FOUNDATIONAL INVESTIGATIONS IN MATHEMATICS AND PHILOSOPHY

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NOTE: This is the text for the public lecture delivered on May 8, 2014 in connection with my appointment as Research Scholar of the Council of the Humanities and the Department of Philosophy at Princeton University, March 24 - May 2, 2014. It has been lightly edited until the Russell Paradox section, which corrects a misstatement and with some improved material.

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I want to express my thanks to the Council of the Humanities and the Philosophy Department for hosting my 6 week visit from March 24 to May 2, 2014. It has been a great pleasure.

I want to especially thank Gideon Rosen, John Burgess, and Hans Halvorson for making this possible.

I greatly enjoyed the interaction with graduate students at the recent Princeton/Rutgers conference, and the Burgess/Halvorson informal seminar in philosophy of mathematics and physics.

I have been holed up for more than 45 years trying to show that Gödel’s Incompleteness Phenomena is not FAKE. I have recently succeeded in establishing this in a clear sense that we will be discussing. I now call this MATHEMATICALLY PERFECT INCOMPLETENESS.

So if you never heard of me before, this is largely what I have been doing for about 100,000 hours.

I retired in 2012 from Ohio State University, in order to start over and rethink everything. (Actually, my pension maxed out).

Thanks to this visit, I am coming out of my shell, and thinking about some wider issues.

FOUNDATIONS OF MATHEMATICS: BIG SUCCESS AND BIG CHALLENGES

F.o.m. is by far the strongest and arguably the only example that we have of a deep foundation for a systematic branch of knowledge. Much credit should be given to great philosophers including Aristotle, Leibniz, Frege, and Russell. F.o.m. has developed powerful rigorous tools for effectively dealing with a wide range of issues in mathematics, and is incomparably more powerful as a scientific tool than what has come out of other parts of philosophy (e.g., not counting math and physics as coming out of philosophy).

Partly because of its success, f.o.m. is not now considered to be a particularly active direction for philosophers. The perception is that we have solved the major problems, and there cannot be a substantial role for philosophers in any problems that remain given the high technical demands.

Although there is plenty of truth here, it is profoundly misguided on many fronts.

First of all, whereas there has been massive progress in f.o.m., there is massive ignorance when we probe more deeply. A sample:

1. A highly organized, detailed, and robust hierarchy of levels of commitments to mathematical objects and mathematical reasoning has emerged. Yet, there has not emerged any seriously convincing arguments that allow us to confidently accept or reject these various levels of commitment.

2. We do not have an understanding of what motivates our acceptance of the fundamental constructions of mathematics

and set theory – e.g., whether or how this is connected with the mind, brain, or evolutionary development.

3. We have no substantial theory of what we mean by a “mathematically natural” or “mathematically perfect” object or statement. This is of critical importance, because there is a wide range of important and striking observed phenomena in f.o.m. that have no adequate explanation. E.g., the comparability of natural systems under interpretability; the comparability of natural decision problems under relative solvability.

4. The history of modern mathematics has been marked by the fruitful elimination from mathematics, of all sorts of informal concepts used in just about every other subject, in favor of relatively abstract mathematical constructions. E.g., moving points and changing objects replaced by mathematical functions. “Infinitely small” replaced by $\in$-δ criteria. Intensional rules replaced by extensional functions. We have not seriously begun the systematic program of putting such notions back into mathematics.

5. The finistist and ultrafinitist positions are among the credible positions in f.o.m., and we have not seriously begun the systematic program of recasting the whole of mathematics in finitist and ultrafinitist terms, clearly establishing what limitations there may be to such an effort.

6. We do not have any in depth understanding of fundamental properties of actual proofs, including their lengths, sizes, or “complexity”.

7. We are only beginning to have any in depth understanding of the nature and scope of the Incompleteness Phenomena.

FOUNDATIONS OF MATHEMATICS: INTERACTION WITH PHILOSOPHY

Instead of focusing on how “dead” foundations of mathematics is, let’s focus on just how incredibly fertile foundations of mathematics can be for interacting with Philosophy.

In any philosophical context whatsoever, be it scintillating one paragraph drafts to whole papers to books to philosophical areas and programs, we can PROJECT ONTO

THE MATHEMATICAL WORLD. I use the word “project”, since when we make such a transfer, we of course only preserve SOME of the features from the philosophical context.

We now have a rich body of material to work with from the f.o.m. perspective. F.o.m. then works its magic, and exciting findings come about, backed up with deep proofs.

Then these developments are presented back to the relevant philosophers.

The philosophers are likely to complain (often bitterly, perhaps with sarcasm) that these f.o.m. developments don’t take into account the essence of the original philosophical context. When pressed, they supply some details to support such valid claims.

Then the process is repeated, with another more refined projection to the mathematical world, inspiring yet more f.o.m. developments, which are again presented to the philosophers.

The philosophers again complain, and so forth.

So we obtain open ended sequences

P0,P1,P2,…
M0,M1,M2,…

of philosophical developments and mathematical developments (through f.o.m.), which are successive refinements.

We call this ***PING PONG***. Philosophers refine and refocus their issues, teasing out fundamental features and difficulties not apparently present in the mathematical world and not immediately subject to effective attack by f.o.m. This includes careful reformulations of their issues.

F.o.m. gets its great power and scope challenged in entirely new ways, with inevitable spin offs along the way that may not be relevant to philosophical issues outside of mathematics, but perhaps of great f.o.m. value.

Every Philosophy Department needs a **Ping Pong table**, and I am selling them!

SIX WEEKS OF PING PONG AT PRINCETON

I have been playing some unexpectedly productive **Ping Pong** here. Some of this Ping Pong started some time ago, and some involves also mathematicians, who notably have an investment in Gödel Incompleteness being ***FAKE***. Here is a summary.

1. **Ping Pong** concerning Incompleteness, with mathematicians and some philosophers for over 45 years. The resulting state of the art I now call PERFECT MATHEMATICAL INCOMPLETENESS. I will show you this a little later in the talk.

2. **Ping Pong** concerning vagueness off and on over the years. Delia told me she works in vagueness, and that was enough to stimulate my rethinking. I came up with ways to use vague concepts – in particularly what I call vague amplification – in order to prove the consistency of mathematics. I started to play some brief intensive **Ping Pong** with Gideon, and he complained, which led to my focusing on new and powerful ways to resolve Russell's Paradox that lead immediately to the consistency of mathematics (ZFC) without the development of any set theoretic infrastructure. I suspect related major breakthroughs are possible with all of the philosophical paradoxes.

3. I have been playing **Ping Pong** with Gideon and John concerning Wittgenstein and Kripkenstein skepticism.

Previous work of mine lays foundations for the first 8 or so levels of the cumulative hierarchy of sets, and the initial segment of integers with 8 or so iterates of base 2 exponentiation from 0, with essentially no axioms (induction, recursion, etc., being provable using pure logic). The development is linear in 8, and can be presented in an actual fully digestible manuscript. This cannot be done by working with names for all of the objects involved, as that is far too large.

I took this further here by adapting the reasonably well known argument that your integers are the "same" as mine, to: your V(8) and your 2^…^2 8 times are the “same” as mine, using essentially no axioms. I think Gideon and John are still complaining, and also Warren Goldfarb is complaining that Gideon and John are complaining.

A natural move for Gideon and John now - although I don't know whether they would make this move - would be to temper the EXTREME skepticism of Wittgenstein and Kripkenstein (although there has been controversy about whether Kripke interpreted Wittgenstein properly) by granting predicate logic only. Then my development in the previous paragraph becomes dead on relevant. In fact, the level of commitment to logic can likely be sharply reduced by yet more careful arguments. A highly productive subject with fully rigorous results has now emerged from this episode of **Ping Pong**.

4. I had already focused on the “your mathematical objects are the same as mine” idea when playing **Ping Pong** a little bit with Beatrice Longuenesse. She gave lectures on the philosophy of mind recently here, and I suggested that there be a **philosophy of mathematical mind**, where a key starting issue is: are your integers the “same” as mine; are your rationals the “same’ as mine; are your real numbers the “same” as mine; etcetera. She responded with interest, and said that “it was very *Kantian*”. I’m not quite sure how complimentary that is, but that isn’t going to stop me. I have since come up with “**philosophy of musical mind**”, without serious ideas (yet). And, of course, the more general idea of “**philosophy of X mind**”, for various X.

5. Some years ago, I played **Ping Pong** at a philosophy conference held at Ohio State, where a philosopher was talking a lot about “most natural numbers …”. It has been known for some time that there is no reasonable finitely additive probability measure on all sets of natural numbers. But “most” is like just working with >1/2, and it seemed to be that there should NOT be a reasonable way to deal with that for all sets of natural numbers. I quickly proved that there is indeed no reasonable way. In fact, there is no reasonable way of dealing with this: in any partition of N into three parts, some two parts, combined, have most numbers. Marc Johnstone has indicated an interest in playing **Ping Pong** concerning “more” in various contexts, including natural numbers, in connection with ethics. I look forward to this **Ping Pong** experience.

6. I'm looking forward to playing **Ping Pong** with Hans, as I said that physical science should be rebuilt from the ground up using only empirically meaningful notions. This led to my emphasizing the fact that if a mathematical relationship between physical quantities takes on some common forms that we see all throughout physical science, then it is reasonable to expect that we can determine that mathematical relationship (perhaps to a high degree of approximation) from surprisingly small amounts of empirical data.

MATHEMATICALLY PERFECT INCOMPLETENESS

This **Ping Pong** has a long history. Here is a short account. A more in depth history can be found on my website in the draft of "Boolean Relation Theory and Incompleteness" - especially the Introduction.

1. Gödel's first incompleteness Theorem.
1'. Complaints.
2. Gödel's second incompleteness theorem.
2'. Complaints.
3. Gödel's continuum hypothesis consistency.
3'. Complaints.
4. Cohen's continuum hypothesis negation consistency.
4'. Complaints.
5. Solovay's no good nonmeasurable sets consistency.
5'. Complaints.
6. California set theorists with projective sets.
6'. Complaints.
7-78,367. My succession of concrete incompleteness results.
7'-78,367'. Complaints.

MPI (mathematically perfect incompleteness).

NOTE: THIS WAS PRESENTED BEFORE THE VERY RECENT PROGRESS THAT IS ABOUT TO BE ANNOUNCED AS PAPER #78 ON https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ THIS #78 WILL BE MORE CONVINCING.

Q is the set of all rationals. Q[0,1] is Q ∩ [0,1].

DEFINITION. Let $E \subseteq Q[0,1]^{2k}$. A square contained in E is a $B^2 \subseteq E$. A maximal square contained in E is a square contained in E that is maximal with respect to inclusion.

DEFINITION. Let $E \subseteq Q[0,1]^n$. and $R \subseteq Q[0,1]^{2n} = Q[0,1]^n \times Q[0,1]^n$. E is R closed if
and only if for all $x,y \in Q[0,1]^{2n}$, $x \mathrel{R} y \wedge x \in E \rightarrow y \in E$.

DEFINITION. Let $R \subseteq Q[0,1]^n$. R is order theoretic iff R can be defined using logical connectives and clauses $v_i < v_j$, $v_i < c$, $c < v_i$, where $1 \leq i,j \leq n$, and $c \in Q[0,1]$.

ORDER THEORETIC TEMPLATE A. Let $R \subseteq Q[0,1]^{2k}$ and $S \subseteq Q[0,1]^{4k}$ be order theoretic. R has an S closed maximal square.

We have partial results on Template A.

THEOREM. There are order theoretic R,S such that "R has an S closed maximal square" is provable using large cardinal hypotheses but not provable in ZFC. Furthermore, we can take k and the number of constants for R,S to be small (probably ≤ 4). In fact, we can take the number of constants for R to be zero (so called order invariant R).

So ZFC is not capable of solving Order Theoretic Template A.

CONJECURE. SRP is sufficient to solve the Order Theoretic Template A.

It is going to be very difficult to get a solution to Order Theoretic Template A, even with large cardinal hypotheses, because the Template fixes order theoretic R.
There is an easier variant of the Template which bundles together the order invariant R as follows.

ORDER THEORETIC TEMPLATE B. Let $S \subseteq Q[0,1]^{4k}$ be order theoretic. Every order invariant subset of $Q[0,1]^{2k}$ has an S closed maximal square.

We have a partial result on Template B which pours over to Template A.

DEFINITION. Let $A \subseteq Q[0,1]$ be finite. $x,y \in Q[0,1]^k$ are A tail shift equivalent iff y (or x) can be obtained by replacing the largest tail of x (or y) lying in A with its shift in A.

PROPOSITION. Every order invariant subset of $Q[0,1]^{2k}$ has an A tail shift closed maximal square.

THEOREM. The Proposition is provably equivalent to the consistency of certain large cardinal hypotheses. In particular, it can be proved in SRP but not in ZFC.

RUSSELL'S PARADOX.

Here is a way of asserting Frege's inconsistent axiom 5:

Every virtual property of things is equivalent to an actual property of things.

This is normally formalized in the language with $\in$, as

$(\exists x)(\forall y)(y \in x \Leftrightarrow \varphi)$
where x is not free in φ.

Here $y \in x$ means that x is a property that holds of the thing y. (Property are automatically things).

We now make some new reflective moves.

Over time, our world enlarges. For any previous world W, we have this:

I. W is viewed as a thing. In fact as a property of things, which is an element of later worlds.

II. Every virtual property of things in W is equivalent to an actual property that exists in any later world.

The universe evolves with enlarging worlds in a coherent way:

III. Anything true of a given world, referring to particular things in that world, remains true of later worlds.

TWO WORLDS

We use the binary relation $\in$ and the constant symbols W1,W2.

$(\exists x \in W2)(\forall y)(y \in x \Leftrightarrow \varphi \wedge y \in W1)$, where φ is any formula in $\in$ in which x is not free.

$x_1,\ldots,x_n \in W1 \wedge \psi \rightarrow \psi[W1/W2]$, where x is not free in φ and W2 is not in ψ.

THEOREM. Two Worlds is mutually interpretable with ZFC + “there exists a subtle cardinal”.

We are looking at certain weakened forms.

MANY WORLDS

Binary $\in$, and constant symbols W1,W2,... .

$(\exists x \in Wk+1)(\forall y)(y \in x \Leftrightarrow \varphi \wedge y \in Wk)$, where φ is any formula in $\in$ in which x is not free.
$x_1,\ldots,x_n \in Wk \wedge \psi \rightarrow \psi^*$, where ψ has free variables among $x_1,\ldots,x_n$, and ψ^* results from replacing each Wi, $i \geq k$, by Wi+1.

THEOREM. Many Worlds is mutually interpretable with SRP.

We are also looking at some weakened forms.

TOTALITY OF WORLDS

Binary $\in,<,=$. Intended interpretation of $z < w$ is that the world w is later than the world z.

$<$ is a strict linear ordering with no greatest element in its nonempty field.

$z < w \rightarrow (\exists x \in w)(\forall y)(y \in x \Leftrightarrow \varphi \wedge y \in z)$, where φ is any formula in $\in$ in which x is not free.

$x_1,\ldots,x_n \in z \wedge \psi \rightarrow \psi\#$, where ψ is a formula in $\in,=$ with all free variables among $x_1,\ldots,x_n$, and $\psi\#$ is obtained by replacing each $W^*(v)$ by $W^*(v) \wedge v \neq z$.

This corresponds to strong large cardinals approaching measurable cardinals. To go further, we add that $<$ has a limit point. The system interprets ZFC + "there exists a measurable cardinal" and is interpretable using standard strengthenings of a measurable cardinal. Further strengthenings into concentrating measurables can be obtained from stronger versions. E.g.,

$x_1,...,x_n \in z < w \wedge \psi \rightarrow \psi\#$, where ψ is a formula in $\in,=$ with all free variables among $x_1,...,x_n$, and $\psi\#$ is obtained by replacing each $W^*(v)$ by $W^*(v) \wedge (v < z \vee v \geq w)$.

again with "< has a limit point".

ALSO MENTIONED IN TALK: the use of the Incompleteness work to support computer confirmations of the consistency of ZFC and large cardinals.