

PRIMITIVE INDEPENDENCE RESULTS

by

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Abstract. We present some new set and class theoretic independence results from ZFC and NBGC that are particularly simple and close to the primitives of membership and equality (see sections 4,5). They are shown to be equivalent to familiar small large cardinal hypotheses. We modify these independent statements in order to give an example of a sentence in set theory with 5 quantifiers which is independent of ZFC (see section 6). It is known that all 3 quantifier sentences are decided in a weak fragment of ZF without power set (see [Fr02a]).

1. SUBTLE CARDINALS.

Subtle cardinals were first defined in a 1971 unpublished paper of Ronald Jensen and Ken Kunen. The subtle cardinal hierarchy was first presented in [Ba75]. The main results of [Ba75] were reworked in [Fr01]. [Fr01] also presents a number of new properties of ordinals (and linear orderings) not mentioning closed unbounded sets, which correspond to the subtle cardinal hierarchy.

The new properties from [Fr01] are not quite in the right form to be applied directly to this context. We need to use some new properties - particularly a property called weakly inclusion subtle.

We follow the usual set theoretic convention of taking ordinals to be epsilon connected transitive sets.

The following definition is used in [Ba75] and [Fr01]. We say that an ordinal α is subtle if and only if

- i) α is a limit ordinal;
- ii) Let $C \subseteq \alpha$ be closed unbounded, and for each $\beta < \alpha$ let $A_\beta \subseteq \beta$ be given. There exists $\gamma, \delta \in C$, $\delta < \gamma$, such that $A_\delta = A_\gamma \cap \delta$.

We need the following new definition for present purposes. We say that an ordinal α is inclusion subtle if and only if

- i) α is a limit ordinal;
- ii) Let $C \subseteq \alpha$ be closed unbounded, and for each $\beta < \alpha$ let $A_\beta \subseteq \alpha$ be given. There exists $\beta, \gamma \in C$, $\beta < \gamma$, such that $A_\beta \subseteq A_\gamma$.

LEMMA 1.1. Every inclusion subtle ordinal is an uncountable cardinal.

Proof: By setting $A_0 = \emptyset$ and $A_{n+1} = \{n\}$, we see that α is not inclusion subtle. Let α be inclusion subtle and not a cardinal. Let $h: \alpha \rightarrow \alpha$ be one-one, where $\beta < \alpha$ is the cardinal of β . Then $\alpha \geq \beta$. Define $A_\beta = \{h(\beta)\}$ for $\beta < \alpha < \alpha$, $A_\beta = \emptyset$ otherwise. Let $C = (\alpha, \alpha)$. This is a counterexample to the inclusion subtlety of α . QED

In light of Lemma 1.1, we drop the terminology "subtle ordinal" in favor of "subtle cardinal".

THEOREM 1.2. A cardinal is subtle if and only if it is inclusion subtle.

Proof: Let α be inclusion subtle. Let $C \subseteq \alpha$ be closed unbounded, and $A_\beta \subseteq \alpha$, $\beta < \alpha$, be given. Since α is an uncountable cardinal, we can assume that every element of C is a limit ordinal. For $\beta \in C$, define $B_\beta = \{2^\beta: \beta \in A_\beta\} \cup \{2^{\beta+1} < \alpha: \beta \in A_\beta\}$. Let $\beta, \gamma \in C$, $\beta < \gamma$, $B_\beta \subseteq B_\gamma$. Then $A_\beta = A_\gamma$. Here 2^β is β copies of 2. QED

For our purposes, we are particularly interested in the following somewhat technical notion.

We say that α is weakly inclusion subtle over β if and only if

- i) β, α are ordinals;
- ii) For each $\beta < \alpha$ let $A_\beta \subseteq \alpha$ be given. There exists $\beta \in \beta < \alpha$ such that $A_\beta \subseteq A_\alpha$.

LEMMA 1.3. If α is weakly inclusion subtle ordinal over $\beta \geq 2$ then $|\alpha| > |\beta|, \beta$.

Proof: Suppose $|\alpha| \leq |\beta|$, and let $f: \alpha \rightarrow \beta$ be one-one. For $\beta < \alpha$, define $A_\beta = \emptyset$. For $\beta \in [\beta, \alpha)$ define $A_\beta = \{h(\beta)\}$. It remains to show that α is uncountable.

Let $g: \alpha \rightarrow \beta$ be one-one. For $\gamma < \alpha$, define $A_\gamma = \emptyset$. For $\gamma \in [\alpha, \beta)$, define $A_\gamma = \{\gamma-1\}$. For $\gamma \in [\beta, \infty)$, define $A_\gamma = \{0, g(\gamma)\}$. QED

LEMMA 1.4. Suppose α is not weakly inclusion subtle over $\alpha \geq 2$. There is a counterexample to α is weakly inclusion subtle over α such that for all $\beta \geq \alpha$, $A_\beta \cap \beta \neq \emptyset$.

Proof: Let $A_\alpha, \alpha < \beta$, be a counterexample to α is weakly inclusion subtle over α . First assume $\alpha \in [2, \beta)$. For $\gamma < \alpha$, define $B_\gamma = \emptyset$, and $B_{\alpha+\gamma} = \{\alpha+1\}$. For $\gamma \in [\alpha, \beta)$, define $B_{\alpha+\gamma} = \{\alpha+\gamma: \alpha \in A_\gamma\} \cup \{0\}$. We are using $\alpha+\alpha = \alpha$ from Lemma 1.3.

Now assume $\alpha \in [\beta, \infty)$. For $\gamma < \alpha$, define $B_\gamma = \emptyset$. For $\gamma < \beta$, define $B_{\alpha+\gamma} = \{\alpha+1\}$. For $\gamma \in [\beta, \alpha)$, define $B_{\alpha+\gamma} = \{0, \gamma\}$. For $\gamma \in [\alpha, \infty)$, define $B_{\alpha+\gamma} = \{\alpha+\gamma: \alpha \in A_\gamma\} \cup \{0\}$. We are again using $\alpha+\alpha = \alpha$ from Lemma 1.3. QED

LEMMA 1.5. The least weakly inclusion subtle ordinal over $\alpha \geq 2$, if it exists, is a cardinal $> |\alpha|$.

Proof: Let α be the least weakly inclusion subtle ordinal over α . By Lemma 1.3, $|\alpha| > |\alpha|, \alpha$, and so it suffices to prove that α is a cardinal. Assume $|\alpha| < \alpha$. Obviously $\alpha < |\alpha|$. Let $h: \alpha \rightarrow |\alpha| \setminus \alpha$, be one-one. Define $B_\alpha = \{h(\alpha)\}$, $\beta \in [|\alpha|, \alpha)$. It remains to define $B_\beta, \beta < |\alpha|$.

By Lemma 1.4, let $B_\beta, \beta < |\alpha|$, be a counterexample to $|\alpha|$ is weakly subtle over α , where for $\gamma \geq \beta$, $B_\gamma \cap \gamma \neq \emptyset$. QED

THEOREM 1.6. The least weakly inclusion subtle ordinal over $\alpha \geq 2$, if it exists, is subtle.

Proof: Let α be the least weakly inclusion subtle ordinal over $\alpha \geq 2$. By Lemma 1.5, α is an uncountable cardinal $> |\alpha|$. According to Theorem 1.2, it suffices to prove that α is inclusion subtle.

Let C and $(A_\alpha), \alpha \in C$, be a counterexample to α is inclusion subtle over α . The $A_\alpha, \alpha \in C$, will be irrelevant. We can assume without loss of generality that every element of C is greater than α , and is of the form α^β , where $\beta \geq \alpha$.

Note that the function $h(\alpha) = \alpha+\alpha+\alpha+2$ is strictly increasing and maps every element of C into itself. Therefore we can

assume that each A_α , $\alpha \in C$, is a nonempty set of infinite ordinals greater than α that are a double successor (i.e., the successor of a successor ordinal).

For $\alpha \in C$, define $B_\alpha = A_\alpha$. We now define B_α for $\alpha \in \alpha \setminus C \setminus \min(C)$. Fix $\beta < \alpha$ to be adjacent elements of C .

Let (D_α) , $\alpha < \beta$, be a counterexample to α is weakly inclusion subtle over β . By adding the ordinal $\beta+1$ on the left, we obtain $E_\alpha \in (\beta, \beta)$, $\alpha \in (\beta, \beta)$, where there are no $\beta+\alpha \in x < y < \beta$ with $E_x \in E_y$. For $0 < \alpha < \beta$, set $B_{\beta+\alpha} = \{\beta, \beta\}$. For $\alpha \in (\beta+\alpha, \beta)$, set $B_\alpha = E_\alpha \cup \{\beta+1\}$.

Note that for all $\{x < y\} \in (\beta, \beta)$, B_x is not a subset of B_y .

We have defined B_α , $\alpha \in \alpha \setminus \min(C)$. We claim that for all $\{x < y\} \in [\min(C), \beta)$, B_x is not a subset of B_y . To see this, let $x < y$ be from $[\min(C), \beta)$, $B_x \in B_y$. It is not the case that $x, y \in C$. Also, every A_z , $z \in C$, consists entirely of double successor ordinals greater than β , every $z \in [\min(C), \beta) \setminus C$ has an element of the form $\beta+1$, $\alpha \in C$. It follows that $x \in C \cap y \in C$. So we can assume that $x, y \in C$.

We have already seen that if x, y lie in the same interval (β, β) , β, β adjacent in C , then B_x is not a subset of B_y .

So x, y lie in different intervals (β, β) , (β', β') , β, β and β', β' adjacent in C . Then $\beta+1 \in A_x \setminus A_y$, contradicting $A_x \in A_y$.

We have now verified the claim. It remains to define B_α , $\alpha < \min(C)$. By Lemma 1.4, take (B_α) , $\alpha < \min(C)$, to be a counterexample to $\min(C)$ is weakly subtle above β , where for all $\alpha \geq \beta$, $B_\alpha \cap \alpha \neq \emptyset$. Since every B_α , $\alpha \geq \min(C)$, consists entirely of infinite ordinals, (B_α) , $\alpha < \beta$ is a counterexample to β is weakly inclusion subtle over β . This is the desired contradiction. QED

It is well known that subtle cardinals are quite large. In particular, every subtle cardinal is an n -Mahlo cardinal for all n ([Fr01], Lemma 12), and the set of totally indescribable cardinals below any given subtle cardinal is stationary ([Fr01], Lemma 11). These two facts are also from [Ba75].

It will also be useful to know that the subtleness condition implies an ostensibly stronger condition involving more than two ordinals.

THEOREM 1.7. Let κ be a subtle cardinal, $C \subseteq \kappa$ be closed unbounded, and for each $\alpha < \kappa$ let $A_\alpha \subseteq \kappa$ be given. For every $\alpha < \kappa$ there exists $E \subseteq C$ of order type α such that for all $\beta, \gamma \in E$, $\beta < \gamma$, we have $A_\beta = A_\gamma \cap \beta$.

Proof: See [Fr01], Lemma 6. This also follows from [Ba75].
QED

2. CHAINS IN TRANSITIVE SETS.

Recall the cumulative hierarchy $V(\alpha)$, which starts with $V(0) = \emptyset$, takes the power set at successor stages, and is the union at limit stages. We write $\text{rk}(x)$ for the rank of x , which is the strict sup of the $\text{rk}(y)$, $y \in x$. Recall that $\text{rk}(x)$ is always one less than the least α such that $x \subseteq V(\alpha)$. Every set and its transitive closure have the same rank. The set of all ranks of elements of a transitive set is exactly the rank of that transitive set. The only set of rank 0 is $\emptyset = 0$, and the only set of rank 1 is $\{\emptyset\} = 1$.

A chain is a set, where any two elements are comparable under inclusion. We say that S contains a chain x if and only if x is a chain with $x \subseteq S$. The order type of a chain is its order type as a linear ordering under inclusion.

THEOREM 2.1. Every transitive set with at least two elements contains a 2 element chain. Every transitive set with an infinite element contains a 3 element chain. There is an infinite transitive set that does not contain a 3 element chain.

Proof: Let S be a transitive set with at least two elements. Then $\emptyset, \{\emptyset\} \subseteq S$. Hence $\emptyset, \{\emptyset\}$ forms a two element chain in S .

Let S be a transitive set with an infinite element, but without a 3 element chain. Clearly $\text{rk}(S) \geq \omega+1$, and S has elements of each rank $\alpha < \omega$.

We prove by induction on integers $n \geq 0$ that the unique element of S of rank n is $W[n] = \{\dots\{\emptyset\}\dots\}$, where there are n pairs of brackets.

Suppose this is true for all $n < k$, where $k \leq \aleph_1$. We wish to show that this is true for k . Clearly this is true for $k = 0$. So we assume $k \geq 1$.

Let $x \subseteq S$ have rank k . Now every element of x is an element of E of rank $< k$. Furthermore, x has an element of rank $k-1$. Hence x consists of $W[k-1]$ together with zero or more of $W[0], \dots, W[k-2]$.

Suppose some $W[i]$, $0 \leq i \leq k-2$, lies in x . Then $W[i+1] \not\subseteq x$, since $W[k-1] \not\subseteq W[i+1]$. Hence $\emptyset, W[i+1], x$ forms a three element chain contained in S . This is a contradiction.

We conclude that x consists entirely of $W[k-1]$, and so $x = W[k]$. This completes the induction.

Now let $y \subseteq S$ be of rank \aleph_1 . Then y has an element of finite rank ≥ 1 . Let $z \subseteq y$ have finite rank ≥ 1 . Since $z \subseteq S$, we see that $z = W[k]$ for some finite $k \geq 1$. Hence $\emptyset, W[k], y$ forms a 3 element chain contained in S .

For the final claim, note that the $W[i]$, $i \geq 0$, are distinct. It follows that $W[i] \not\subseteq W[j] \not\subseteq W[i]$ ($i = 0 \leq i = j$). Hence $S = \{W[0], W[1], W[2], \dots\}$ has no 3 element chain. QED

THEOREM 2.2. Every transitive set whose cardinality is subtle contains a chain of every order type less than its cardinality.

Proof: Let S be a transitive set of cardinality \aleph_1 , where \aleph_1 is subtle. Then \aleph_1 is a strongly inaccessible cardinal, and $\text{rk}(S) \geq \aleph_1$. Therefore S has elements of every rank $< \aleph_1$.

Let $h: V(\aleph_1) \rightarrow \aleph_1$ be a bijection. Let $C = \{\alpha < \aleph_1: \alpha \text{ is a limit ordinal and } h[V(\alpha)] \subseteq \alpha\}$. Then C is closed and unbounded in \aleph_1 .

For each $\alpha < \aleph_1$, let A_α be $h[x]$, where x is any element of S of rank α , provided $h[x] \subseteq \alpha$; \emptyset otherwise.

Note that for each $\alpha \in C$, A_α is $h[x]$, where x is some element of S of rank α .

By Lemma 1.7, for any $\alpha \in C$, there exists $E \subseteq C$ be of order type α , where for all $\beta, \gamma \in E$, $\beta < \gamma$, we have $A_\beta = A_\gamma \cap \beta$.

Hence for all $\alpha, \beta \in E$, $\alpha < \beta$, we have that $h^{-1}[A_\alpha], h^{-1}[A_\beta]$ are elements of S of ranks α, β , with $h^{-1}[A_\alpha] \in h^{-1}[A_\beta]$. QED

LEMMA 2.3. Let α be an ordinal which is not weakly inclusion subtle over $\beta \geq 2$. There exists a counterexample (A_α) , $\alpha < \beta$, to α is weakly inclusion subtle over β such that the strict sup of each A_α is β .

Proof: Let A_α , $\alpha < \beta$, be a counterexample to α is weakly inclusion subtle over β . We define another counterexample, B_α , $\alpha < \beta$.

Define $B_0 = \emptyset$. For successor ordinals α , define $B_\alpha = \{\alpha-1, 0\}$. For limit ordinals α , define $B_\alpha = \{2x+2: x \in A_\alpha\} \cup \{2x+1: x < \alpha\}$. It is clear that the strict sup of each A_α is β .

Let $\alpha \in \alpha < \beta$, $B_\alpha \in B_\beta$. We want to obtain a contradiction. Clearly not both α, β are successors. If α is a successor then β is a limit, and we have $0 \in B_\alpha$, $0 \in B_\beta$, which is impossible. If α is a limit and β is a successor, then B_α is infinite and B_β is finite, which is also impossible. Hence α, β are limits. Therefore $A_\alpha \in A_\beta$, which is impossible. This shows that (B_α) , $\alpha < \beta$, is a counterexample to α is weakly inclusion subtle over β . QED

LEMMA 2.4. For all $\alpha \geq \beta$, every transitive set of rank $\geq \alpha$ has a transitive subset of rank β and cardinality $|\alpha|$.

Proof: We first show that if x is transitive, $\text{rk}(x) \geq \alpha$, then x has a transitive subset of rank β . To see this, just take $x \in V(\alpha)$.

It is now clear that it suffices to prove the Lemma with "rank $\geq \alpha$ " replaced by "rank α ".

We prove this by transfinite induction on α . Suppose this is true for all infinite $\beta < \alpha$, where α is infinite. We will show that this is true for α .

First suppose that $\alpha = \beta$. It is obvious that every transitive set of rank α is already of cardinality α . So assume that $\alpha \geq \beta+1$.

Let A be transitive, $\text{rk}(A) = \alpha$. Then for all infinite $\beta < \alpha$, A has a transitive subset A_β of rank β and cardinality $|\alpha|$. Choose one such A_β for each infinite $\beta < \alpha$. Let B be the union

of these A_α , $\alpha < \beta < \gamma$. Then $B \subseteq A$ is transitive. Note that B is the union of \aleph_β many sets each of nonzero cardinality \aleph_α . Hence $|B| = \aleph_\beta$. So if β is countable then since B is nonempty, we have $|B| = \aleph_1$. And if β is uncountable, then B is the union of \aleph_β sets each of nonzero cardinality \aleph_β . Hence again $|B| = \aleph_\beta$.

Note that if β is a limit ordinal, then $\text{rk}(B) = \beta$, and we are done. Suppose $\beta = \alpha+1$. Then $\text{rk}(B) = \alpha$. Take $B' = B \cup \{B\}$, which is transitive, of cardinality \aleph_β , and of rank β . QED

THEOREM 2.5. Let $\beta > \alpha \geq 2$ be ordinals. The following are equivalent (all 7 forms).

- i) There is a subtle cardinal in $(\alpha, \beta]$;
- ii) Every transitive set of the same cardinality as \aleph_β (of rank α) contains a chain of order type $\beta+2$;
- iii) Every transitive set of the same cardinality as \aleph_β (of rank α) contains a 2 element chain whose elements are of rank $\geq \alpha$;
- iv) Every transitive set of the same cardinality as \aleph_β (of rank α) contains a chain of every order type less than the least subtle cardinal $> \alpha$.

Proof: Note that we have given two forms each of ii) - iv). By Lemma 2.4, the form with "cardinality" implies the form with "rank".

Suppose i), and let $\alpha < \beta$, where β is the least subtle cardinal $> \alpha$. We claim that every transitive set of cardinality \aleph_β has a transitive subset of cardinality \aleph_β . To see this, note that β is a strongly inaccessible cardinal, and intersect the set with $V(\beta)$. By Theorem 2.2, we immediately obtain ii), iv), and therefore iii).

We now assume i) is false. We wish to refute ii), iii), iv) with "rank".

By Theorem 1.6, β is not weakly inclusion subtle over α . By Lemma 2.3, let (A_α) , $\alpha < \beta$, be a counterexample to β is weakly inclusion subtle over α such that the strict sup of each A_α is α .

We define $f: \beta \rightarrow V(\beta)$ as follows. $f(\alpha) = \{f(\beta) : \beta \in A_\alpha\}$. By transfinite induction, each $f(\alpha)$ has rank α . Let S be the range of f . Then S is a transitive set of rank β , where S has exactly one element of each rank $< \beta$.

We claim that $f(\alpha) \cap f(\beta) \in A_\alpha \cap A_\beta$. To see this, suppose $f(\alpha) \cap f(\beta) \notin A_\alpha$. Then $\{f(\alpha) : \alpha \in A_\alpha\} \cap \{f(\beta) : \beta \in A_\beta\}$. Since f is a bijection, $A_\alpha \cap A_\beta$.

Suppose there exists a 2 element chain $\{x \neq y\} \in S$, $\text{rk}(x), \text{rk}(y) \geq \alpha$. Since $x \neq y$, we have $\text{rk}(x) \neq \text{rk}(y)$, and hence $\alpha \leq \text{rk}(x) < \text{rk}(y)$. Write $\text{rk}(x) = \beta$ and $\text{rk}(y) = \gamma$. Then $x = f(\beta)$ and $y = f(\gamma)$. Therefore $A_\beta \cap A_\gamma$. Since $\beta \leq \alpha < \gamma$, this contradicts the choice of (A_α) , $\alpha < \beta$.

Suppose there exists a chain in S of order type $\alpha+2$. By the argument in the previous paragraph, the ranks of the elements of this chain form a set of ordinals of order type $\alpha+2$. Hence the last two of these ranks are $\geq \alpha$. Again this contradicts the choice of (A_α) , $\alpha < \alpha$.

Obviously iv) implies iii), and the proof is complete. QED

COROLLARY 2.6. Let α be an ordinal. The following are equivalent.

- i) There is a subtle cardinal α ;
- ii) Every transitive set of the same cardinality as α (of rank α) contains a 4 element chain;
- iii) Every transitive set of the same cardinality as α (of rank α) contains a 2 element chain not using $\emptyset, \{\emptyset\}$;
- iv) Every transitive set of the same cardinality as α (of rank α) contains a chain of every order type less than the first subtle cardinal.

Proof: Immediate from Theorem 2.5 by setting $\alpha = 2$. QED

We can use this development to give a striking definition of a set of cardinality and rank that of the first subtle cardinal.

THEOREM 2.9. The transitive sets that do not contain a 4 element chain form a proper class or form a set of cardinality and rank that of the least subtle cardinal. If there is a subtle cardinal, then the transitive sets that do not contain a 4 element chain form a set of cardinality and rank that of the least subtle cardinal. The converse holds.

Proof: Suppose they form a set of rank α . Let x be a transitive set of rank α . Then x cannot lie in the set, and

so must have a 4 element chain. This verifies ii) in Corollary 2.6. Hence there is a subtle cardinal \aleph_α .

Now suppose there is a subtle cardinal, and let \aleph_α be the least subtle cardinal. Then $\aleph_\alpha \leq \aleph_\beta$. Suppose $\aleph_\alpha < \aleph_\beta$. Then there must be a transitive set y without a 4 element chain which is of rank $\geq \aleph_\alpha$. This contradicts Corollary 2.6 with \aleph_α set to $\text{rk}(y)$. QED

3. ALMOST INEFFABLE CARDINALS.

We say that \aleph_α is almost ineffable if and only if

- i) \aleph_α is a limit ordinal;
- ii) For each $\aleph_\beta < \aleph_\alpha$ let $A_\beta \subseteq \aleph_\beta$ be given. There exists an unbounded $E \subseteq \aleph_\alpha$ such that for all $\aleph_\beta, \aleph_\gamma \in E$, $\aleph_\beta < \aleph_\gamma$, we have $A_{\aleph_\beta} = A_{\aleph_\gamma} \cap \aleph_\beta$.

Almost ineffability is discussed in [Fr01]. It implies subtlety, and in fact is much stronger (see Lemma 8). Such results are also in [Ba75].

We say that an ordinal \aleph_α is almost inclusion ineffable if and only if

- i) \aleph_α is an infinite ordinal;
- ii) For each $\aleph_\beta < \aleph_\alpha$ let $A_\beta \subseteq \aleph_\beta$ be given. There exists $E \subseteq \aleph_\alpha$ of the same cardinality as \aleph_α such that for all $\aleph_\beta, \aleph_\gamma \in E$, $\aleph_\beta < \aleph_\gamma$, we have $A_{\aleph_\beta} \subseteq A_{\aleph_\gamma}$.

LEMMA 3.1. Any almost inclusion ineffable ordinal is uncountable.

Proof: Let \aleph_α be an almost inclusion ineffable ordinal which is countable. Let $f: \aleph_\alpha \rightarrow \aleph_\alpha$ be one-one. For $\aleph_\beta \in [\aleph_\alpha, \aleph_\alpha)$, define $A_{\aleph_\beta} = \{f(\aleph_\beta)+1\}$. For $\aleph_\beta \in [1, \aleph_\alpha)$ define $A_{\aleph_\beta} = \{0, \aleph_\beta-1\}$. Define $A_0 = \emptyset$. Now any infinite $E \subseteq \aleph_\alpha$ must contain infinitely many elements from \aleph_α , or infinitely many elements from $[\aleph_\alpha, \aleph_\alpha)$. Thus we have constructed a counterexample to \aleph_α is almost inclusion ineffable. QED

THEOREM 3.2. An ordinal is almost inclusion ineffable if and only if its cardinal is almost ineffable.

Proof: Let κ be an almost inclusion ineffable ordinal. By Lemma 3.1, $|\kappa|$ is an uncountable cardinal. Let $f: \kappa \rightarrow |\kappa|$ be one-one.

Towards showing $|\kappa|$ is almost ineffable, let (A_α) , $\alpha < |\kappa|$, be given. For limits $\alpha < |\kappa|$, define $B_\alpha = \{2\beta: \beta \in A_\alpha\} \cup \{2\beta+1 < \alpha: \beta \in A_\alpha\}$. For successors $\alpha < |\kappa|$, define $B_\alpha = \{\alpha-1\}$. Define $B_0 = \emptyset$. For $\alpha \geq |\kappa|$, define $B_\alpha = \{f(\alpha)\}$.

Since κ is almost inclusion ineffable, let $E \subseteq \kappa$, $|E| = |\kappa|$, where for all $\beta < \alpha$ from E , $B_\beta \subseteq B_\alpha$. Observe that E cannot have more than 1 element $\geq |\kappa|$. Also E has at most one successor ordinal. Hence the set E' of limit ordinals in E is unbounded in κ .

Let $\beta < \alpha$ be from E' . Since $B_\beta \subseteq B_\alpha$, we see that $A_\beta = A_\alpha \cap \beta$. We have thus verified that κ is almost ineffable. QED

THEOREM 3.3. Every transitive set whose cardinality is almost ineffable contains a chain of that cardinality.

Proof: Let S be a transitive set of cardinality κ , where κ is almost ineffable. Then κ is a strongly inaccessible cardinal. It follows that S has rank $\geq \kappa$. Let $S' = S \cap V(\kappa)$. Then S' is transitive and $\text{rk}(S') = \kappa$.

Let $h: V(\kappa) \rightarrow \kappa$ be a bijection. Let $C = \{\alpha < \kappa: \alpha \text{ is a limit ordinal and } h[V(\alpha)] \subseteq \alpha\}$. Then C is closed and unbounded in κ .

For each $\alpha \in [\min(C), \kappa)$, let A_α be $h[x]$, where x is any element of S' whose rank is the greatest element of C that is $\leq \alpha$. Note that $A_\alpha \subseteq \alpha$. For $\alpha < \min(C)$, let $A_\alpha = \emptyset$.

Let $E \subseteq \kappa$ be unbounded, such that for all $\beta, \alpha \in E$, $\beta < \alpha$, we have $A_\beta = A_\alpha \cap \beta$. Let $E' \subseteq E$ be such that there is at least one element of C strictly between any two adjacent elements of E' . Then for all $\beta, \alpha \in E'$, $\beta < \alpha$, we have $A_\beta \subseteq A_\alpha$, and $h[x] \subseteq h[y]$ for some unique x, y . For these x, y , we have $\text{rk}(x) < \text{rk}(y)$. Hence $x \neq y$. QED

LEMMA 3.4. Let κ be an infinite ordinal which is not almost inclusion ineffable. There exists a counterexample (A_α) , $\alpha < \kappa$, to κ is almost inclusion ineffable such that the strict sup of each A_α is κ .

Proof: Let $(A_\alpha), \alpha < \kappa$, be a counterexample to κ is almost inclusion ineffable. We define another counterexample, $B_\alpha, \alpha < \kappa$.

Define $B_0 = \emptyset$. For successor ordinals α , define $B_\alpha = \{\alpha-1, 0\}$. For limit ordinals α , define $B_\alpha = \{2x+2: x \in A_\alpha\} \cup \{2x+1: x < \alpha\}$.

Let $S \subseteq \kappa$ be of cardinality κ , where for all $\alpha < \beta$ from S , $B_\alpha \subseteq B_\beta$. We want to obtain a contradiction. Clearly at most one element of S is a successor. For limits $\alpha, \beta \in S$, since $B_\alpha \subseteq B_\beta$, we have $A_\alpha \subseteq A_\beta$. Hence $(A_\alpha), \alpha < \kappa$, is not a counterexample to κ is almost inclusion ineffable. This is a contradiction. So we see that $(B_\alpha), \alpha < \kappa$, is a counterexample to κ is almost inclusion ineffable of the desired kind. QED

THEOREM 3.5. Let κ be an infinite cardinal. The following are equivalent.

- i) κ is almost ineffable;
- ii) Every transitive set of cardinality κ contains a chain of cardinality κ .

Proof: i) \Rightarrow ii) is immediate from Theorem 3.3. Now suppose ii). Since κ is a cardinal, by Theorem 3.2 it suffices to prove that κ is almost inclusion ineffable. Let $(A_\alpha), \alpha < \kappa$, be given. By Lemma 3.4, it suffices to assume that the strict sup of each A_α is κ .

We define $f: \kappa \rightarrow V(\kappa)$ as follows. $f(\alpha) = \{f(\beta): \beta \in A_\alpha\}$. By transfinite induction, each $f(\alpha)$ has rank α .

Suppose $f(\alpha) \subseteq f(\beta)$. Then $\{f(\gamma): \gamma \in A_\alpha\} \subseteq \{f(\gamma): \gamma \in A_\beta\}$. Since f is a bijection, $A_\alpha \subseteq A_\beta$.

Let S be the range of f . Then S is a transitive set of cardinality κ , where S has exactly one element of each rank $< \kappa$.

By ii), let $C \subseteq S$ be a chain of cardinality κ . Let $C = f[E]$, where $E \subseteq \kappa$. Note that E is unbounded in κ . Clearly for all $\alpha < \beta$ from E , $f(\alpha) \not\subseteq f(\beta)$ or $f(\beta) \not\subseteq f(\alpha)$. From rank considerations, for all $\alpha < \beta$ from E , $f(\alpha) \not\subseteq f(\beta)$. Since f is one-one, we have for all $\alpha < \beta$ from E , $A_\alpha \not\subseteq A_\beta$. Hence κ is almost inclusion ineffable. QED

4. PRIMITIVE INDEPENDENCE RESULTS IN SET THEORY.

From the point of view of combinatorial set theory, the clearest statement coming out of this development is the following.

PROPOSITION 4.1. Every transitive set of sufficiently large cardinality (or rank) contains a 4 element chain.

However, from the point of view of simplicity in primitive notation, the following is preferable because it does not involve the concept of cardinality or rank.

PROPOSITION 4.2. There is a set E such that every transitive set not in E contains a 4 element chain.

In section 5, we will see that the following modification reduces the number of quantifiers in primitive notation.

PROPOSITION 4.3. For every set A there exists a set E such that every transitive set B not in E , $B \setminus A$ contains a 2 element chain.

THEOREM 4.4. Proposition 4.1 (both forms) and Proposition 4.2 are provably equivalent to the existence of a subtle cardinal over ZFC. Proposition 4.3 is provably equivalent to the existence of arbitrarily large subtle cardinals over ZFC. These results hold using "infinite" or "uncountable" instead of 4.

Proof: First suppose that there is a subtle cardinal. Then we obtain Proposition 4.1 (both forms) from Theorem 2.2. For Proposition 4.2, let $E = V(\aleph)$, where \aleph is the least subtle cardinal. Then every transitive set not in E is of cardinality $\geq \aleph$. Now apply Theorem 2.2.

Now suppose that there are arbitrarily large subtle cardinals. For Proposition 4.3, let A be given. Let $\aleph = \text{rk}(A)$, $x = V(\aleph)$, and $\aleph > \text{rk}(A)$ be subtle. Then by Theorem 2.5, for all transitive x of cardinality \aleph , $x \setminus V(\aleph)$ contains a 2 element chain. By Lemma 2.4, for all transitive $x \subseteq V(\aleph)$, $x \setminus A$ contains a 2 element chain. Set $E = V(\aleph)$.

For the reverse direction, assume Proposition 4.1. Let \aleph be a cardinal such that any transitive set of cardinality $\geq \aleph$ has a 4 element chain. By Corollary 2.6, there is a subtle

cardinal. We can replace "cardinality" by "rank", again by Corollary 2.6.

Assume Proposition 4.2. Let E be such that every transitive set not in E contains a 4 element chain. Let κ be a cardinal greater than the cardinality of $TC(E)$. Then every set of cardinality κ is not in E . Hence every transitive set of cardinality κ contains a 4 element chain. By Corollary 2.6, there is a subtle cardinal.

Assume Proposition 4.3. Let κ be an ordinal. We want to show that there is a subtle cardinal $> \kappa$. Let E be such that for every transitive set $B \subseteq E$, $B \setminus V(\kappa)$ contains a 2 element chain. Let λ be a cardinal greater than the cardinality of $TC(E), \kappa$. Then for every transitive set B of cardinality λ , $B \setminus V(\kappa)$ contains a 2 element chain. By Theorem 2.5, there is a subtle cardinal $> \lambda$.

We can replace "4 element" or "2 element" throughout by, e.g., "infinite" or "uncountable", using Theorem 2.5 and Corollary 2.6. QED

In section 5, we will see that we can yet further reduce the number of quantifiers using the following more technical statement.

PROPOSITION 4.5. There is a set E such that
 i) every element of E has at least two elements;
 ii) for all transitive $x \subseteq E, \subseteq E$, there exists $y, z \subseteq E, x$, such that $y \not\subseteq z$.

THEOREM 4.6. Proposition 4.5 is provably equivalent to the existence of a subtle cardinal over ZFC.

Proof: Let κ be the least subtle cardinal. Let E be the set of all elements of $V(\kappa)$ with at least two elements. Let $x \subseteq E, \subseteq E$ be transitive. Note that $\kappa E = V(\kappa)$. Then $rk(x) \geq \kappa$, and so by Corollary 2.6, let $w, u, y, z \subseteq x \subseteq V(\kappa)$, and $w \not\subseteq u \not\subseteq y \not\subseteq z$. Then $y, z \subseteq E$.

Now let E have the properties of Proposition 4.5. It suffices to verify iii) in Corollary 2.6. Let κ be a strong limit cardinal greater than $rk(E)$, and x be a transitive set of cardinality κ . Let $y, z \subseteq E, x$, be such that $y \not\subseteq z$. Then y, z have at least two elements. So Corollary 2.6 iii) has been verified. Hence there is a subtle cardinal. QED

PROPOSITION 4.7. There is an infinite cardinal \aleph such that every transitive set of cardinality \aleph contains a chain of cardinality \aleph .

THEOREM 4.8. Proposition 4.7 is provably equivalent to the existence of an almost ineffable cardinal over ZFC.

Proof: Immediate from Theorem 3.5. QED

5. PRIMITIVE INDEPENDENCE RESULTS IN CLASS THEORY.

The statements in class theory are arguably even simpler than those in section 4 in set theory. On is the class of all ordinal numbers.

PROPOSITION 5.1. Every transitive proper class contains a \aleph_1 element chain.

PROPOSITION 5.2. Every transitive proper class contains arbitrarily long set chains.

We will see that the following modification reduces the number of quantifiers in primitive notation.

PROPOSITION 5.3. For every set, every transitive proper class contains a \aleph_1 element chain disjoint from that set.

Actually, we use a slight modification.

PROPOSITION 5.4. Every transitive proper class contains a two element chain that is disjoint from any specified element of that class.

We let NBG be the usual von Neumann, Bernays, Gödel theory of classes. Let NBGC be NBG with the global axiom of choice.

In the present context of class theory, a set ordinal is an epsilon connected transitive set, and a class ordinal is an epsilon connected transitive class. The only proper class ordinal is On , which is the class of all set ordinals. On is also a class cardinal.

Note that the notions of subtle, inclusion, weakly subtle, weakly inclusion subtle, almost ineffable, and almost inclusion ineffable make perfectly good sense for On .

All of the proofs in sections 1-3 work for On with only trivial modifications.

THEOREM 5.5. Proposition 5.1 is provably equivalent to "there is a subtle class cardinal" over NBGC. Propositions 5.2 - 5.4 are provably equivalent to "for every set ordinal α there is a subtle class cardinal $\geq \alpha$ " over NBGC.

Proof: Assume there is a subtle class cardinal. The proof of Theorem 2.2 can obviously be adapted to obtain Proposition 5.1.

Now assume that for all set ordinals α there is a subtle class cardinal $\geq \alpha$. If On is subtle then we can adapt the proof of Theorem 2.2 to obtain Proposition 4.2. Now suppose there are arbitrarily large subtle set cardinals. Let X be a transitive proper class. For each subtle set cardinal α , $X \cap V(\alpha)$ is a transitive set of cardinality α . Hence by Theorem 2.2, X contains a chain of every order type $< \alpha$. Proposition 5.2 follows. Also note that the implications 5.2 \Rightarrow 5.3 \Rightarrow 5.4 are trivial.

For the reverse direction, first assume Proposition 5.1. The proof of Corollary 2.6 can obviously be adapted to On to obtain that there is a subtle class cardinal.

Now assume Proposition 5.4. We adapt Theorem 2.5 to the case $\alpha = \text{On}$. It suffices to verify iii) of Theorem 2.5 for an arbitrary set ordinal $\alpha \geq 2$. Fix a set ordinal $\alpha \geq 2$, and let x be a transitive proper class. If $x = V$ then we are done. Now assume $x \neq V$ and let $\alpha' > \alpha$ be such that $V(\alpha') \cap x$. Let $y = x \cap V(\alpha') \cap \{V(\alpha')\}$. Then y is also a transitive proper class. By Proposition 5.4, let $u, v \in y \setminus V(\alpha')$, $u \neq v$.

We claim that $u \neq \{V(\alpha')\}$. To see this, suppose $u = \{V(\alpha')\}$. Then $v \in V(\alpha') \cap \{V(\alpha')\}$, and so $V(\alpha') \in v \in x$. But this contradicts $V(\alpha') \cap x$.

We now conclude that $u \in x$. Hence again $v \in V(\alpha') \cap \{V(\alpha')\}$. Therefore $v \in x$. So $\{u, v\}$ is a two element chain in $x \setminus V(\alpha)$. This completes the verification of Theorem 2.5 iii) for $\alpha = \text{On}$ and α . Hence by Theorem 2.5, there is a subtle cardinal in $[\alpha, \text{On}]$. QED

PROPOSITION 5.6. Every transitive proper class contains a proper class chain.

THEOREM 5.7. Proposition 5.6 is provably equivalent to "On is almost ineffable" over NBGC.

Proof: By adapting the proof of Theorem 3.5. QED

6. FORMALIZATIONS.

Normally, mathematical logicians count the number of quantifier alternations rather than the actual number of quantifiers. Properly counting the actual number of quantifiers is a little more delicate. We will use the following principle in predicate calculus.

LEMMA 6.1. Let $\phi = (\exists x_1^*) (\phi_1) \dots (\exists x_k^*) (\phi_k)$, where each x_i^* is a block of zero or more distinct variables. Then ϕ is logically equivalent to a formula $(\exists x^*) (\phi_1[x_1^*/x^*] \dots \phi_k[x_k^*/x^*])$, where x^* is a block of distinct variables not appearing in ϕ , of length the maximum of the lengths of the x_i^* , and $\phi_i[x_i^*/x^*]$ denotes the result of replacing the variables x_i^* by (an initial segment of) the variables x^* , in order from left to right. It follows that the dual holds, with \exists replaced by \forall , and \forall replaced by \exists .

Proof: Let x^* be any list of distinct variables not appearing in ϕ , of length the maximum of the lengths of the x_i^* . Then ϕ is logically equivalent to $(\exists x^*) (\phi_1[x_1^*/x^*]) \dots (\exists x^*) (\phi_k[x_k^*/x^*])$, which in turn is logically equivalent to $(\exists x^*) (\phi_1[x_1^*/x^*] \dots \phi_k[x_k^*/x^*])$. QED

We are now prepared to count quantifiers. We began our study with Proposition 4.2, which yields a 7 quantifier sentence. This was our original primitive independence result from ZFC. To reduce the number, we discovered Proposition 4.3. This reduces the number to 6. Finally, we discovered the more technical Proposition 4.5, which yields a 5 quantifier sentence.

THEOREM 6.2. There is a $\forall\forall\forall\forall\forall\forall\forall$ sentence (7 set quantifiers) in $\mathcal{L}_{\in,=}$, which is independent of ZFC. In particular, it is provably equivalent to the existence of a subtle cardinal over ZFC.

Proof: We formalize Proposition 4.2 as follows.

$(\exists x)(\exists y \subseteq x)(y \text{ transitive} \wedge y \text{ contains a 4 element chain}).$

$(\exists x)(\exists y)(y \subseteq x \wedge y \text{ not transitive} \wedge y \text{ contains a 4 element chain}).$

$(\exists x)(\exists y)(y \subseteq x \wedge \dots \dots \dots).$

Proposition 4.2 has been put in form $\dots \dots \dots$. QED

THEOREM 6.3. There is a $\dots \dots \dots$ sentence (6 set quantifiers) in $\mathcal{L}_{\in,=}$, which is independent of ZFC. In particular, it is provably equivalent to the existence of arbitrarily large subtle cardinals over ZFC.

Proof: We formalize Proposition 4.3 as follows.

$(\exists x)(\exists y)(\exists z \subseteq y)(z \text{ transitive} \wedge z \setminus x \text{ contains a 2 element chain}).$

$(\exists x)(\exists y)(\exists z)(z \subseteq y \wedge z \text{ not transitive} \wedge z \setminus x \text{ contains a 2 element chain}).$

$(\exists x)(\exists y)(\exists z)(z \subseteq y \wedge \dots \dots \dots).$

Proposition 4.3 has been put in form $\dots \dots \dots$. QED

THEOREM 6.4. There is a $\dots \dots \dots$ sentence (5 set quantifiers) in $\mathcal{L}_{\in,=}$, which is independent of ZFC. In particular, it is provably equivalent to the existence of a subtle cardinal over ZFC.

Proof: We formalize Proposition 4.5 by verifying that the formalization of the property of E is $\dots \dots \dots$.

$(\exists x)((x \subseteq E \wedge ((x \subseteq E \wedge x \text{ transitive}) \wedge \dots \dots \dots)) \wedge (x \subseteq E \wedge \dots \dots \dots)).$

$(\exists x)((x \subseteq E \wedge ((\dots \dots \dots) \wedge \dots \dots \dots)) \wedge (x \subseteq E \wedge \dots \dots \dots)).$

$(\exists x)((x \subseteq E \wedge (\dots \dots \dots)) \wedge (x \subseteq E \wedge \dots \dots \dots)).$

$(\exists x)((x \subseteq E \wedge (\dots \dots \dots)) \wedge (x \subseteq E \wedge \dots \dots \dots)).$

$(\exists x)((x \subseteq E \wedge \dots \dots \dots) \wedge (x \subseteq E \wedge \dots \dots \dots)).$

$(\exists x)((x \in E \cap \emptyset) \rightarrow (x \in E \cap \emptyset)).$

$(\exists x)(\emptyset \in \emptyset).$

$(\exists x)(\emptyset \in \emptyset).$

$\emptyset \in \emptyset.$

Proposition 4.5 has been put in form $\emptyset \in \emptyset$. QED

We now consider the formalization of the statements in section 5 in class theory. We will use the same primitive language of set theory, where all variables range over classes. In this setup, the sets are taken to be the classes that are a member of some class. Thus the standard "model" of class theory consists of all subclasses of the class of all sets, under membership.

Our original primitive independence result from NBGC was Proposition 5.1, with six class quantifiers. With more effort, we discovered Proposition 5.4, which reduces the number of class quantifiers to five.

THEOREM 6.4. There is a $\emptyset \in \emptyset$ sentence (6 class quantifiers) in $\emptyset, =$ which is independent of NBGC. In particular, it is provably equivalent to "there is a subtle class cardinal" over NBGC.

Proof: We formalize Proposition 5.1.

$(\emptyset \text{ transitive proper class } x)(x \text{ contains a 4 element chain}).$

$(\exists x)(x \text{ is a set } \rightarrow x \text{ contains a 4 element chain}).$

$(\exists x)(\emptyset \in \emptyset).$

Proposition 5.1 has been put in form $\emptyset \in \emptyset$. QED

THEOREM 6.5. There is a $\emptyset \in \emptyset$ sentence (6 class quantifiers) in $\emptyset, =$ which is independent of NBGC. In particular, it is provable equivalent to "there is a subtle class cardinal" over NBGC.

Proof: We formalize Proposition 5.4.

[Go79] presents a proof that all three quantifier sentences are provable or refutable in ZFC. In the author's words, not all of the details are presented:

"It is tedious but involves no difficulty to verify that if...". p. 5, line 14.

"This can be verified by considering all the possible cases, but is quite clear if considered carefully. So we omit what would be a very long verification." p.8.

We give a fully detailed proof in [Fr02a], where we show that every three quantifier sentence (and a bit beyond) is decided in a weak fragment of ZF without the power set axiom.

[Go79], p.3, conjectures that all 7 quantifier sentences are provable or refutable in ZF. The axiom of choice is displayed there, and it is remarked that it has 8 quantifiers:

$$(\exists x_1) ((\exists x_2) (\exists x_3) (x_2 \neq x_1 \wedge x_3 \neq x_1 \wedge (\exists x_4) (x_4 \neq x_2 \wedge x_4 \neq x_3)) \\ \wedge (\exists x_5) (\exists x_6) (x_6 \neq x_1 \wedge (\exists x_7) (x_7 \neq x_6 \wedge x_7 \neq x_5 \wedge (\exists x_3) (x_8 \neq x_6 \\ \wedge x_8 \neq x_5 \wedge x_3 = x_7))))).$$

The last two x_3 's are typographical errors, so this should read:

$$(\exists x_1) ((\exists x_2) (\exists x_3) (x_2 \neq x_1 \wedge x_3 \neq x_1 \wedge (\exists x_4) (x_4 \neq x_2 \wedge x_4 \neq x_3)) \\ \wedge (\exists x_5) (\exists x_6) (x_6 \neq x_1 \wedge (\exists x_7) (x_7 \neq x_6 \wedge x_7 \neq x_5 \wedge (\exists x_8) (x_8 \neq x_6 \\ \wedge x_8 \neq x_5 \wedge x_8 = x_7))))).$$

In fact, this standard version of the axiom of choice can be put into 7 quantifier form as follows:

$$\exists (A \subseteq \mathbb{R} \mid \forall x \in A \exists y \in \mathbb{R} \dots)$$

$$\exists (A \subseteq \mathbb{R} \mid \forall x \in A \exists y \in \mathbb{R} \dots)$$

$$\dots$$

So the example presented in [Go79], the axiom of choice, already refutes the conjecture made there that all 7 quantifier sentences are provable or refutable in ZF.

In fact, by Theorem 6.4, there is a 5 quantifier sentence that is neither provable nor refutable in ZFC + V = L.

The most obvious question left open is whether every four quantifier sentence is decided in ZFC. We conjecture that this is true.

8. FURTHER RESULTS.

We mention some further results. These are among the results that will appear in [Fr02b].

One the key notions in [Fr02b] is that of relative chains. Let S, x be classes. We say that S contains a relative chain x if and only if

- i) $(\exists a, b \in x)(a = b \vee x \cap a \cap b \vee x \cap b \cap a)$;
- ii) $x \subseteq S$.

Note that if we remove " $x \cap$ ", then this is the definition of " S contains a chain x ".

PROPOSITION 8.1. Every set of sufficiently large cardinality contains a relative four element (infinite, uncountable) chain.

PROPOSITION 8.2. Every proper class contains a relative four element chain.

PROPOSITION 8.3. Every proper class contains relative chains of every set cardinality.

PROPOSITION 8.4. Every proper class contains a relative proper class chain.

THEOREM 8.5. All three forms of Proposition 8.1 are equivalent, over ZFC, to there existence of a subtle cardinal. Propositions 8.2 and 8.3 are equivalent, over NBGC, to "On is subtle". Proposition 8.4 is equivalent, over NBGC, to "On is almost ineffable".

By way of comparison with the results in this paper, note that Proposition 8.1 is more "rudimentary" in that it does not involve transitivity, but it does involve cardinality. The use of cardinality prevents it from being directly formalized as a simple sentence in primitive notation. We also rely heavily on transitivity in order to give the striking example of a set of rank and cardinality the first subtle cardinal in Theorem 2.9.

Propositions 8.2 - 8.4 drop transitivity in favor of the notion of relative chain, without bringing in new cardinality considerations. However, this approach does not seem to lead to a reduction in the number of quantifiers in the primitive notation of class theory.

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