but not in WZC. It is provable in PRA that Proposition C is equivalent to Con(WZC).

Here Z = Zermelo set theory, and WZC = Zermelo set theory with only bounded separation, with the axiom of choice. The equivalence of Con(WZC) and the consistency of Russell's type theory with infinity is provable in PRA.

THEOREM 2.3. Proposition D, for any fixed $r \geq 1$, is provable in PA. The set of all instances of Proposition D, for fixed $r \geq 1$, is provably equivalent, over PRA, to {Con($PA_r$: $r \geq 1$}. For fixed $r \geq 1$, Proposition D corresponds to roughly $PA_r$; i.e., r quantifier induction.