MATHEMATICALLY NATURAL CONCRETE INCOMPLETENESS

by
Harvey M. Friedman
Distinguished University Professor of
Mathematics, Philosophy, and Computer Science Emeritus
Ohio State University
June 21, 2015

Abstract. Every order invariant $R \subseteq Q^{2k}$ has a maximal nonnegative root, where $S_{1...n}|>n = S_{0...n-1}|>n$. Every order invariant $R \subseteq Q^{2k}$ has a # basis containing its upper shift. These are proved here by going well beyond the usual ZFC axioms for mathematics, using the consistency of the system SRP (Con(SRP)). Here SRP is the extension of ZFC by certain well studied large cardinal hypotheses (existence of cardinals with the Stationary Ramsey Property). We show that both statements are actually equivalent to Con(SRP). These statements represent the first Perfectly Mathematically Natural Concrete Incompleteness from ZFC. The first statement also provides partial information concerning the Perfectly Mathematically Natural Concrete Template "R has a T invariant maximal nonnegative root", where R,T are order theoretic relations on Q^k. We show that each instance of this Template is combinatorial, in the sense of asserting that an associated algorithm does not terminate. We show that no finite fragment of SRP is sufficient to prove or refute every instance of the Template. We conjecture that every instance can be proved or refuted in SRP. We also present a simple sufficient condition on R,T so that R has a T invariant maximal clique, where the sufficiency is again equivalent to Con(SRP). We also present a strong form of the second statement involving that is equivalent to the much stronger hypothesis Con(HUGE) – an hypothesis that is presently considered problematic among set theorists. In addition, we give some explicitly Π^0_1 forms with the same metamathematical properties. We also give natural characterizations of the provable ordinals of SRP and HUGE in the sense of proof theory. We close with a report on some current ongoing developments.

TABLE OF CONTENTS

1. CONCRETE MATHEMATICAL INCOMPLETENESS

In [Fr14a], we presented the then state of the art in our Concrete Mathematical Incompleteness program, initiated in 1967. Here we report on major progress in the CMI program since then. The CMI effort has recently crossed from the Arguably Mathematically Natural Concrete in [Fr14a] to the Perfectly Mathematically Natural Concrete here, with the discovery of Propositions 4.3.1 and 5.3.3. This kind of development in the Gödel Incompleteness Phenomenon is essential for its ongoing relevance to mathematics.

[Fr15] concerns the much earlier Boolean Relation Theory, which illustrates Arguably Natural Concrete Incompleteness, where the concreteness is one major level away from the computational compared to the results reported on here (i.e., Π^0_2 versus Π^0_1). However, Boolean Relation Theory also has considerable merits in the way it has interacted and promises to interact with many areas of mathematics.

Here are the two featured examples of Perfectly
Mathematically Natural Concrete Incompleteness from the
usual ZFC axioms for mathematics, cited in the Abstract:

PROPOSITION 4.3.1. Every order invariant $R \subseteq Q^{2k}$ has a
maximal nonnegative root, where $S_{1...n}|>n = S_{0...n-1}|>n$.

PROPOSITION 5.3.3. Every order invariant $R \subseteq Q^{2k}$ has a #
basis containing its upper shift.

We direct the reader's attention to section 11 which
indicates that a next generation set of examples is
emerging which may be yet more compelling.

At this point, we make some remarks about the logical
features of Propositions 4.3.1 and 5.3.3 given what we know
at this point. The independence from ZFC kicks in for
Proposition 4.3.1 when $8 \leq n \leq k-4$ (perhaps quite a bit
earlier). From then on, provided $k \geq n+4$, Proposition 4.3.1
corresponds to roughly level n in the SRP hierarchy of
large cardinals (see section 10). Larger k for the same n
does not gain any strength.

Proposition 5.3.3 corresponds to the entire SRP hierarchy
as long as k is sufficiently large (say at least 8).

We view Propositions 4.3.1 and 5.3.3 as Perfectly
Mathematically Natural and Concrete. All of the notions are
fully defined in section 2, but for the reader's
convenience, we explain the notation here. Both of these
Propositions have advantages and disadvantages over the
other.

Both statements live in the fundamental category of order
invariant $R \subseteq Q^{2k}$. These R are defined in full detail in
section 2. They can be defined abstractly as the $R \subseteq Q^{2k}$
that are fixed under all automorphisms of $(Q,<)$. More
concretely, these are the $R \subseteq Q^{2k}$, where membership in R of
$x \in Q^{2k}$ depends only on the relative order $(<)$ of the 2k
coordinates of x. There are only finitely many order
invariant $R \subseteq Q^{2k}$ for any fixed k.

For Proposition 4.3.1, a nonnegative root in $R \subseteq Q^{2k}$ is an S
$\subseteq Q^k \cap [0,\infty)^k$ with $S^2 \subseteq R$. A maximal nonnegative root in R
$\subseteq Q^{2k}$ is a nonnegative root in $R \subseteq Q^{2k}$ which is not a proper

subset of any nonnegative root in R. For $S \subseteq Q^k$ and $x \in Q^n$, $S_x = \{y: (x,y) \in S\}$. Thus $S_{1...n}$, $S_{0...n-1}$ are the k-n dimensional sets resulting from fixing the first n coordinates to be 1,...,n and 0,...,n-1, respectively. For $A \subseteq Q^m$, $A|{>}p = A \cap (p,\infty)^m$. All ≤ 0 dimensional sets are considered empty.

For Proposition 5.3.3, let $R \subseteq Q^{2k}$. x R-reduces to y if and only if x R y and $x >_{lex} y$. A basis for $R \subseteq Q^{2k}$ is an $S \subseteq Q^k$ where for all $x \in Q^k$, $x \in S$ if and only if x does not R-reduce to any $y \in S$. Bases generally do not exist because $Q^k \backslash S$ is too large. For # bases, replace Q^k by #(S), where #(S) is the least $D^k \supseteq S \cup \{0\}^k$. The upper shift of $S \subseteq Q^k$ is obtained by adding 1 to all nonnegative coordinates of elements of S. E.g., ush({(1,-1/2),(0,1/3)}) = {(2,-1/2),(1,4/3)}.

Maximal roots in sets R of ordered pairs are closely related to maximal cliques in graphs. The connection is discussed in detail in section 4.5.

The sense in which Proposition 4.3.1 and 5.3.3 are concrete is addressed here in six ways.

a. The statements call for a (usually) infinite set of rational vectors (the maximal nonnegative root or the # basis), satisfying a requirement referring to finite objects (rational vectors). In the usual complexity hierarchies used in mathematical logic, the statements are explicitly Σ^1_1. This is of far lower complexity than all of the well known general set theoretic incompleteness from ZFC (e.g., continuum hypothesis), which are Σ^2_1 and higher. The well known incompleteness from ZFC in the projective hierarchy are Π^1_3 and higher. Furthermore, Σ^1_1 statements are subject to various strong forms of absoluteness and recursion theoretic witnessing which the aforementioned incompleteness are not. In particular, our statements are independent of ZFC together with Gödel's Axiom of Constructibility (V = L), in contrast to the well known independent set theoretic statements.

b. If the (usually) infinite set of rational vectors (the maximal clique) is required to be reasonably concrete, then the resulting modified statements are equivalent to their original formulations. Here we cannot go so far as to require that the set of rational vectors be recursive, for

then both statements can be refuted. We can, instead, require that the set of rational vectors be first order definable in the field of rationals. By [Ro49], this requirement is the same as being arithmetical in the sense of recursion theory. This results in statements of complexity just beyond the arithmetical, which is much lower than $\sum^1_1$. In fact, we can require that the sets of rational vectors be in class Δ^0_2 (recursive in 0'), which puts the statements in arithmetic form (Π^0_1). See section 6. This eliminates the use of infinite sets in the statements. Of course the use of an infinite ambient space (Q) is required.

c. Proposition 4.3.1 can be a priori put in the form that certain sentences in first order predicate calculus have a countable model. By Gödel's Completeness Theorem, this is equivalent to consistency under the usual axioms and rules of first order predicate calculus. This puts Proposition 4.3.1 in Π^0_1 form, much lower than the forms arising from a,b. A variant of Proposition 5.3.3, specifically Proposition 5.3.9, can also be a priori put into Π^0_1 form. See Theorems 4.1.4 and 5.3.10.

d. Both statements have associated finite forms involving finite approximations to the required (usually) infinite set of rational vectors. We obtain explicitly Π^0_1 and Π^0_2 forms. Upper bounds are placed on the Π^0_2 forms to put them in explicitly Π^0_1 form. These are discussed in sections 7.1 - 7.3.

e. Both statements are provably falsifiable. This means that it is provable that if the statement is false then it is automatically provably false. This can be viewed as a familiar kind of concreteness. See section 6.

f. Finally, there is an adjustment of the notion of # basis used for Proposition 5.3.3, to Ω bases, where Ω bases can be finite. This leads to natural equivalents of the Proposition 5.3.3 that are explicitly Π^0_2 and even explicitly Π^0_1. See section 7.4.

Mathematicians have been fully content to operate under an entirely informal notion of Perfectly Mathematically Natural when formulating theorems and conjectures. They recognize the Perfectly Mathematically Natural when they

see it. There may be disagreements, but there is a considerable agreement on the thrust of the idea.

There has generally been no particular need to precisely analyze the notion of Perfectly Mathematically Natural. This is a major challenge that will require important new ideas of a foundational and philosophical nature. Although we feel that a deep and productive analysis can be achieved, the matter lies well beyond the scope of this paper.

In particular, at least some Perfectly Mathematically Natural statements are in that category independently of the present stage in the history of mathematics. Whereas some mathematical statements may be Perfectly Mathematically Natural in virtue of the particular path that mathematics has taken, others have a Perfect Mathematical Naturalness that either transcends existing mathematical developments altogether or depends on very minimal aspects of it. We contend that Propositions 4.3.1 and 5.3.3 are in this category.

This contention for Proposition 4.3.1 can at this point be justified in a clearer way than for Proposition 5.3.3. This is because of the particularly vivid Master Template/≥ 0 mentioned in the abstract: Does R have a T invariant maximal nonnegative root? Here $R, T \subseteq Q^{2k}$ are order theoretic. This covers the Max Template/≥ 0: Does every order invariant R on Q^k have a T invariant maximal nonnegative root? Here $T \subseteq Q^{2k}$ is order theoretic. See Theorem 4.1.7. Motivation for the passage from the first of these to the second is provided in section 4.1. Proposition 4.3.1 is an instance of the Max Template/≥ 0. See section 9 for more discussion of these and other Templates.

For our CMI (Concrete Mathematical Incompleteness) program, originated in 1967, the Perfectly Mathematically Natural idea plays a central role. This is because we already know from Gödel's Second Incompleteness theorem that Con(ZFC) is an example of both

PERFECTLY INTELLECTUALLY NATURAL CONCRETE INCOMPLETENESS OF ZFC
MATHEMATICALLY UNNATURAL CONCRETE INCOMPLETENESS OF ZFC

Here ZFC Incompleteness refers to not being provable or refutable in ZFC. This is a special case of the Gödel Second Incompleteness Theorem, which is (a general form of what is) proved in [Go31]. Concrete here reflects that Con(ZFC) involves only finite mathematical objects (finite strings of symbols).

But Con(ZFC) is very far from being Mathematically Natural. Contrast the subject matter and notions involved in understanding the significance of Con(ZFC) with that of typical well known results in analysis, algebra, topology, number theory, differential equations, or combinatorics. Of course, Con(ZFC) is obviously mathematical when formulated using only the Perfectly Mathematically Natural finite strings of symbols. However, when so formulated, the statement loses any mathematical naturalness.

For a particularly famous example of the

PERFECTLY MATHEMATICALLY NATURAL CONCRETE COMPLETENESS OF ZFC

there is of course Fermat's Last Theorem (FLT), which was proved in [Wi95] using somewhat more than ZFC, with the proof subsequently modified to lie well within ZFC. FLT asserts that the equation $x^n + y^n = z^n$ has no solutions in positive integers x, y, z, n with $n \geq 3$. Here Completeness simply refers to the fact that FLT is provable or refutable in ZFC – in this case, provable in ZFC. See [Mc10], [Mc11].

From the work of Gödel and Cohen on the continuum hypothesis (CH), [Go40], [Co63,64], we have the first example of

PERFECTLY MATHEMATICALLY NATURAL NONCONCRETE INCOMPLETENESS OF ZFC

CH is Perfectly Mathematically Natural. However, CH is very far from being concrete in the normal mathematical sense, as it involves arbitrary sets of real numbers. As a rough guide, the level of concreteness corresponds to how far away one is from actual computation. In the hierarchy of concreteness in mathematics, CH is a large number of major levels more remote from computation than FLT. See [Fr15], Introduction, for an extended and detailed discussion of the hierarchy of concreteness.

The focus of CMI is on uncovering

PERFECTLY MATHEMATICALLY NATURAL CONCRETE INCOMPLETENESS OF ZFC

and the first examples of such are being presented here as Propositions 4.3.1 and 5.3.3 - as discussed in the Abstract and earlier in this Introduction - along with some variants in sections 4,5.

The future of the Gödel Incompleteness Phenomena, initiated in [Go31], and more generally, much of the entire area of foundations of mathematics, crucially depends on having Perfectly Mathematically Natural Concrete Incompleteness of ZFC. Here is an elaboration on this point.

Since at least the middle of the 20th century, there has been a steady move in mathematics toward the concrete (toward actual computation) and away from the overtly set theoretic (away from the non computational). We are referring particularly to the statements of celebrated theorems and conjectures.

However, mathematicians will use what it takes to prove results, and normally, concrete theorems are given concrete proofs – or proofs that can be readily adjusted to be concrete. This is not always the case. Most notably, the current proofs of FLT, although well within ZFC, are far from concrete. It has been conjectured by many that FLT has a concrete proof, although this has not yet been satisfactorily established. See [Mc10], [Mc11]. In particular, we have conjectured that FLT can be proved in EFA (see [Av03]). However, no proof of FLT even in Peano Arithmetic has been adequately documented.

This move to focus on the concrete and away from the overtly set theoretic has been accelerating. We believe that this move is inevitable and compelling from foundational and philosophical considerations alone - but it is also being spurred by other factors.

One major factor is the overwhelming effect of computer technology on mathematics. Concrete mathematics often – arguably usually – suggests rich computer investigations of various kinds. In the early years, these involved computations in theory only. But with the development of

great computing power, these rich computer investigations involve actual computation using real computers.

In fact, the first Perfectly Mathematically Natural Concrete Incompleteness of ZFC, presented here, has already led to associated rich computer investigations of a fully practical nature, yet to be carried out. They are in the design stages. See [Fr14b].

Another factor is the Gödel Incompleteness Phenomena itself, initiated by [Go31]. The Gödel Incompleteness Phenomena is generally regarded by the mathematics community as something that originally threatened to have a spectacular impact on mathematics in terms of throwing its foundations into utter chaos - but has receded into obscurity as a harmless distraction, of interest only to specialists working in the rather esoteric branch of mathematics known as set theory, and more broadly, mathematical logic.

Mathematicians generally do not think that set theoretic issues genuinely pertain to real mathematical concerns. They find set theoretic formulations very convenient when they do not distract from serious mathematical issues, and rather easy to happily avoid when they cause their own difficulties.

Furthermore, set theoretic issues, if taken seriously, require a reconsideration of the appropriate axioms for mathematics, and therefore a reconsideration of what constitutes a genuine mathematical proof – something that would push mathematics into highly controversial and unfamiliar foundational and philosophical territory. This is viewed as a distraction that should be avoided at all reasonable costs.

Thus it is inevitable that Gödel Incompleteness will always remain an obscure and unwanted side issue for mathematics as long as it does not enter into the realm of the Perfectly Mathematically Natural Concrete.

From the mathematician's perspective, the most natural move is to regard discoveries like Propositions 4.3.1 and 5.3.3 as isolated curiosities that can be safely ignored. This justifies our major effort to put these and related statements into Perfectly Natural strategic contexts.

Here the strategic contexts take the form of Perfectly Natural Templates, or families of concrete statements. The strategic programs aim to determine which of these statements are true. As indicated in the Abstract, we can start strategically with the vivid templates

MASTER TEMPLATE. R HAS A T INVARIANT MAXIMAL ROOT
$R, T \subseteq Q^{2k}$ are order theoretic.

MASTER TEMPLATE/≥0. R HAS A T INVARIANT MAXIMAL NONNEGATIVE ROOT
$R, T \subseteq Q^{2k}$ are order theoretic.

MAX TEMPLATE. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T INVARIANT MAXIMAL ROOT
$T \subseteq Q^{2k}$ is order theoretic.

MAX TEMPLATE/≥0. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T INVARIANT MAXIMAL NONNEGATIVE ROOT
$T \subseteq Q^{2k}$ is order theoretic.

Using Gödel's Completeness Theorem, we can put every instance of all four of these Templates into a priori Π^0_1 form - see section 4.1.

Results concerning Proposition 4.3.1 show that there are instances of the Master Template/≥0 and the Max Template/≥0 which cannot be proved or refuted in ZFC. Here ZFC can be replaced by any SRP[n] fixed in advance. Here the strong instances involve reasonably simple T, and in the case of Master Template/≥0, artificial R. Note how the Max Template and the Max Template/≥0 Template wash out artificial R.

We conjecture that every instance of all four of these Templates is provable or refutable in SRP, which is ZFC augmented with the existence of certain well studied large cardinal hypotheses, generally accepted by the set theory community. See section 9 for discussion.

We give two simple sufficient conditions on T for the Max Template and the Max/≥0 Template (section 4.2). We also view these as partial results on the Master Template and the Master Template/≥0. The proof of sufficiency uses Con(SRP). In fact, sufficiency for the Max Template/≥0 is equivalent to Con(SRP). See section 4.2.

In section 5.4, we use ≤# bases, which are weaker than # bases, and support requirements on their projections. This propels us into HUGE instead of SRP, with Proposition 5.4.3.

In section 8, we use # and ≤# bases to give a reasonably simple mathematical characterization of the provable ordinals of SRP and HUGE in the sense of proof theory.

2. SOME PRELIMINARIES

DEFINITION 2.1. N is the set of all nonnegative integers. We use i,j,k,n,m,r,s,t for positive integers unless indicated otherwise. Q is the set of all rational numbers. For $x,y \in Q^k$, $x <_{lex} y$ if and only if for the least i such that $x_i \neq y_i$, we have $x_i < y_i$ (the usual lexicographic ordering). As usual, subsequences of sequences pick out terms going forward, where terms may be skipped over. Blocks in sequences are subsequences going forward consecutively. All sequences are nonempty.

DEFINITION 2.2. Let $S \subseteq Q^k$, $S^= = \{x \in S: x_1 \leq ... \leq x_k\}$, and analogously for $S^<$, $S^=$, $S^>$. $S|>p = S \cap (p,\infty)^k$, and analogously for $S|<p$, $S|\leq p$, $S|\geq p$. For $x \in Q^n$, $S_x = \{y: (x,y) \in S\}$. Note that for $n \geq k$, $S_x = \emptyset$. S_x is called the projection of S at x. The projections of S are the S_x for finite sequences x from Q. S below p, and S above p, are $S|<p$, and $S|>p$, respectively. $fld(S) \subseteq Q$ is the set of all coordinates of elements of S.

It is simplest to treat relations, roots, maximal roots, and invariance, without reference to any ambient space. We adopt a convenient setup for this. However, order theoretic does refer to the ambient spaces Q^k.

DEFINITION 2.3. Let R be a set of ordered pairs. $x R y \leftrightarrow (x,y) \in R$. S is a root in R if and only if $S^2 \subseteq R$. S is a maximal root in R if and only if S is a root in R which is not a proper subset of any root in R. Let $R \subseteq Q^{2k}$. S is a nonnegative root in R if and only if $S^2 \subseteq R|\geq 0$. S is a maximal nonnegative root in R if and only if S is a nonnegative root in R which is not a proper subset of any nonnegative root in R.

DEFINITION 2.4. S is T invariant if and only if S,T are sets of ordered pairs, where $(\forall x \in S)(\forall y)(x\ T\ y \rightarrow y \in S)$. T is an equivalence relation on X if and only if $T \subseteq X^2$ is symmetric, transitive, reflexive on X. T is an equivalence relation if and only if T is an equivalence relation on some X.

THEOREM 2.1. (RCA_0) Let S,T be sets of ordered pairs. S is T invariant if and only if $T[S] \subseteq S$. If, furthermore, T is an equivalence relation, then S is T invariant if and only if S is the union of equivalence classes of T.

In section 4, we focus on maximal roots in sets R of ordered pairs. This is closely related to maximal cliques in graphs G, and in section 4.5, we recast the results of section 4 in terms of graphs.

DEFINITION 2.5. A graph is a pair G = (V,E), where V is a set of vertices, and $E \subseteq V^2$ is the edge relation, which is required to be irreflexive and symmetric. S is a clique in G if and only if any two distinct elements of S are related by E. S is a maximal clique in G if and only if S is a clique in G which is not a proper subset of any clique in G.

Note that graphs do require ambient spaces - the set of vertices.

The two featured statements in the abstract, Propositions 4.3.1 and 5.3.3, correspond to the stationary Ramsey property hierarchy (SRP), otherwise known as the subtle or ineffable cardinal hierarchy. See [Fr14a], section 10.1 for background on the SRP hierarchy of large cardinals.

DEFINITION 2.6. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP is read "stationary Ramsey property".

DEFINITION 2.7. A Π^0_1 sentence is a sentence asserting that some given Turing machine never halts starting with the empty input tape. A Π^0_2 sentence is a sentence asserting that some given Turing machine halts starting with every

given finite input tape. A sentence φ is implicitly Π^0_1 over a theory T if and only if there is a Π^0_1 sentence ψ such that T proves $\varphi \leftrightarrow \psi$.

A full list of the formal systems that we will use are presented in the section 10. In section 5.4 we use the vastly stronger formal system HUGE.

3. ORDER THEORETIC AND ORDER INVARIANT

The results in this section are well known and the proofs can be safely left to the reader. Model theorists know the order theoretic subsets of Q^k as the definable subsets of Q^k over $(Q,<)$, with parameters, and the order invariant subsets of Q^k as the definable subsets of Q^k over $(Q,<)$, without parameters. This is one of the most elementary logic textbook contexts for elimination of quantifiers.

We begin with the notion of order theoretic, which is particularly robust. With order theoretic, we use the ambient spaces Q^k.

DEFINITION 3.1. $E \subseteq Q^k$ is order theoretic if and only if E is of the form $\{x \in Q^k: \varphi\}$, where φ is a finite propositional combination of formulas $x_i < x_j$, $x_i < p$, $p < x_i$, with $1 \leq i,j \leq k$ and $p \in Q$.

DEFINITION 3.2. An automorphism of $(Q,<)$ is a strictly increasing bijection $h:Q \to Q$. Each such $h:Q \to Q$ lifts to the bijection $h:Q^k \to Q^k$ by acting on coordinates. Each such $h:Q \to Q$ then also lifts to the bijection $h:\wp(Q^k) \to \wp(Q^k)$ by forward imaging. h fixes $x \in Q^k$ if and only if $h(x) = x$. h fixes $E \subseteq Q^s$ if and only if $h[E] = E$.

DEFINITION 3.3. More generally, let $A \subseteq Q$. An automorphism of $(Q,<,A)$ is a strictly increasing bijection $h:Q \to Q$ such that for all $p \in Q$, $h(p) \in A \leftrightarrow p \in A$. Note that if A is finite then h fixes every element of A.

THEOREM 3.1. (RCA_0) Let $E \subseteq Q^k$. The following are equivalent.
i. There exists finite $A \subseteq Q$ such that E is fixed under all automorphisms of $(Q,<,A)$.
ii. $E \subseteq Q^k$ is order theoretic.

The p's used in Definition 3.1 for order theoretic $E \subseteq Q^k$ are called parameters. It is sometimes important to keep track of them.

DEFINITION 3.4. $E \subseteq Q^k$ is order theoretic in $A \subseteq Q$ if and only if E is of the form $\{x \in Q^k: \varphi\}$, where φ is a finite propositional combination of formulas $x_i < x_j$, $x_i < p$, $p < x_i$, with $1 \leq i,j \leq k$ and $p \in A$.

THEOREM 3.2. (RCA$_0$) Let $E \subseteq Q^k$ and $A \subseteq Q$ be finite. The following are equivalent.
i. E is fixed under all automorphisms of $(Q,<,A)$.
ii. $E \subseteq Q^k$ is order theoretic in A.

THEOREM 3.3. (EFA) For finite $A \subseteq Q$, there are finitely many $E \subseteq Q^k$ order theoretic in A.

We now focus on the crucial case of $A = \emptyset$. Theorem 3.2 provides two equivalent definitions, but there is another important definition involving invariance with respect to certain fundamental equivalence relations.

DEFINITION 3.5. Order equivalence on Q^k is the equivalence relation E on Q^k defined by: x E $y \leftrightarrow (\forall i,j)(1 \leq i,j \leq k \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$. $S \subseteq Q^k$ is order invariant if and only if S is E invariant, where E is order equivalence on Q^k.

THEOREM 3.4. The number of equivalence classes in order equivalence on Q^k is finite.

THEOREM 3.5. (RCA$_0$) Let $E \subseteq Q^k$. The following are equivalent.
i. E is fixed under all automorphisms of $(Q,<)$.
ii. $E \subseteq Q^k$ is order theoretic in $\emptyset$.
iii. E is of the form $\{x \in Q^s: \varphi\}$, where φ is a finite propositional combination of formulas $x_i < x_j$, with $1 \leq i,j \leq k$.
iv. $E \subseteq Q^k$ is order invariant.

4. INVARIANT MAXIMAL NONNEGATIVE ROOTS

4.1. BACKGROUND

THEOREM 4.1.1. (ZF) The following are equivalent.
i. Every set of ordered pairs has a maximal root.
ii. In every set of ordered pairs, every root extends to a maximal root.
iii. The Axiom of Choice.

Proof: iii → i,ii is by a well known application of the axiom of choice through Zorn's Lemma. Now assume i. Let A be a pairwise disjoint set of nonempty sets. Let R be the relation on $\cup A$, where x R y if and only if $x,y \in \cup A \wedge \neg (\exists z \in A)(x,y \in z)$. Any maximal clique in R is a set which has exactly one element from each element of A. QED

THEOREM 4.1.2. (RCA_0) In every set of ordered pairs from a countable set, every finite root extends to a maximal root. The following are equivalent.
i. In every set of ordered pairs from a countable set, every root extends to a maximal root.
ii. In every order invariant $R \subseteq Q^2$, every root extends to a maximal root.
iii. In every order invariant $R \subseteq Q^2$, every nonnegative root extends to a maximal nonnegative root.
iv. ACA_0.

Proof: The first claim is obvious from an inductive construction using an enumeration of the countable set. iv → i → ii ∧ iii is also obvious by such an inductive construction. It remains to show ii → iv, and iii → iv. Assume ii. We claim that the range of every f:Q → Q without a fixed point exists. Let R be the relation on Q^2 given by (p,q) R (r,s) if and only if

i. $p \neq q \wedge r \neq s \wedge (p = r \rightarrow q = s)$; or
ii. $p = q \wedge r = s$; or
iii. $p = q \wedge r \neq s \wedge p \neq s$; or
iv. $p \neq q \wedge r = s \wedge r \neq q$.

It is clear that f is a root in R. Let $f \subseteq S$ where S is a maximal root in R. Then $S \cap \{(p,q): p \neq q\}$ is a function extending f, and hence is f. Clearly $(p,p) \in S \leftrightarrow p \notin rng(f)$. Hence Q\rng(f) exists. Therefore rng(f) exists.

In order to derive iv, we need that the range of every g:Q → Q exists. Let g:Q → Q. Let f:Q → Q be defined by f(2n) = 2g(n)+1; f(2n+1) = 0. Then f has no fixed points and rng(f)

= 2rng(g)+1 ∪ {0}. By the claim, rng(f) exists, and therefore rng(g) exists.

Now assume iii. We argue as above that the range of every $f:Q|{\geq}0 \to Q|{\geq}0$ without a fixed point exists. Here f is a nonnegative root in R. We then argue as above that the range of every $f:Q|{\geq}0 \to Q|{\geq}0$ exists, which implies iv. QED

We now consider the Perfectly Mathematically Natural Concrete Templates

MASTER TEMPLATE. R HAS A T INVARIANT MAXIMAL ROOT
$R,T \subseteq Q^{2k}$ are order theoretic.

MASTER TEMPLATE/≥0. R HAS A T INVARIANT MAXIMAL NONNEGATIVE ROOT
$R,T \subseteq Q^{2k}$ are order theoretic.

THEOREM 4.1.4. Let $R,T \subseteq Q^k$ be order theoretic. "R has a T invariant maximal root" and "R has a T invariant maximal nonnegative root" are implicitly Π^0_1 over WKL_0.

Proof: First write R,T as $\{x \in Q^{2k}: \varphi\}$, $\{x \in Q^{2k}: \psi\}$, where φ,ψ are quantifier free formulas in $<,x_1,...,x_k$, with combined parameters $p_1 < ... < p_r$. $\alpha(R,T)$ uses $<$, constant symbols $c_1,...,c_r$, and the 2k-ary predicate symbol S. The axioms of $\alpha(R,T)$ are as follows.

1. $<$ is a strict dense linear ordering with no endpoints.
2. $c_1 < ... < c_r$.
3. $S(x) \land x \neq y \land S(x) \land S(y) \to \varphi(x,y)) \land (\forall x)(\neg S(x) \to (\exists y)(x \neq y \land S(y) \land (\neg\varphi(x,y) \lor \neg\varphi(y,x))))$.
4. $S(x) \land \psi(x,y) \to S(y)$.

Here x abbreviates $x_1,...,x_k$, and y abbreviates $y_1,...,y_k$, where $x_1,...,x_k,y_1,...,y_k$ are 2k distinct variables. Also φ,ψ use $c_1,...,c_r$ in place of $p_1,...,p_r$.

Suppose there is a T invariant maximal root in R. Let S be such. Then $\alpha(R,T)$ has an obvious model. On the other hand, suppose $\alpha(R,T)$ has a model $(D,<,c_1,...,c_r,S)$. We can assume D is countably infinite. The model is clearly isomorphic to $(Q,<,p_1,...,p_r,S')$, for some $S' \subseteq Q^{2k}$. Clearly S' is a T invariant maximal clique in R. So far we have only used

RCA_0.

We now use the formalized Gödel Completeness Theorem, which is available in WKL_0. There is a T invariant maximal clique in R if and only if $\alpha(R,T)$ is formally consistent. This is a Π^0_1 assertion.

The argument for "R has a T invariant maximal nonnegative root" is the same, except that we use an additional constant symbol 0 for the left endpoint. QED

So all instances of the two Master Templates are concrete in the sense of Theorem 4.1.4. In section 7.1 we present a graded form of the Master Templates where the instances are all explicitly Π^0_1.

A natural idea for obtaining substantial sufficient conditions is to find a variety of T's such that an extensive class of R's have T invariant maximal cliques. In its crudest form, the idea is to find order theoretic $T \subseteq Q^{2k}$ such that every order theoretic $R \subseteq Q^{2k}$ has a T invariant maximal root.

THEOREM 4.1.5. (RCA_0) Let $T \subseteq Q^{2k}$. The following are equivalent.
i. Every order theoretic $R \subseteq Q^{2k}$ has a T invariant maximal root.
ii. $(\forall x, y \in Q^k)(x \, T \, y \rightarrow x = y)$.

Proof: Let $T \subseteq Q^k$. Clearly ii $\rightarrow$ i. Suppose i, and let x T y, $x \neq y$. Define $R \subseteq Q^{2k}$ by $z \, R \, w \leftrightarrow z = x$. Then R has the unique maximal root {x}, which is not T invariant. QED

THEOREM 4.1.6. (RCA_0) Let $T \subseteq Q^{2k}|{\geq}0$. The following are equivalent.
i. Every order theoretic $R \subseteq Q^{2k}$ has a T invariant maximal nonnegative root.
ii. $(\forall x, y \in Q^k|{\geq}0)(x \, T \, y \rightarrow x = y)$.

Proof: See the proof of Theorem 4.1.5. QED

Note how the definition of R in the above proofs utilize parameter x in an essential way. This suggests a more refined plan. Look for order theoretic $T \subseteq Q^{2k}$ such that every order invariant $R \subseteq Q^{2k}$ has a T invariant maximal

(nonnegative) clique. Thus we arrive at the templates

MAX TEMPLATE. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T
INVARIANT MAXIMAL ROOT
$T \subseteq Q^{2k}$ is order theoretic.

Our main results use nonnegative roots rather than general roots. Thus in the next section, we focus on the template

MAX TEMPLATE/≥0. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T
INVARIANT MAXIMAL NONNEGATIVE ROOT
$T \subseteq Q^{2k}$ is order theoretic.

THEOREM 4.1.6. Every instance of the Max Template is provably equivalent, over RCA_0, to a conjunction of finitely many instances of the Master Template. Every instance of the Max/≥0 Template is provably equivalent, over RCA_0, to a conjunction of finitely many instances of the Master Template/≥0.

Proof: This is an immediate consequence of the fact that there are only finitely many order invariant $R \subseteq Q^{2k}$. QED

4.2. SUFFICIENT CONDIIITONS

In this section, we give two sufficient conditions for an order invariant R to have a T invariant maximal nonnegative root. The first condition links up well with [Fr14a], and we adapt it to give a proof of sufficiency here in WKL0 + Con(SRP). The second sufficient condition is couched in terms of partial embeddings, and will be proved elsewhere.

DEFINITION 4.2.1. Let $T \subseteq Q^{2k}$. T is order preserving if and only if $(\forall x,y)(x\ R\ y \rightarrow x,y$ are order equivalent). T has property * if and only if $\{x_i \leq x_j < y_j: x\ T\ y\}$ is a finite subset of Q^3.

PROPOSITION 4.2.1. Let $T \subseteq Q^{2k}|{\geq}0$ be an order preserving equivalence relation with property *. Every order invariant $R \subseteq Q^{2k}$ has a T invariant maximal nonnegative root.

Note that there are continuumly many T obeying the hypothesis of Proposition 4.2.1, and so Proposition 4.2.1

is prima facie far away from the concrete. However, we shall see that any such T is contained in some such order theoretic T'. So we give the following more down to earth equivalent formulation.

PROPOSITION 4.2.2. Let $T \subseteq Q^{2k}|{\geq}0$ be an order theoretic order preserving equivalence relation with property *. Every order invariant $R \subseteq Q^{2k}$ has a T invariant maximal nonnegative root.

We prove Propositions 4.2.1 and 4.2.2 by necessarily going far beyond the usual ZFC axioms for mathematics. We first show that the T involved are contained in a specific class of order theoretic equivalence relations.

DEFINITION 4.2.2. Let $A \subseteq Q$ be finite and $x,y \in Q^k$. The bottom A part of x is the subsequence of x consisting of all x_i such that every $x_j \leq x_i$ lies in A. The off-bottom A part of x is the subsequence of x consisting of the x_i not on the bottom part of x. The equivalence relation $BEQ(A,Q^k)$ on Q^k is defined as follows. x $BEQ(A,Q^k)$ y if and only if x,y are order equivalent $\land$ the off-bottom A parts of x,y are identical. $BEQ(A,Q^k|{\geq}0) = BEQ(A,Q^k)|{\geq}0$ is the corresponding equivalence relation on $Q^k|{\geq}0$.

Here BEQ abbreviates "bottom equivalence relation".

LEMMA 4.2.3. (RCA$_0$) Let $T \subseteq Q^{2k}|{\geq}0$ be an order preserving equivalence relation with property *. There exists finite A $\subseteq Q|{\geq}0$ such that $T \subseteq BEQ(A,Q^k|{\geq}0)$.

Proof: Let T be as given. Let A be the set of coordinates of elements of $\{x_i \leq x_j < y_j: x$ T $y\} \subseteq Q^3|{\geq}0$. Obviously A $\subseteq$ $Q|{\geq}0$ is finite. Fix x T y. We show that x $BEQ(A,Q^k|{\geq}0)$ y.

Let $x_j < y_j$. Obviously $x_j,y_j \in A$. If $x_i < x_j$ then $x_i \in A$ by the definition of A. Therefore x_j is in the bottom A part of x. To show that y_j is in the bottom A part of y, suppose $y_i < y_j$. Then $x_i < y_i$, and so $y_i \in A$. Hence y_j is in the bottom A part of y.

We have shown that $x_j \neq y_j \rightarrow x_j$ is in the bottom A part of x $\land$ y_j is in the bottom A part of y.

Suppose x_i is off the bottom A part of x. By the previous

paragraph, $x_i = y_i$ and y_i is in the off-bottom A part of y. By the same argument, if y_i is in the off-bottom A part of y then x_i is in the off-bottom A part of x and $x_i = y_i$. Hence x,y have the same off-bottom A parts. Therefore x BEQ(A,Q[0,∞)^k) y. QED

PROPOSITION 4.2.4. Let A ⊆ Q|≥0 be finite. Every order invariant R ⊆ Q^{2k} has a BEQ(A,Q^k|≥0) invariant maximal nonnegative root.

LEMMA 4.2.5. (RCA$_0$) Propositions 4.2.1, 4.2.2, 4.2.4 are equivalent. This also holds for any fixed k.

Proof: 4.2.1 → 4.2.2 → 4.2.4 since the BEQ(A,Q^k|≥0) from Proposition 4.2.4 obeys the condition in Proposition 4.2.1. Also 4.2.4 → 4.2.1 follows from Lemma 4.2.3. QED

We now use [Fr14a] and [Fr11] to prove Proposition 4.2.4 in WKL$_0$ + Con(SRP). The result that we use from [Fr14a] lives in Q^k, and uses a strengthened kind of maximal clique called step maximal clique. It uses the equivalence relation Nteq which is closely related to our BEQ(N,Q^k|≥0). It is thus convenient to convert to graphs with the following Proposition.

PROPOSITION 4.2.6. Let A ⊆ Q|≥0 be finite. Every order invariant graph on Q^k has a BEQ(A,Q^k|≥0) invariant maximal clique.

THEOREM 4.2.7. (RCA$_0$) Propositions 4.2.1, 4.2.2, 4.2.4, 4.2.6 are equivalent. This also holds for any fixed k.

Proof: It suffices to show that Propositions 4.2.4 and 4.2.6 are equivalent. This will be proved in section 4.6. QED

DEFINITION 4.2.3. Let A ⊆ Q and x,y ∈ Q^k|≤n. The top A part of x is the subsequence of x consisting of all x_i such that every $x_j ≥ x_i$ lies in A. The off-top A part of x is the subsequence of x consisting of the x_i not on the top part of x. The equivalence relation TEQ(A,Q^k|≤n) on Q^k|≤n is defined as follows. x TEQ(A,Q^k|≤n) y if and only if x,y ∈ Q^k|≤n are order equivalent ∧ the off-top A parts of x,y are identical.

PROPOSITION 4.2.8. Let $A \subseteq Q|{\leq}n$ be finite. Every order invariant graph on $Q^k|{\leq}n$ has an TEQ$(A,Q^k|{\leq}n)$ invariant maximal clique.

PROPOSITION 4.2.9. Every order invariant graph on $Q^k|{\leq}n$ has an TEQ$(\{0,\ldots,n\},Q^k|{\leq}n)$ invariant maximal clique.

LEMMA 4.2.10. (RCA$_0$) TEQ$(A,Q^k|{\leq}n) \subseteq$ TEQ$(A \cup \{n\},Q^k|{\leq}n)$.

Proof: Suppose x TEQ$(A,Q^k|{\leq}n)$ y. Then x,y are order equivalent, and x,y are identical off their top A parts. Suppose x_i is on the off-top $A \cup \{n\}$ part of x. Let $x_j \geq x_i$ lie outside $A \cup \{n\}$. Then x_i and x_j are on the off-top A part of x, and so $x_i = y_i \wedge x_j = y_j$. Hence y_i is on the off-top $A \cup \{n\}$ part of y because $y_j \geq y_i$ and $y_j = x_j \notin A \cup \{n\}$. Thus we have shown that if x_i is on the off-top $A \cup \{n\}$ part of x then $y_i = x_i$ is on the off-top $A \cup \{n\}$ part of y. By symmetry, we see that x,y have the same off-top A parts. QED

LEMMA 4.2.11. (RCA$_0$) Propositions 4.2.1, 4.2.2, 4.2.4, 4.2.6, 4.2.8, 4.2.9 are equivalent. This also holds for any fixed k.

Proof: It suffices to show 4.2.6, 4.2.8, 4.2.9 are equivalent. Proposition 4.2.6 and 4.2.8 for $n = 0$ are equivalent by duality. Using arbitrary $n \geq 1$ clearly makes no difference for these order theoretic statements. With Proposition 4.2.9, we are only considering $A = \{0,\ldots,n\}$. This amounts to restricting to the case where the right endpoint, n, lies in A. By Lemma 4.2.10, this restriction is inconsequential. QED

The following is from [Fr14a] and the earlier [Fr11].

DEFINITION 4.2.4. Let R be a graph on Q^k. S is a step maximal clique if and only if $S \subseteq Q^k$ and each $S|{\leq}n$, $n \geq 0$, is a maximal clique in $R|{\leq}n$. Nteq is the equivalence relation on Q^k given by x Nteq $y \leftrightarrow x,y$ are order equivalent and x,y have the same off-top N parts. I.e., Nteq is TEQ(N,Q^k).

The following Proposition appears in [Fr14a] as ISMR(Nteq).

PROPOSITION 4.2.12. Every order invariant graph on Q^k has a TEQ(N,Q^k) invariant step maximal clique.

LEMMA 4.2.13. (WKL$_0$) Proposition 4.2.12 is equivalent to Con(SRP).

Proof: For the derivation of Proposition 4.2.12 from Con(SRP), see [Fr14a]. For the reversal, use the statement IMCT in [Fr11], p. 19. In [Fr11], Theorem 5.9.3, IMCT is shown to imply Con(SRP) over ACA'. Subsequent ideas reduce the base theory to WKL$_0$. QED

THEOREM 4.2.14. (WKL$_0$) Propositions 4.2.1, 4.2.2, 4.2.4, 4.2.6, 4.2.8, 4.2.9, 4.2.12 are equivalent to Con(SRP).

Proof: It remains only to derive Proposition 4.2.9 from Proposition 4.2.12. Let G be an order invariant graph on $Q^k|{\leq}n$. Lift G to G' on Q^k, and let S be a TEQ(N,Q^k) invariant step maximal clique. Then S is a TEQ({0,...,n},$Q^k|{\leq}n$) invariant maximal clique in G'$|{\leq}n$ = G. QED

We now give our second sufficient condition.

DEFINITION 4.2.5. Let S $\subseteq$ Q[0,∞)^k. h is a partial self embedding of S if and only if h:Q $\to$ Q is a one-one partial function such that for all $p_1,\ldots,p_k \in$ dom(h), $(p_1,\ldots,p_k) \in$ S $\leftrightarrow$ (h(p_1),...,h(p_k)) $\in$ S. For partial h:Q $\to$ Q, diag(h,k):$Q^k \to Q^k$ has domain dom(h)k sending ($p_1,\ldots,p_k$) to (h(p_1),...,h(p_k)).

THEOREM 4.2.15. h partially self embeds S $\subseteq$ Q[0,∞)^k if and only if S is diag(h,k) $\cup$ diag(h^{-1},k) invariant.

Proof: Suppose h partially self embeds S. Let $(p_1,\ldots,p_k) \in$ S. Then the at most two images of the relation diag(h,k) $\cup$ diag(h^{-1},k) lie in S. So S is diag(h,k) $\cup$ diag(h^{-1},k) invariant. Suppose S is diag(h,k) $\cup$ diag(h^{-1},k) invariant. Let $p_1,\ldots,p_k \in$ dom(h). We have $(p_1,\ldots,p_k) \in$ S $\to$ (h(p_1),...,h(p_k)) by diag(h,k). Applying diag(h^{-1},k) to (h(p_1),...,h(p_k)), we have (h(p_1),...,h(p_k)) $\in$ S $\to$ $(p_1,\ldots,p_k) \in$ S. Therefore h self embeds S. QED

THEOREM 4.2.16. Let h:Q $\to$ Q be partial. Suppose every order invariant R $\subseteq$ Q^{2k} has an h self embedded maximal nonnegative

root. Then h is strictly increasing.

PROPOSITION 4.2.17. Let $h:Q \rightarrow Q$ be partial, strictly increasing, and move finitely many points. Every order invariant R co Q^{2k} has an h self embedded maximal nonnegative root.

PROPOSITION 4.2.18. Let $h:Q \rightarrow Q$ be partial, strictly increasing, and order theoretic. Every order invariant R co Q^{2k} has an h self embedded maximal nonnegative root.

THEOREM 4.2.19. Proposition 4.2.17 and 4.2.18 are provably equivalent over RCA_0.

Proof: It is obvious that every order theoretic partial $h:Q \rightarrow Q$ moves finitely many points. QED

THEOREM 4.2.20. Propositions 4.2.17 and 4.2.18 are provably equivalent to Con(SRP) over WKL_0. They are provable in SRP for any fixed k.

Theorems 4.2.16 and 4.2.20 will be proved elsewhere. It is easily seen that Proposition 4.2.18 is a implicitly $\Pi01$ over WKL0 via Gödel's Completeness Theorem.

4.3. PROJECTIONS

We now present two immediate applications of Proposition 4.2.1 that represent Perfectly Mathematically Natural Concrete Incompleteness.

PROPOSITION 4.3.1. Every order invariant $R \subseteq Q^k$ has a maximal nonnegative root, where $S_{1...n}|>n = S_{0...n-1}|>n$.

PROPOSITION 4.3.2. Every order invariant $R \subseteq Q^{2k}$ has a maximal nonnegative root, where projections at any two equal length subsequences of $(0,...,k)$ agree above k.

PROPOSITION 4.3.3. Every order invariant $R \subseteq Q^{2k}$ has a maximal nonnegative root, where projections at any two equal length subsequences of $(0,...,n)$ agree above their combined terms.

We also use blocks, which are subsequences whose terms are consecutive.

PROPOSITION 4.3.4. Every order invariant R $\subseteq$ Q^{2k} has a maximal nonnegative root, where projections at any two equal length blocks in (0,...,k) agree above their combined terms.

Note that Propositions 4.3.1 - 4.3.4 are based on order theoretic equivalence relations on the Q^k, and therefore are subject to Theorem 4.1.4. It follows that Propositions 4.3.1 - 4.3.4 are implicitly Π^0_1 over WKL$_0$.

THEOREM 4.3.5. Propositions 4.3.1 - 4.3.4 are provably equivalent to Con(SRP) over WKL$_0$. For fixed k, they are provable in SRP. For fixed n, Propositions 4.3.1 and 4.3.3 are provable in SRP. Furthermore, Propositions 4.3.1 - 4.3.4 prove Con(SRP) over RCA$_0$. For all sufficiently large k,n, there is an order invariant R $\subseteq$ Q^{2k} such that Proposition 4.3.1 for R implies Con(SRP[n]) over RCA$_0$.

Proof: We claim that Propositions 4.3.1 - 4.3.4 are special cases of Proposition 4.2.1. To see this, let T be the following equivalence relation on Q^k|$\geq$0. x T y if and only if there exists 1 $\leq$ i $\leq$ k such that (x$_1$,...,x$_i$) and (y$_1$,...,y$_i$) are subsequences of (0,...,n) and (x$_{i+1}$,...,x$_k$) = (y$_{i+1}$,...,y$_k$) $\in$ (max(x$_i$,y$_i$),∞)$^{k-i}$. T obeys the hypotheses of Proposition 4.2.1. The first two claims now follow from Theorem 4.2.14. The remaining claims will be proved elsewhere. The reversals in [Fr11] are closely related to those of Propositions 4.3.3 and 4.3.4, but farther away from those of the weaker Propositions 4.3.1 and 4.3.2. QED

4.4. MAXIMAL ROOTS

For sections 4.2 and 4.3, the difference between maximal nonnegative roots and maximal roots is substantial. The endpoint 0 is heavily used in the derivations of Con(SRP).

In Q, we find it convenient to reverse order. Compare property # to property *.

DEFINITION 4.4.1. Let T $\subseteq$ Q^{2k}. T is order preserving if and only if ($\forall$x,y)(x R y $\rightarrow$ x,y are order equivalent). T has property #) if and only if {x$_i$ < y$_i$ $\leq$ y$_j$: x T y} is a finite subset of Q^3.

THEOREM 4.4.1. Let T $\subseteq$ Q^{2k} be an order preserving

equivalence relation on Q^k with property * or #. Every order invariant $R \subseteq Q^{2k}$ has a T invariant maximal root.

Theorem 4.4.1 will be proved elsewhere, in Z = Zermelo set theory.

We can apply Theorem 4.4.1 with property # to Propositions 4.3.1 - 4.3.4 with "nonnegative root" replaced by "root". The following is immediate from Theorem 4.4.1.

THEOREM 4.4.2. Every order invariant $R \subseteq Q^{2k}$ has a maximal root, where projections at any two equal length subsequences of (0,...,n) agree below their combined terms.

We now use (0,1,2,...).

PROPOSITION 4.4.3. Every order invariant $R \subseteq Q^{2k}$ has a maximal root, where projections at any two equal length subsequences of (0,1,2,...) agree below their combined terms.

THEOREM 4.4.4. Proposition 4.4.3 is provable in WKL_0 + Con(SRP).

Proof: Use the step maximal statement Proposition 4.2.12 from [Fr14a], as we did in section 4.2. The application is more direct since there is no endpoint in Q. QED

We do not know if Proposition 4.4.3 is provable in ZFC. In fact, we do not know if the following weaker statement is provable in ZFC.

PROPOSITION 4.4.5. Every order invariant $R \subseteq Q^{2k}$ has a maximal root, where projections at length n blocks in (0,1,2,...) agree below 0.

4.5. USING GRAPHS

In general, the relation/root formulations we have been using are equivalent to the corresponding graph/clique formulations.

DEFINITION 4.5.1. Let $R \subseteq X^2$. graph(R,V) is the graph (V,E), where x E y $\leftrightarrow$ x $\neq$ y $\wedge$ ((x,x) $\notin$ R $\vee$ (y,y) $\notin$ R $\vee$ {x,y}^{2} $\subseteq$ R). Let G = (V,E). rel(G) = {(x,y): x = y $\in$ V $\vee$ x E y}.

THEOREM 4.5.1. (RCA$_0$) Let R $\subseteq$ V^2 and G be a graph on V. The roots in R are the cliques in graph(R,V) contained in {x: (x,x) $\in$ R}. The roots in rel(G) are the cliques in G.

Proof: Let R,V,G be as given. Let S be a root in R. then S $\subseteq$ {x: (x,x) $\in$ R}. Let x $\neq$ y be from S. Since S^2 $\subseteq$ R, clearly x E y. So S is a clique in graph(R,V).

Let S $\subseteq$ {x: (x,x) $\in$ R} be a clique in graph(R,V). Let x,y $\in$ S. If x = y then (x,y) $\in$ R. If x $\neq$ y then x E y, and so {x,y}^{2} $\subseteq$ R, and in particular, (x,y) $\in$ R. Hence S is a root in R.

Let S be a root in rel(G). Then S $\subseteq$ V. Let x $\neq$ y be from S. Since S^2 $\subseteq$ R, clearly x E y. So S is a clique in G.

Let S be a clique in G. Let x,y $\in$ S. If x = y then (x,y) $\in$ rel(G). If x $\neq$ y then x E y, and so (x,y) $\in$ rel(G). Hence S is a root in rel(G). QED

THEOREM 4.5.2. (RCA$_0$) Let R $\subseteq$ V^2 and G be a graph on V. The maximal cliques in graph(R,V) are the maximal roots in R unioned with {x $\in$ V: (x,x) $\notin$ R}. The maximal roots in rel(G) are the maximal cliques in G.

Proof: Let R,V,G be as given. Let S be a maximal clique in graph(R,V). Clearly S $\supseteq$ {x $\in$ V: (x,x) $\notin$ R}. We claim that S $\cap$ {x: (x,x) $\in$ R} is a maximal root in R. By Theorem 4.5.2, S $\cap$ {x: (x,x) $\in$ R} is a root in R. Suppose S $\cap$ {x: (x,x) $\in$ R} is a proper subset of the root in R, S'. Then S is a proper subset of the clique S' $\cup$ {x $\in$ V: (x,x) $\notin$ R} in graph(R,V). This is a contradiction.

Let S be a maximal root in R. Then S $\cup$ {x $\in$ V: (x,x) $\notin$ R} is a clique in graph(R,V). We claim that S $\cup$ {x $\in$ V: (x,x) $\notin$ R} is a maximal clique in graph(R,V). Suppose S $\cup$ {x $\in$ V: (x,x) $\notin$ R} is a proper subset of the clique S' in graph(R,V). Then S is a proper subset of the root S' $\cap$ {x: (x,x) $\in$ R}. This is a contradiction.

The second claim is left to the reader. QED

THEOREM 4.5.3. (RCA$_0$). Let R $\subseteq$ Q^{2k} and G be a graph on Q^k.

The maximal nonnegative cliques in graph(R,Q^k) are the maximal nonnegative roots in R unioned with $\{x \in Q^k | \geq 0: (x,x) \notin R\}$. The maximal nonnegative roots in rel(G) are the maximal nonnegative cliques in G.

Proof: Let R,G be as given. Let S be a maximal nonnegative clique in graph(R,Q^k). Clearly $S \supseteq \{x \in Q^k | \geq 0: (x,x) \notin R\}$. We claim that $S \cap \{x \in Q^k | \geq 0: (x,x) \in R\}$ is a maximal nonnegative root in R. Then $S \cap \{x \in Q^k | \geq 0: (x,x) \in R\}$ is a root in R. Suppose $S \cap \{x \in Q^s | \geq 0: (x,x) \in R\}$ is a proper subset of the nonnegative root in R, S'. Then S is a proper subset of the nonnegative clique $S' \cup \{x \in Q^k | \geq 0: (x,x) \notin R\}$ in graph(R,Q^k). This is a contradiction.

Let S be a maximal nonnegative root in R. Then $S \cup \{x \in Q^k | \geq 0: (x,x) \notin R\}$ is a nonnegative clique in graph(R,Q^k). We claim that $S \cup \{x \in Q^k | \geq 0: (x,x) \notin R\}$ is a maximal clique in graph(R,Q^k). Suppose $S \cup \{x \in Q^k | \geq 0: (x,x) \notin R\}$ is a proper subset of the clique S' in graph(R,V). Then S is a proper subset of the root $S' \cap \{x \in Q^k | \geq 0: (x,x) \in R\}$. This is a contradiction.

The second claim is left to the reader. QED

THEOREM 4.5.4. (RCA$_0$). Let $T \subseteq Q^{2k}$ be order preserving, $R \subseteq Q^{2k}$ be order invariant, and G be an order invariant graph on Q^k.
i. R has a T invariant maximal root if and only if graph(R,Q^k) has a T invariant maximal clique.
ii. G has a T invariant maximal clique if and only if rel(G) has a T invariant maximal root.
These results hold for any fixed k.

Proof: Let T,R,G be as given. Let S be a T invariant maximal root in R. Then $S \cup \{x \in Q^k: (x,x) \notin R\}$ is a maximal clique in graph(R,Q^k). Now S is T invariant. Since $\{x \in Q^k: (x,x) \notin R\} \subseteq Q^k$ is order invariant, $T \subseteq Q^{2k}$, and T is order preserving, $\{x \in Q^k: (x,x) \notin R\}$ is T invariant.

Let S be a T invariant maximal clique in graph(R,Q^s). Then $S \cap \{x: (x,x) \in R\}$ is a maximal root in R. Now S is T invariant. Since $\{x: (x,x) \in R\}$ is order invariant, $T \subseteq Q^{2k}$, and T is order preserving, $S \cap \{x: (x,x) \in R\}$ is T invariant.

ii is immediate. In addition, the results hold for fixed k as k is unchanged throughout. QED

THEOREM 4.5.5. (RCA_0). Let $T \subseteq Q^{2k}|{\geq}0$ be order preserving, $R \subseteq Q^{2k}$ be order invariant, and G be an order invariant graph on Q^k.
i. R has a T invariant maximal nonnegative root if and only if graph(R,Q^k) has a T invariant maximal nonnegative clique.
ii. G has a T invariant maximal nonnegative clique if and only if rel(G) has a T invariant maximal nonnegative root. These results hold for any fixed k.

Proof: Let T,R,G be as given. Let S be a T invariant maximal root in R. Then $S \cup \{x \in Q^k|{\geq}0: (x,x) \notin R\}$ is a maximal nonnegative clique in graph(R,Q^k). Now S is T invariant. Since $\{x \in Q^k|{\geq}0: (x,x) \notin R\} \subseteq Q^k|{\geq}0$ is relatively order invariant, $T \subseteq Q^{2k}|{\geq}0$, and T is order preserving, $\{x \in Q^k|{\geq}0: (x,x) \notin R\}$ is T invariant. Hence $S \cup \{x \in Q^k|{\geq}0: (x,x) \notin R\}$ is T invariant.

Let S be a T invariant maximal nonnegative clique in graph(R,Q^s). Then $S \cap \{x \in Q^k|{\geq}0: (x,x) \in R\}$ is a maximal nonnegative root in R. Now S is T invariant. Since $\{x \in Q^k|{\geq}0: (x,x) \in R\}$ is relatively order invariant, $T \subseteq Q^{2k}|{\geq}0$, and T is order preserving, $S \cap \{x \in Q^k|{\geq}0: (x,x) \in R\}$ is T invariant.

ii is immediate. In addition, the results hold for fixed k as k is unchanged throughout. QED

The following was promised in the proof of Theorem 4.2.7.

THEOREM 4.5.6. (RCA_0) Propositions 4.2.4 and 4.2.6 are equivalent. This also holds for any fixed k.

Proof: Apply Theorem 4.5.5 with order preserving T = BEQ($A,Q^k|{\geq}0$), A finite. QED

THEOREM 4.5.7. (RCA_0) Propositions 4.2.1, 4.2.2, 4.3.1 - 4.3.4, 4.4.3 and Theorems 4.4.1, 4.4.2 are each equivalent to their graph formulations. This is true for any fixed k.

Proof: Use Theorems 4.5.4 and 4.5.5. QED

5. INVARIANT BASES

5.1. #,≤# BASES

DEFINITION 5.1.1. Let $R \subseteq Q^{2k}$. x R-reduces to y if and only if x R y ∧ y $>_{lex}$ x. S is a basis for R if and only if S $\subseteq$ Q^k, no x $\in$ S R-reduces to any y $\in$ S, and every x $\in Q^k$\S R-reduces to some y $\in$ S. I.e., S is a basis for R if and only if S $\subseteq Q^k$, and for all x $\in Q^k$, x $\in$ S if and only if x does not R-reduce to any y $\in$ S.

This follows the general pattern of bases in mathematics. Everything in a basis "reduces" to something in the basis, and everything in the space outside the basis "reduces" to something in the basis.

Unfortunately, this notion of basis is far too strong for our purposes.

THEOREM 5.1.1. The relation Q^2 on Q has no basis.

Appropriate notions of basis are obtained by restricting the set Q^k\S to a smaller X\S.

DEFINITION 5.1.2. Let S $\subseteq Q^k$. #(S) is the least $D^k \supseteq$ S $\cup$ {0}. ≤#(S) = #(S)$_≤$.

Note that #(S) = (fld(S) $\cup$ {0})^k, ≤#(S) = (fld(S) $\cup$ {0})$^{k_≤}$.

DEFINITION 5.1.2. Let R $\subseteq Q^k$. S is a # basis for R if and only if S $\subseteq Q^k$, and for all x $\in$ #(S), x $\in$ S if and only if x does not R-reduce to any y $\in$ S. S is a ≤# basis for R if and only if S $\subseteq Q^k$, and for all x $\in$ ≤#(S), x $\in$ S if and only if x does not R-reduce to any y $\in$ S.

We use # bases in section 5.3 and ≤# bases in section 5.4.

5.2. MODELS IN Q

The following result is formulated so that it can be conveniently used in sections 5.3 and 5.4.

THEROEM 5.2.1. (RCA$_0$) Let (N,<',c,F) be given, where
i. <' is a strict linear ordering on N with left endpoint

c.

ii. F:N → N obeys x <' y → F(x) <' F(y) ∧ x <' F(x).

iii. (∀n ∈ N)(∃k ≥ 0)(n <' F^k(c)).

iv. rng(F) exists.

(N,<',c,F) is isomorphic to some (D,$<_D$,-1,SC), where -1 ∈ D ⊆ Q and SC:D → D is the successor function, +1.

Proof: Let (N,<',c,F) be as given. We use RCA_0 to construct h:N → Q|≥-1 such that

i. h(c) = -1.

ii. x <' y → h(x) < h(y).

iii. h(F(x)) = h(x)+1.

iv. rng(h) exists.

Obviously F is one-one. Using rng(F), we see that the partial function F^{-1} and its domain exist. Hence the set α(n) of iterates of n by F exists. I.e., α(n) = {n <' F(n) <' F^2(n) <' ...} exists. The maximal α(n) are called the threads, and they are the α(n) such that n ∉ rng(F). It is also clear that N is partitioned into one or more pairwise disjoint threads. We write γ(n) for the unique thread that contains n. We also write γ(n,i) for the i-th term of the thread γ(n), i ≥ 0. It is also clear that the binary function γ and the sets {(n,m): n is on γ(m)}, {(n,m): γ(n) = γ(m)}, exist. This uses that rng(F) and F^{-1} exist.

We define h successively on the threads γ(0),γ(1),... . After stage n, we will have conditions i-iv, with h defined at γ(0) ∪ ... ∪ γ(n). To keep a record of h at stages 0,...,n, we need only keep a record of h defined at the initial terms γ(0,0),γ(1,0),...,γ(n,0), since we are maintaining iii. Thus records are finite.

We start by defining h at the thread γ(c) = c,F(c),..., by h(c) = -1. We now successively define h at γ(n), n = 0,1,..., as follows. We use interval notation with respect to <'.

If γ(n) is not new, then we do nothing. Suppose γ(n) is new. Choose m so large that each γ(n'), n' ≤ n, meets [F^m(c),F^{m+1}(c)). Choose r so that γ(n,r) ∈ [F^m(c),F^{m+1}(c)). Clearly m,r exist by assumptions ii,iii. Define h(γ(n,r)) so that condition ii holds for h on [F^m(c),F^{m+1}(c)), where the denominator of h(γ(n,r)) in reduced form is greater than n.

Extend h to the rest of γ(n) so that conditions ii,iii continue to hold. We need only record h(γ(n,0)).

The construction is provably well defined, with conditions i-iii preserved. Condition iv is clear because any rational with a given denominator r in reduced form lying in rng(h) must be a value that is assigned in the first r stages - and that can be explicitly ascertained.

We now use h to isomorphically transfer from $(N,<',c,F,...)$ to the required $(D,<_D,-1,SC)$, with $D = $ rng(h). QED

We need a refinement of the usual formalized completeness theorem.

5.3. UPPER SHIFT

We have already defined the # bases for $R \subseteq Q^{2k}$ in section 5.1. We begin by presenting an alternative definition of # basis, which is useful conceptually - e.g., for section 7.3.

DEFINITION 5.3.1. Let R be a relation on Q^k. S is R free if and only if $S \subseteq Q^k$ and there is no $x \neq y$ from S with x R y.

THEOREM 5.3.1. Let R be a relation on Q^k. S is a # basis for R if and only if S is R free and every $x \in Q^k \backslash S$ R-reduces to an element of S.

Proof: Let R be a relation on Q^k. Suppose S is a # basis for R. Let $x \in S$. No $x \in S$ R-reduces to an element of S. Let $x \neq y$. If x R y then x R-reduces to y or y R-reduces of x since the lexicographic ordering is a strict linear ordering. Hence S is R free. The other direction is immediate. QED

DEFINITION 5.3.2. The upper shift of $x \in Q^k$, ush(x), results from adding 1 to all nonnegative coordinates. The upper shift of $S \subseteq Q^k$, ush[S], is {ush(x): $x \in S$}.

THEOREM 5.3.2. Every relation on Q^k has the # basis $\{0\}^k$. The following is false in each dimension. Every relation on Q^k has a # basis containing its upper shift.

Proof: The first claim is immediate since #($\{0\}^k$) = $\{0\}^k$. Let R = Q\$\{0\}^{2k}$, and let S be a # basis for R containing its upper shift. Then (0,...,0) ∈ S, and hence (1,...,1),(2,...,2) ∈ S. But (2,...,2) R-reduces to (1,...,1), which is a contradiction. QED

PROPOSITION 5.3.3. Every order invariant R ⊆ Q^{sk} has a # basis containing its upper shift.

THEOREM 5.3.4. Proposition 5.3.3 is provably equivalent to Con(SRP) over WKL_0. For all fixed k, Proposition 5.3.3 is provable in SRP. There is a fixed k such that Proposition 5.3.3 is provably equivalent to Con(SRP) over WKL_0.

We now prove Proposition 5.3.3 from WKL_0 + Con(SRP). Extra details are needed to accommodate WKL_0 rather than ACA or ACA_0. The reversal in Theorem 5.3.4 will be proved elsewhere.

We apply Theorem 5.2.1 to the following theory purely universal theory T(k,R), where R is an order invariant relation on Q^k, given as a propositional combination of formulas $x_i < x_j$, $1 \le i,j \le k$. We use <',c,F as in Theorem 5.2.1, together with the unary function symbols j,G, the k-ary relation symbol P, and the k-ary function symbols $H_1,...,H_k$. The idea is that the points are ordinals ordered by <', P represents the unique basis for R, and j is an outside elementary embedding. F adjusts j only below the critical point, where it is simply right ordinal addition by the critical point. This allows F to be used in Theorem 5.2.1, as j cannot be used there for F.

The axioms of T(k,R) are

i. <' is a strict linear ordering on N with left endpoint c.
ii. x <' y → F(x) <' F(y).
iii. x <' F(x).
iv. x <' F(c) → j(x) = x.
v. x ≥' F(c) → j(x) = F(x).
vi. P($x_1,...,x_k$) ↔ ¬(∃$y_1,...,y_k$)(P($y_1,...,y_k$) ∧ ($y_1,...,y_k$) $<_{lex}$ ($x_1,...,x_k$)).
vii. P($x_1,...,x_k$) ↔ P(j(x_1),...,j(x_k)).

For background on the SRP hierarchy, see [Fr14a], section

10.1.

LEMMA 5.3.5. (WKL$_0$ + Con(SRP)) There is a model of SRP, (N,∈',λ), with constant λ, such that
i. For all standard n, λ internally has n-SRP.
ii. The complete diagram of (N,∈',λ) exists.

Proof: The theory ZFC + {λ has n-SRP: n ≥ 1} is consistent. By the formalized completeness theorem, this theory has a model with domain N and a complete diagram. QED

LEMMA 5.3.6. (WKL$_0$ + Con(SRP)) Each T(k,R) has a model (N,<',c,F,j,P), where
i. ($\forall$n ∈ N)($\exists$m ≥ 0)(n <' F^m(c)).
ii. rng(F) exists.

Proof: Fix order invariant R on Q^k. Let M = (N,∈',λ) be given by Lemma 5.3.5. There is a nonstandard integer r such that in M, λ has the r-SRP. By the standard combinatorics of the SRP hierarchy in M, let E ⊆ λ be an internally stationary set which forms a set of strong indiscernibles for M in the following sense. According to M,

$$\varphi(x,u) \leftrightarrow \varphi(y,u)$$

where φ is an M first order formula over M without parameters, x,y are order equivalent r tuples from E, and u is an M finite sequence strictly below all coordinates of x,y.

In particular, E ⊆ λ is internally stationary, forming a set of strong indiscernibles for M in the following weaker sense that we will exploit. According to M,

$$\varphi(x,u) \leftrightarrow \varphi(y,u)$$

where φ is a standard first order formula over M without parameters, x,y are order equivalent standardly finite length tuples from E, and u is a standardly finite sequence below x,y.

We define Δ to be the set of all definitions, in M, of M elements of λ, with constants c_i, i ∈ N. We will interpret each c_i as the i-th element E_i of E. We caution the reader that whereas the map from c_i to E_i exists, its range may not

exist, since we are working in WKL_0. Using the complete diagram of M, Δ exists.

Define val:$\Delta \rightarrow$ N, where val(φ) is the M element of λ uniquely defined by φ. Since the same M element of λ may have many such definitions, for each $\varphi \in \Delta$, we let φ' be the $\psi \in \Delta$ with least Gödel number such that val(φ) = val(ψ). We let $\Delta' = \{\varphi': \varphi \in \Delta\}$. The subscripts of the c_i are of course used to evaluate the Gödel numbers. Clearly $\Delta,\Delta',$val exist using the complete diagram of M. We do not claim that the range of val exists.

We define a model M* = ($\Delta',<',c,F,j,P$) of T(k,R) as follows. $\varphi <' \psi \leftrightarrow \varphi,\psi \in \Delta' \wedge$ val(φ) $\in'$ val(ψ). c is the unique element of Δ' with val(c*) = $\emptyset$ of M. Internally in M, let P* be the unique # basis for R, in the sense of axiom vi. Note that P* is M definable without parameters. We define P $\subseteq \Delta'$k by P($\varphi_1,...,\varphi_k$) $\leftrightarrow$ P*(val(φ_1),...,val(φ_k)).

We now define F,j. For $\varphi \in \Delta$, let $\alpha(\varphi)$ be the result of replacing each ci by ci+1. By E indiscernibility, $\alpha(\varphi) \in \Delta$. For $\varphi \in \Delta'$, define j(φ) = $\alpha(\varphi)'$. Define F:$\Delta' \rightarrow \Delta'$ so that F,j agree on the $\varphi \in \Delta'$ with val(φ) $\notin'$ E_0. Uniquely define F on the $\varphi \in \Delta'$ with val(φ) $\in'$ E_0 so that val(F(φ)) = E_0 + val(φ), where + is ordinal addition in M.

In M, we use x $\in'$= y for x $\in'$ y $\vee$ x = y.

Claim 1. Axiom i holds. Use that $\in'$ is a strict linear ordering in M on the val(φ), $\varphi \in \Delta'$, and that any two elements of Δ' are equal if and only they have same val.

Claim 2. val(F(c)) = E_0. Since val(c) = $\emptyset \in'$ E_0, we have val(F(c)) = E_0 by the definition of F.

Claim 3. Let $\varphi \in \Delta'$. Then val(φ) < E_0 $\rightarrow$ val($\alpha(\varphi)$) = val(j(φ)) = val(φ). Assume val(φ) = μ < E_0. Apply E indiscernibility to the statement in M, val(φ) = μ. We get val($\alpha(\varphi)$) = μ.

Claim 4. Axiom iv holds. Let $\varphi \in \Delta'$, and assume val(φ) $\in'$ val(F(c)). By claim 2, val(φ) $\in'$ E_0. By claim 3, val(j(φ)) = val(F(φ)) = val(φ).

Claim 5. Axiom v holds. Let $\varphi,\psi \in \Delta'$, $\mathrm{val}(\varphi) \notin' \mathrm{val}(F(c))$. By claim 2, $\mathrm{val}(F(c)) = E_0$. Hence $\mathrm{val}(\varphi) \notin' E_0$. Therefore $\mathrm{val}(F(\varphi)) = \mathrm{val}(j(\varphi))$.

Claim 6. Let $\varphi,\psi \in \Delta$. Then $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi) \to \mathrm{val}(\alpha(\varphi)) \in' \mathrm{val}(\alpha(\psi))$. Apply E indiscernibility to the statement in M, $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi)$.

Claim 7. Let $\varphi,\psi \in \Delta'$. Then $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi) \to \mathrm{val}(j(\varphi)) \in' \mathrm{val}(j(\psi))$. By claim 6.

Claim 8. Let $\varphi \in \Delta'$. $\mathrm{val}(\varphi) \in E_0 \to \mathrm{val}(F(\varphi)) \in' E_0 + E_0$. $\mathrm{val}(\varphi) = E_0 \to \mathrm{val}(F(\varphi)) = E_1$. $\mathrm{val}(\varphi) \notin' E_0 \to \mathrm{val}(F(\varphi)) \notin' E_1$. The first statement is immediate by the definition of F. For the second statement, suppose $\mathrm{val}(\varphi) = E_0$. Apply E indiscernibility to the statement in M, $\mathrm{val}(\varphi) = E_0$ to obtain $\mathrm{val}(\alpha(\varphi)) = E_1$. Hence $\mathrm{val}(F(\varphi)) = E_1$. The third statement follows from the second statement and claim 6 or 7.

Claim 9. Let $\varphi,\psi \in \Delta'$. Then $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi) \to \mathrm{val}(F(\varphi)) \in' \mathrm{val}(F(\psi))$. Assume $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi)$. Suppose $\mathrm{val}(\varphi), \mathrm{val}(\psi) \notin' E_0$. Apply claim 7. Suppose $\mathrm{val}(\varphi), \mathrm{val}(\psi) \in' E_0$. Use that ordinal addition in M is strictly increasing. Suppose $\mathrm{val}(\varphi) \in' E_0 \wedge \mathrm{val}(\psi) \notin' E_0$. By claim 8, $\mathrm{val}(F(\varphi)) \in' E_0 + E_0 < E_1$, and $\mathrm{val}(F(\psi)) = \mathrm{val}(j(\psi)) \notin E_1$. Suppose $\mathrm{val}(\varphi) \notin' E0 \wedge \mathrm{val}(\psi) \in' E_0$. This contradicts $\mathrm{val}(\varphi) \in' \mathrm{val}(\psi)$.

Claim 10. Axiom ii holds. By claim 9.

Claim 11. Let $\varphi \in \Delta$ and $n \geq 0$. Suppose all ci present in φ have $i < n$. Then $\mathrm{val}(\varphi) \in' E_n$. Assume hypotheses. Suppose $E_n \in'= \mathrm{val}(\varphi)$. By E indiscernibility, we can replace E_n by any element of E. This is impossible since $\mathrm{val}(\varphi) \in' \lambda$.

Claim 12. Let $\varphi \in \Delta$, where for all ci present in φ, $\mathrm{val}(\varphi) \in' E_i$. Then $\mathrm{val}(\varphi) \in' E_0$. Assume the hypotheses. It is simplest to dispense with the case where no c_i is present in φ. Let $\mathrm{val}(\varphi) \in' \gamma \in E$. By E indisernibility, $\mathrm{val}(\varphi) \in' E_0$. Now assume some c_i is present in φ. By E indiscernibility applied to the statement $\mathrm{val}(\alpha(\varphi)) = \mu$, we obtain $\mathrm{val}(\alpha(\varphi))$

= val(φ) = μ. By more E indiscernibility, we can work backward, and change φ to ψ, where either ψ has c_0 present, and where val(φ) = val(ψ) < E_0.

Claim 13. Let $\varphi \in \Delta$. Then val(φ) $\in'$= val($\alpha(\varphi)$). Suppose val($\alpha(\varphi)$) $\in'$ val(φ). By E indiscernibility, val($\alpha\alpha(\varphi)$) $\in'$ val($\alpha(\varphi)$). We can continue in this way, working inside M, and get an internally infinite descending sequence of internal ordinals, which is a contradiction.

Claim 14. Let $\varphi \in \Delta$. Then val(φ) $\notin'$ E_0 $\rightarrow$ val(φ) $\in'$ val($\alpha(\varphi)$). Assume val(φ) $\notin'$ E_0. By claim 11, val(φ) $\in'$= val($\alpha(\varphi)$). Suppose val(φ) = val($\alpha(\varphi)$).

Claim 15. Axiom iii holds. Let $\varphi \in \Delta'$. By claims 5,6.

Claim 16. Axiom vi holds. The forward direction clearly holds. For the converse, let $\neg P(\varphi_1,...,\varphi_k)$. We can find $\psi_1,...,\psi_k$ with $P(\psi_1,...,\psi_k)$ $\wedge$ (val(ψ_1),...,val(ψ_k)) $<_{lex}$ (val(φ_1),...,val(φ_k)), by choosing witnesses in M definably.

Claim 17. Axiom vii holds. let $\varphi_1,...,\varphi_k \in \Delta'$. $P(\varphi_1,...,\varphi_k)$ $\leftrightarrow$ $P^*($val(φ_1),...,val(φ_k)) $\leftrightarrow$ $P^*($val($j(\varphi_1)$),...,val($j(\varphi_k)$)) $\leftrightarrow$ $P(j(\varphi_1),...,j(\varphi_k))$.

Claim 18. For all $n \geq 0$, val($F^n(c)$) = E_n. By induction on n.

Claim 19. Condition i holds. By claims 11 and 18.

We now verify condition ii, that rng(F) exists.

Claim 20. Let $\varphi \in \Delta'$, where φ does not mention c_0. Then $\varphi \in$ rng(j). To see this, let $\alpha(\psi)$ = φ. Then val($j(\psi')$) = val($\alpha(\psi')$). By E indiscernibility, val($\alpha(\psi')$) = val($\alpha(\psi)$) = valφ). Hence $j(\psi')$ = φ.

For claims 21,22, we fix $\varphi \in \Delta'$, and let $\beta(\varphi)$ be obtained from $\alpha(\varphi)$ by replacing all occurrences of c_1 in $\alpha(\varphi)$ by c_0.

Claim 21. val($\alpha(\varphi)$) = val($\beta(\varphi)$) $\rightarrow$ $\varphi \in$ rng(j). Assume val($\alpha(\varphi)$) = val($\beta(\varphi)$). We work in M, and view $\varphi,\alpha(\varphi),\beta(\varphi)$ as definitions in M with the c_i replaced by the ordinals E_i. Let $\mu \in'$ λ be internally least such that if we replace all

occurrences of E_1 in $\alpha(\varphi)$ by μ then we get the same value
$val(\alpha(\varphi))$. Since E_0 is among these μ's, we have $\mu \in'= E_0$. By
E indiscernibility, $\mu = E_0 \rightarrow \mu = E_1$, and so $\mu \in' E_0$. This
gives an internal definition of μ without using E_0,E_1. By
substitution, we obtain an internal definition in Δ of
$val(\alpha(\varphi))$ without using c0,c1. By E indiscernibility, we
have an internal definition $\psi \in \Delta$ of $val(\varphi)$ without using
c0. Now apply claim 20 to $\psi' = \varphi$ writing $j(\rho) = \psi' = \varphi$.
Hence $\varphi \in rng(j)$.

Claim 22. $\varphi \in rng(j) \rightarrow val(\alpha(\varphi)) = val(\beta(\varphi))$. Let $\varphi = j(\psi)$.
Then $\varphi = \alpha(\psi)'$, $val(\varphi) = val(\alpha(\psi))$, $val(\alpha(\varphi)) =$
$val(\alpha(\alpha(\psi)))$. Since c0,c1 do not appear in $\alpha(\alpha((\psi))$, by E
indiscernibility, we can replace c1 by c0 in this last
equation, obtaining $val(\beta(\varphi)) = val(\alpha(\alpha(\psi))) = val(\alpha(\varphi))$.

Claim 23. $rng(F)$ exists. By claims 21,22, $\varphi \in rng(j)$ if and
only if $val(\alpha(\varphi)) = val(\beta(\varphi))$. Using the complete diagram of
M, $rng(j)$ exists. Using the complete diagram of M and how F
adjusts j, we finally see that $rng(F)$ exists.

For the required model of $T(k,R)$, we have only to convert
the domain of M from Δ' to N in the standard way. QED

THEOREM 5.3.7. WKL_0 + Con(SRP) proves Proposition 5.3.3.

Proof: Let R be an order invariant relation on Q^k. Let
$(N,<',c,F,j,P)$ be given by Lemma 5.3.6. By Theorem 5.2.1,
$(N,<',c,F)$ is isomorphic to some $(D,<_D,-1,SC)$, $D \subseteq Q$. Hence
$(N,<',c,F,j,P)$ is isomorphic to some $(D,<_D,-1,SC,ush,S)$ by
the same isomorphism. Note that $N \subseteq D$. Now the isomorphism
preserves the axioms of $T(k,R)$, and so $S = \{x \in D^k: \neg(\exists y \in$
$S)(x$ R-reduces to $y \wedge y <_{lex} x)\}$. Since $fld(S) \subseteq D$, we have
$S = \{x \in S\#: \neg(\exists y \in S)(x$ R-reduces to $y \wedge y <_{lex} x))\}$. Also S
$\supseteq ush(S)$. Hence S is a # basis for R. QED

We gave a straightforward a priori proof that Proposition
4.3.1 is implicitly Π^0_1 via Theorem 4.1.4 (the $\alpha(R,T)$
construction). There is a glitch in giving such an easy
priori proof that Proposition 5.3.3 is implicitly Π^0_1.
Theorem 5.3.7 proves much more, but is not proved a priori.
The glitch is due to the fact that if y lies in a given
initial segment of < and $x <_{lex} y$, then it is not
necessarily the case that x lies in that initial segment.

However, this problem does not arise if we use some well known variants of $<_{lex}$.

DEFINITION 5.3.2. Let $x,y \in Q^k$. $x <_{mlex} y$ if and only if $\max(x) < \max(y) \vee (\max(x) = \max(y) \wedge x <_{lex} y)$. Using $<_{mlex}$, we define mR-reduce and m# bases accordingly, with $<_{lex}$ replaced by $<_{mlex}$. For the end of section 5.4, we also define m≤# bases in the obvious way.

We arrive at the following variant of Proposition 5.3.3 of independent interest. We give an a priori proof that this variant is implicitly Π^0_1 over ACA_0 (Theorem 5.3.11).

PROPOSITION 5.3.8. Every order invariant $R \subseteq Q^{2k}$ has an m# basis containing its upper shift.

THEOREM 5.3.9. Proposition 5.3.9 is provably equivalent to Con(SRP) over WKL_0. There is a fixed k such that Proposition 5.3.3 is provably equivalent to Con(SRP) over WKL_0.

The proof of Proposition 5.3.9 in WKL_0 + Con(SRP) is a trivial modification of the proof given above for Proposition 5.3.3. The reversal in Theorem 5.3.9 (and Theorem 5.3.4) will appear elsewhere.

THEOREM 5.3.10. Proposition 5.3.8 is provably equivalent to a Π^0_1 sentence over ACA_0 via Gödel's Completeness Theorem.

Proof: It suffices to put Proposition 5.3.8 in Π^0_1 form for any fixed order invariant relation R on Q^k, uniformly in R. Let K(k,R) be the following theory in $<,0,-1,F,j,P$.

i. $<$ is a dense linear ordering with left endpoint -1 and no right endpoint.
ii. F is a strictly increasing map from [-1,∞) onto [0,∞).
iii. $F(x) > x \wedge F(-1) = 0$.
iv. $j(x) = F(x)$ if $x \geq 0$; x if $x < 0$.
v. P is an m# basis for R.
vi. $P(x_1,...,x_k) \rightarrow P(j(x_1),...,j(x_k))$.

Note that axiom v uses the # construction, #(P), with 0.

If Proposition 5.3.8 holds for R then obviously K(k,R) has a model with domain Q[-1,∞), using ordinary $<,-1,0$, and F =

SC, j = ush, P = S. Now suppose K(k,R) has a model (E,<',-1',0',F',j',P'). Let (E*,<*,-1',0',F*,j*,P*) be its initial segment with E* = {x: ($\exists$i)(x < F^i(0') $\land$ x > F^{-i}(-1'))}. Because we are using $<_{mlex}$ rather than $<_{lex}$, this initial segment is also a model of K(k,R). Let h be any isomorphism from (E*,<*,-1',0',F*,j*) onto (Q,<,-1,0,SC,ush). Extend h to an isomorphism from (E*,<*,-1',0',F*,j*,P*) onto (Q|$\geq$-1,<,-1,0,SC,ush,S). Then S is an m# basis for R containing its upper shift.

Thus we have shown that Proposition 5.3.8 is equivalent to "for every order invariant R on every Q^k, K(k,R) is consistent", using the formalized completeness theorem, available in WKL_0. QED

5.4. TRANSLATED PROJECTIONS

Nothing new arises from simply replacing # bases with ≤# bases.

PROPOSITION 5.4.1. Every order invariant R $\subseteq$ Q^{2k} has a ≤# basis containing its upper shift.

THEOREM 5.4.2. Proposition 5.4.1 is provably equivalent to Con(SRP) over WKL_0.

We now place an additional requirement on the ≤# basis.

DEFINITION 5.4.1. The bounded N^k translates of S $\subseteq$ Q^k are the sets S+x, x $\in$ N^k, which are contained in some Q^k|<p.

PROPOSITION 5.4.3. Every order invariant R $\subseteq$ Q^{2k} has a ≤# basis containing its upper shift, whose bounded N^k translated projections are projections.

THEOREM 5.4.4. Proposition 5.4.3 is provably equivalent to Con(HUGE) over WKL_0. There is a fixed k such that Proposition 5.4.3 is provably equivalent to Con(HUGE) over WKL_0.

We now prove Proposition 5.4.3 from WKL_0 + Con(HUGE). The reversal in Theorem 5.4.4 will appear elsewhere.

We apply Theorem 5.2.1 to the following theory $W(k,R)$, where R is an order invariant relation on Q^k. The language of $W(k,R)$ is $<'$,c,F,j,P. The rough idea that needs to be defined is as follows. The points are ordinals ordered by $<'$, P represents the canonical $\leq\#$ basis for R, j is an internal elementary embedding, and F adjusts j below the critical point of j for use in Theorem 5.2.1. More precisely, we start with a nontrivial elementary embedding from a rank into itself, where the rank is augmented with one of its well orderings. We then take the canonical $\leq\#$ basis for R as defined in this rank using the well ordering. The points are elements of the rank. The H's are Skolemizations for P, j is the internal elementary embedding, and F adjust j.

Now of course, we do not want to use something as strong as a nontrivial elementary embedding from a rank into itself. Instead we use what amounts to an elementary embedding from a nonstandard rank into a higher nonstandard rank, and restricting its domain to the appropriate proper initial segment. This results in an appropriate embedding that is appropriately locally elementary. This is executed via a compactness argument using Con(HUGE). The crucial point is that j is internal in the sense that its restriction to any internal rank is internal. In section 5.3, the elementary embedding j was purely external.

Here j corresponds to the +1 function above 1 = the crucial point of j. F corresponds to the +1 function everywhere. F is used to execute translations. I.e., F^i corresponds to the +i function.

The axioms of $W(k,R)$ are

i. $<'$ is a strict linear ordering on N with left endpoint c.
ii. $x <' y \rightarrow F(x) <' F(y)$
iii. $x <' F(x)$.
iv. $x <' F(c) \rightarrow j(x) = x$.
v. $x \geq' F(c) \rightarrow j(x) = F(x)$.
vi. Let $x_1 \leq' ... \leq' x_k$. $P(x_1,...,x_k) \leftrightarrow \neg (\exists (y_1,...,y_k) <_{lex} (x_1,...,x_k))(P(y_1,...,y_k) \wedge (x_1,...,x_k) R (y_1,...,y_k))$.
vii. $P(x_1,...,x_k) \leftrightarrow P(j(x_1),...,j(x_k))$.
viii. Let $b_1,...,b_i \in N$, $1 \leq i \leq k-1$. The image of every bounded i-dimensional projection of P under $F^{b_1},...,F^{b_i}$ is an i-dimensional projection of P.

In order to show that each W(k,R) is consistent, we need some lemmas concerning elementary embeddings. Below, κ,λ,μ always denote limit ordinals.

DEFINITION 5.4.2. < is a good well ordering of $V(\lambda)$ if and only if < is a well ordering of $V(\lambda)$ where
i. $rk(x) \in rk(y) \rightarrow x < y$.
ii. The < least element of each $V(\lambda)\backslash V(\alpha)$, $\alpha \in \lambda$, is α.
Here $rk(x)$ is the least α such that $x \in V(\alpha)$.

DEFINITION 5.4.3. Let < be a good well ordering of $V(\mu)$ and R be an order invariant relation on $V(\mu)^k$. We define the $\leq$-basis $B(R,V(\mu),<) \subseteq V(\mu)^k$ by transfinite induction as follows. Let $x \in V(\mu)^{k_<}$. $x \in B(R,V(\mu),<) \leftrightarrow \neg(\exists y \in B(R,V(\mu),<))(x >_{lex} y \wedge x \mathrel{R} y)$. Let $x \in V(\mu)^k\backslash V(\mu)^{k_<}$. $x \in B(R,V(\mu),<) \leftrightarrow <x_2,...,x_k> \in x_1$.

Note that $B(R,V(\mu)),<)$ is definable in $(V(\mu),\in,<)$ without parameters.

DEFINITION 5.4.4. $EE(n;j,\kappa,\lambda,\mu)$ if and only if $j:V(\lambda) \rightarrow V(\mu)$ is an elementary embedding with critical point κ, where $j^n(\kappa) < \lambda$.

DEFINITION 5.4.5. $EE(n;<,j,\kappa,\lambda,\mu)$ if and only if
i. < is a good well ordering of $V(\mu)$.
ii. $j:(V(\lambda),\in,<) \rightarrow (V(\mu),\in,<)$ is an elementary embedding with critical point κ, where $j^n(\kappa) < \lambda$.
Here the < on the left side is the restriction of < (on the right side) to $V(\lambda)$.

LEMMA 5.4.5. (ZFC) Let $EE(n+1;j,\kappa,\lambda,\mu)$. There is a good well ordering < on $V(j^n(\kappa))$ which is an initial segment of the good well ordering $j(<)$ on $V(j^{n+1}(\kappa))$.

Proof: Let $EE(n+1;j,\kappa,\lambda,\mu)$. Start with any good well ordering < on $V(j_0(\kappa)) = V(\kappa)$. We claim that $<,j(<),...,j^n(<)$ are good well orderings on $V(\kappa),V(j(\kappa)),...,V(j^n(\kappa))$, where each is an initial segment of the next. To see this, since < is a good well ordering on $V(\kappa)$, by j elementarity, each $j^i(<)$ is a good well ordering on $V(j^i(\kappa))$. Clearly $\alpha < \beta \leftrightarrow j(\alpha) \mathrel{j(<)} j(\beta) \leftrightarrow \alpha$

j(<) β, provided α,β < κ. Hence < is an initial segment of j(<). By j elementarity, j(<) is an initial segment of j^2(<), and so forth. QED

LEMMA 5.4.6. (ZFC) Let EE(n+2;j,κ,λ,μ). There exist <,j',κ',λ',μ' such that EE(n;<,j',κ',λ',μ').

Proof: We have an elementary embedding j:V(λ) → V(μ) with critical point κ, where j^{n+3}(κ) < λ. Set κ' = κ, λ' = jn+1(κ), μ' = jn+2(κ). By Lemma 5.4.5, let < be a good well ordering on V(jn+2(κ)) = V(μ') which is an initial segment of the good well ordering j(<) on V(jn+3(κ)). We claim that the restriction j' of j to V(λ'), written j:(V(λ'),∈,<) → (V(μ'),∈,<), is an elementary embedding. To see this, let φ(x_1,...,x_r) be a first order statement that holds in (V(λ'),∈,<), where x_1,...,x_r ∈ V(λ'). We can also view φ(x_1,...,x_r) naturally as a first order statement ψ(x_1,...,x_r,< ∩ V(λ')) in (V(λ),∈) with parameters x_1,...,x_r,< ∩ V(λ')^2. Hence ψ(j(x_1),...,j(x_r),j(< ∩ V(λ')^2) holds in (V(μ),∈). I.e., ψ(j(x_1),...,j(x_r),< ∩ V(μ')^2) holds in (V(μ),∈). Hence φ(j(x_1),...,j(x_r),<) holds in (V(μ'),∈,<). QED

LEMMA 5.4.7. (WKL$_0$ + Con(HUGE) There exists M = (N,∈',<',F,j,κ,λ,μ), with binary relation <', unary function j, and constants κ,λ,μ, such that
i. (N,∈') satisfies ZFC.
ii. For all standard n, EE(n;<',j,κ,λ,μ).
iii. F(x) = j(x) if x ∉ V(κ); x' otherwise, where the position of x' in <' is κ plus the position of x in <'.
iiv. The complete diagram of M exists.

Proof: Consider the theory K = ZFC + {EE(n;<',j,κ,λ,μ): n ≥ 1}, with symbols ∈',<',j,κ,λ,μ. By Lemma 5.4.6, every finite fragment of K can be proved to be consistent in HUGE. Hence, assuming WKL$_0$ + Con(HUGE), we see that K is consistent. Insert the interpretation of F according to the explicit definition given in iii. The result now follows by the formalized completeness theorem available in WKL$_0$. QED

LEMMA 5.4.8. (WKL$_0$ + Con(HUGE) Each W(k,R) has a model (N,<*,c*,F*,j*,P*), where
i. (∀n ∈ N)(∃k ≥ 1)(n <* F*^{k}(c*)).

ii. rng(F*) exists.

Proof: Let $M = (N, \in', <', F, j, \kappa, \lambda, \mu)$ be given by Lemma 5.4.7.
In particular, for all standard n, $j^n(\kappa) < \lambda$.

Let Γ consist of the (x,n), where n is the least
nonnegative integer such that $x \in' V(j^n(\kappa))$, in the sense of
M. We think of x as internal to M and n as external to M.
We now give the interpretations of <',c,F,j,P on the domain
Γ. The structure will have the required properties except
that the domain is Γ. We then convert the domain from Γ to
N in the standard way. We will be verifying condition i for
Γ instead of N, before we make this conversion.

(x,n) <* (y,m) $\leftrightarrow$ x <' y.
c* = $(\varnothing, 0)$.
j*((x,0)) = (x,0).
j*((x,n)) = (j(x),n+1), n $\geq$ 1.
F*((x,n)) = (F(x),1) if n = 0; (j(x),n+1) if n $\geq$ 1.
P*($(x_1, i_1), \ldots, (x_k, i_k)$) $\leftrightarrow$ B(R,V(μ),<)($x_1, \ldots, x_k$).

Here $\varnothing$ is the empty set of M. Also x' is chosen so that in
the sense of M, the ordinal position of x' in <' is κ plus
the ordinal position of x in <'. The given
$(x,n),(y,m),(x,0)),(x_1,i_1),\ldots,(x_k,i_k)$ are assumed to be in
Γ. Note that Γ and these interpretations exist, using the
complete diagram of M.

We need to first check that the pairs occurring here, other
than those given, are elements of Γ. We work internally in
M. Clearly $\varnothing \in' V(j^0(\kappa)) = V(\kappa)$, and so $(\varnothing, 0) \in \Gamma$. Also
(x,n) $\in \Gamma' \leftrightarrow$ (j(x),n+1) $\in \Gamma'$ by j elementarity, provided n
$\geq$ 1. Let (x,0) $\in \Gamma$. Then $x \in' V(\kappa)$. Then clearly $x' \notin V(\kappa)$,
where x' is as defined above, as V(κ) comprises the first κ
points under <. As x' has position < $\kappa+\kappa$ in <, clearly x'
$\in' V(\kappa+1) \subseteq V(j(\kappa))$, and so (x',1) $\in \Gamma$.

Henceforth, whenever we write (x,n), we are implicitly
asserting that (x,n) $\in \Gamma$.

Since <' is good in M, for each n $\geq$ 0, the <* least (x,n+1)
$\in \Gamma$ has x = $j^n(\kappa)$, and the least (x,0) $\in \Gamma$ has x = 0^M. Also
the fixed points of j* are the (x,0) with j(x) = x. The

critical point of j* is $(\kappa,1) = F((0,0))$.

For axiom i of $W(k,R)$, it is clear that $<^*$ is a strict linear ordering, with least element $(0,0)$.

For axiom ii, it is clear by construction that $(x,0) <^*$ $(y,0) \rightarrow F^*((x,0)) <^* F^*((y,0))$. Also for $n \geq 1$, $(x,0) <^*$ $(y,1) \rightarrow F^*((x,0) <^* (j(\kappa),2) \leq^* (j(y),2) = F^*((y,1))$. Now let $(x,n),(y,m) \in \Gamma$, $n \geq 1$, where $(x,n) <^* (y,m)$. Then $m \geq 1$, and $(j(x),n+1) <^* (j(y),m+1)$ holds by j elementarity.

For axiom iii, clearly by construction, $(x,0) <^* F^*((x,0))$. Also $(x,n) <^* (j(x),n+1)$, $n \geq 1$, by the well known property in M of j that $x <' j(x)$ for $x \in V(j^n(\kappa))\backslash V(\kappa)$.

For axiom iv, note that $F^*(c^*) = (\kappa,1)$. Let $(x,n) <^* (\kappa,1)$. Then $n = 0$ and $j^*((x,0)) = (x,0)$.

For axiom v, let $(x,n) \geq^* F^*(c^*)$. Then $(x,n) \geq^* (\kappa,1)$, and so $n \geq 1$ and $j^*((x,n)) = F^*((x,n))$.

For axiom vi, let $(x_1,a_1) \leq^* \ldots \leq^* (x_k,a_k)$. We verify $P^*((x_1,a_1),\ldots,(x_k,a_k)) \leftrightarrow \neg(\exists(y_1,b_1),\ldots,(y_k,b_k) <_{lex}$ $((x_1,a_1),\ldots,(x_k,a_k)))(P^*((y_1,b_1),\ldots,(y_k,b_k)) \wedge$ $((x_1,a_1),\ldots,(x_k,a_k)) R ((y_1,b_1),\ldots,(y_k,b_k)))$. This is very close to $P(x_1,\ldots,x_k) \leftrightarrow \neg(\exists(y_1,\ldots,y_k) <_{lex}$ $(x_1,\ldots,x_k))(P(y_1,\ldots,y_k) \wedge (x_1,\ldots,x_k) R (y_1,\ldots,y_k))$, which we have. Here P abbreviates $B(R,V(\mu),<)$. This is according to the construction of $B(R,V(\mu),<)$ in M, since $x_1 \leq \ldots \leq x_k$. An issue arises because the existential quantifiers may be witnessed with ranks above the $j^n(\kappa)$, $n \in N$. However, if we use $<'$ least witnesses in M, then by j elementarity, the ranks are appropriately controlled.

For axiom vii, $P^*((x_1,n_1),\ldots,(x_k,n_k)) \leftrightarrow$ $P^*(j^*(x_1,n_1)),\ldots,j^*(x_k,n_k))$ is equivalent to $B(R,V(\mu),<)(x_1,\ldots,x_k) \leftrightarrow B(R,V(\mu),<)(j(x_1),\ldots,j(x_k))$, which holds by j elementarity.

For axiom viii, let $b_1,\ldots,b_i \in N$, $1 \leq i \leq k-1$. Assume the projection E of P^* at $(x_1,a_1),\ldots,(x_{k-i},a_{k-i})$ lies entirely below $(j^n(\kappa)),n+1$ in $<^*$. We show that the translate of E by $F^{*b\_1},\ldots,F^{*b\_i}$ is a projection of P^* in the sense of $(\Gamma,<^*,c^*,F^*,j^*,P^*)$. Let E' be the result of just taking the

first coordinates of the pairs. Then E' is a projection of B(R,V(μ),<) living in the V($j^n(\kappa)$) of M, using j elementarity (use least witnesses for membership in the projection). The translate of E by $F\star^{b\_1},\ldots,F\star^{b\_i}$ corresponds to taking the translate of E' by $F^{b\_1},\ldots,F^{b\_i}$ in M. The translate of E' by $F^{b\_1},\ldots,F^{b\_i}$ in M is a set D internal to M lying in its V($j^{n+1}(\kappa)$), of dimension i. Therefore, in M, D is the projection of B(R,V(μ),<) at (u,0,...,0), where u = {(0,...,0,v): v $\in$ D}. Note that u $\in$ the V($j^{n+1}(\kappa)$) of M. Now take the corresponding projection of P* in (Γ,<*,c*,F*,j*,P*), which is at ((u,m),(0,0),...,(0,0)), for some unique m $\in$ N.

For condition i, let (x,n) $\in$ Γ. Then (x,n) <* $F\star^n$(c*). So we can set k = n.

For condition ii, note that rng(F*) = {(y,m) $\in$ Γ: ($\exists$(x,n) $\in$ Γ)(F(x) = y)} = {(y,m) $\in$ Γ: ($\exists$(x,n) $\in$ Γ)(F(x) = y $\land$ m < n)}. Since Γ exists and the relation F(x) = y exists, both from the existence of the complete diagram for M, rng(F*) must exist.

Now convert from domain Γ to domain N in the standard way. QED

THEOREM 5.4.9. WKL$_0$ + Con(HUGE) proves Proposition 5.4.3.

Proof: Let R be an order invariant relation on Q^k. Let (N,<*,c*,F*,j*,P*) be given by Lemma 5.4.8. By Theorem 5.2.1, (N,<*,c*,F*) is isomorphic to some (D,<$_D$,-1,SC), D $\subseteq$ Q. Hence (N,<*,c*,F*,j*,P*) is isomorphic to some (D,<$_D$,-1,SC,ush,S) by the same isomorphism. We just use (D,<$_D$,ush,S). By the isomorphism, S is the desired $\leq$# basis that contains its upper shift, and whose projections are closed under N^k translation. QED

Proposition 5.3.8 is a natural modification of Proposition 5.3.3 which we showed was a priori Π^0_1 over WKL$_0$ (Theorem 5.3.10). We now give an analogous treatment for Proposition 5.4.3.

PROPOSITION 5.4.10. Every order invariant relation on Q^k has an m$\leq$# basis containing its upper shift, whose bounded N^k translated projections are projections.

THEOREM 5.4.11. Proposition 5.4.10 is provably equivalent to Con(HUGE) over WKL_0. This also holds for some fixed dimension k.

The proof of Proposition 5.4.10 in WKL_0 + Con(HUGE) is a straightforward modification of the proof given above for Proposition 5.4.3. The reversal in Theorem 5.4.11 will be proved elsewhere.

THEOREM 5.4.12. Proposition 5.4.10 is provably equivalent to a Π^0_1 sentence over ACA_0 via Gödel's Completeness Theorem.

Proof: Follow the proof of Theorem 5.3.11. QED

6. PROVABLE FALSIFIABILITY AND ARITHMETICITY

There is a well recognized key property of certain mathematical propositions. Informally,

IF φ IS FALSE THEN IT IS AUTOMATICALLY REFUTABLE

We say that φ is "provably falsifiable". E.g., before FLT was proved, it was well recognized that FLT is provably falsifiable. After FLT was proved, FLT was seen to be provably falsifiable by default.

DEFINITION 6.1. A sentence φ in the language of set theory is provably falsifiable over a theory T if and only if the sentence "if φ is false then φ is refutable in T" is itself provable in T.

THEOREM 6.1. Let φ be a sentence in the language of second order arithmetic, and T prove RCA_0. Suppose φ is implicitly Π^0_1 over T. Then φ is provably falsifiable over T.

Proof: Left to the reader. QED

Theorem 6.1, with T = ACA_0, applies to most of the propositions we have presented in sections 4,5 above, as they are all implicitly Π^0_1 over WKL_0. These include the following list:

    Propositions 4.2.1, 4.2.2, 4.2.4, 4.2.6, 4.2.8, 4.2.9,
 4.2.12, 4.2.17, 4.2.18, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 5.3.3,
            5.3.8, 5.4.1, 5.4.3, 5.4.10.
 Each instance of the Master Template, Master Template/≥0,
          Max Template, Max Template/≥0.

The condition "provably falsifiable" is related to
falsifiability of physical theories. Generally speaking,
physical theories that are not falsifiable by observations
have rather controversial reputations. Such physical
theories are often outright rejected as not being
meaningful by many physical scientists.

Will this kind of attitude be adopted by mathematicians?
I.e., that in order for a mathematical question to be
regarded as truly significant, must it be first seen to be
provably falsifiable? This kind of attitude has already
been perhaps adopted by a significant segment of applied
mathematicians.

The work on Concrete Mathematical Incompleteness provides
the only current examples of Mathematically Natural
Provably Falsifiable Incompleteness. The first
Intellectually Natural Provably Falsifiable Incompleteness
is of course due to Gödel with his consistency statements.

The list of Propositions above remain provably equivalent,
over ACA, if we require that the clique or basis be
arithmetical (in the sense of recursion theory).

In fact, we can strengthen the requirement that the maximal
clique or basis be arithmetical to it being recursive in
0', or, equivalently, Δ^0_2 (notions from recursion theory),
with the same results. When we make this stronger
requirement, we obviously put the statements into
arithmetical form - although certainly not in Perfectly
Mathematical Natural form.

7. FINITE FORMS

Most of the results in this section will be proved
elsewhere. They are based on the following standard notion
of height.

DEFINITION 7.1. For $x \in Q^k$, the height of x, $\eta(x)$, is the
least n such that x can be written with numerators and

denominators of magnitude ≤ n. Let $S \subseteq Q^k$ and $p \in Q$. $\eta(S) =$ max$\{\eta(x): x \in S\}$ if S is finite; ∞ otherwise. $S[p] = \{x \in S: \eta(x) \leq p\}$.

7.1. GRADED

We now give a graded form for the Master Template.

R HAS A T INVARIANT MAXIMAL ROOT
$R,T \subseteq Q^{2k}$ order theoretic.

DEFINITION 7.1.1. Let $R \subseteq Q^{2k}$. S is an r,t-graded maximal root in R if and only if $S \subseteq Q^k[kt]$ is a root in R $\wedge$ ($\forall x \in Q^k \backslash S$)($r \leq \eta(x) \leq t \rightarrow$ ($\exists y \in S$)($\eta(y) \leq k\eta(x) \wedge \{x,y\}^2 \neg\subseteq R$)). S is a t-graded maximal nonnegative root in R if and only if $S \subseteq Q^k[kt]|{\geq}0$ is a nonnegative root in R $\wedge$ ($\forall x \in Q^k|{\geq}0 \backslash S$)($r \leq \eta(x) \leq t \rightarrow$ ($\exists y \in S$)($\eta(y) \leq k\eta(x) \wedge \{x,y\}^2 \neg\subseteq R$)).

THEOREM 7.1.1. (WKL$_0$) Let R,T be order theoretic in Q[r]. The following are equivalent.
i. $R \subseteq Q^k$ has a T invariant maximal root.
ii. For all t, $R \subseteq Q^{2k}$ has a r,t-graded maximal nonnegative root whose T image is a nonnegative root in R.

Note that this graded finite form, ii, is explicitly Π^0_1.

Here is our graded finite form of Proposition 4.3.1.

PROPOSITION 7.1.2. Every order invariant $R \subseteq Q^{2k}$ has a t-graded maximal nonnegative root with $S_{1...n}|{>}n = S_{0...n-1}|{>}n$.

PROPOSITION 7.1.3. Every order invariant $R \subseteq Q^{2k}$ has a t-graded maximal root, where projections at any two equal length subsequences of (0,...,k) agree above their combined terms.

We now give our graded finite form of Proposition 5.3.3.

DEFINITION 7.1.2. Let $R \subseteq Q^{2k}$. S is a t-graded # basis for R if and only if $S \subseteq Q[kt]^k$ is R free $\wedge$ ($\forall x \in \#(S) \backslash S$)($\eta(x) \leq t \rightarrow$ ($\exists y \in S$)(x R-reduces to y $\wedge \eta(y) \leq k\eta(x)$)).

PROPOSITION 7.1.4. Every order invariant $R \subseteq Q^{2k}$ has a t-

graded # basis containing its upper shift below k.

THEOREM 7.1.5. Propositions 7.1.2 - 7.1.4 are provably equivalent to Con(SRP) over WKL_0.

Note that Propositions 7.1.2 - 7.1.4 are explicitly Π^0_1.

7.2. **EXTENDABILITY**

We now present a different kind of finite form for Propositions 4.3.1 and 5.3.3.

DEFINITION 7.2.1. Let $R \subseteq Q^{2k}$ and $E \subseteq Q$. S is an E maximal nonnegative root in R if and only if S is a nonnegative root in R which is not a proper subset of any nonnegative root in R contained in $S \cup E^k$.

THEOREM 7.2.1. (RCA_0) Every order invariant $R \subseteq Q^{2k}$ has a finite $\{0,...,k\}$ maximal nonnegative root, where $S_{1...n}|>n = S_{0...n-1}|>n$.

PROPOSITION 7.2.2. Every order invariant $R \subseteq Q^{2k}$ has a $\{0,...,k\}$ maximal nonnegative root S, extendable to a finite fld(S) maximal nonnegative root with $S'_{1...n}|>n = S'_{0...n-1}|>n$.

PROPOSITION 7.2.3. Every order invariant $R \subseteq Q^{2k}$ has a finite $\{0,...,k\}$ maximal nonnegative root S extendable to a finite fld(S) maximal nonnegative root whose projections at any two equal length subsequences of $(0,...,k)$ agree above their combined terms.

DEFINITION 7.2.2. Let $R \subseteq Q^{2k}$ and $E \subseteq Q$. S is an E,# basis for R if and only if S is R free and every $x \in E^k \backslash S$ R-reduces to an element of S.

THEOREM 7.2.4. (RCA_0) Every order invariant $R \subseteq Q^{2k}$ has a finite $\{0,...,k\}$,# basis containing its upper shift below k.

PROPOSITION 7.2.5. Every order invariant $R \subseteq Q^{2k}$ has a $\{0,...,k\}$,# basis extendable to a finite fld(S),# basis containing its upper shift below k.

THEOREM 7.2.6. Propositions 7.2.2, 7.2.3, 7.2.5 are each

provably equivalent to Con(SRP) over WKL_0.

Note that these Propositions are explicitly Π^0_2. They become explicitly Π^0_1 when the second root or basis is required to be contained in $Q^k[(8k)!!]$.

7.3. SEQUENTIAL CONSTRUCTIONS

We now give a sequential construction associated with Proposition 5.3.3.

PROPOSITION 7.3.1. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic. Every order invariant $R \subseteq Q^{2k}$ has a free $\{x_1,...,x_k,ush(x_1),..., ush(x_k)\}$, where each $F(x_i,x_{i+1}) \neq x_{i+2}$ R-reduces to x_{i+2}.

PROPOSITION 7.3.2. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic. Every order invariant $R \subseteq Q^{2k}$ has a free $\{x_1,...,x_k,ush(x_k)\}$, where each $F(x_i,x_{i+1}) \neq x_{i+2}$ R-reduces to x_{i+2}.

Propositions 7.3.1 and 7.3.2 are explicitly Π^0_2. Using elimination of quantifiers for $(Q,<)$, we see that they are in Π^0_1 form. Alternatively, we can give a quantitative bound.

PROPOSITION 7.3.3. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic in $Q[n]$. Every order invariant $R \subseteq Q^{2k}$ has a free $\{x_1,...,x_k, ush(x_1),...,ush(x_k)\} \subseteq Q^k[(8kn)!]$, where each $F(x_i,x_{i+1}) \neq x_{i+2}$ R-reduces to x_{i+2}.

PROPOSITION 7.3.4. Let $F:Q^{2k} \rightarrow Q^k$ be order theoretic in $Q[n]$. Every order invariant $R \subseteq Q^{2k}$ has a free $\{x_1,...,x_k, ush(x_k)\} \subseteq Q^k[(8kn)!]$, where each $F(x_i,x_{i+1}) \neq x_{i+2}$ R-reduces to x_{i+2}.

THEOREM 7.3.5. Propositions 7.3.1 – 7.3.4 are provably equivalent to Con(SRP) over WKL_0.

Scott Aaronson has been interested in finding simple computer programs where the halting question is independent of ZFC. Our explicitly Π^0_1 sentences provide such examples, although the associated computer programs are not as simple as one would hope for. So this must be viewed as an ongoing project.

In his Master's Thesis under Scott Aaronson, Adam Yedidia

has programmed an earlier sequential statement of ours involving graphs on $[Q]^{\leq k}$. From a strictly mathematical point of view, we prefer to use finite sequences, roots, and order theoretic functions, as in Propositions 7.3.1 - 7.3.4. However, from the point of view of programming, it is less clear just which is preferred.

We first give a version using finite sequences, with roots, and without the F. This corresponds closely to the variant that Yedidia programs. We then give the actual version Yedidia programmed, which uses graphs on finite sets.

DEFINITION 7.1. Let $R \subseteq Q^{2k}$ and $A, B \subseteq Q^k$. A is reducing B if and only if $(\forall x \in B)(\exists y \in A)(x = y \lor (x \, R \, y \land x >_{lex} y))$.

We caution the reader that R-reduces is strict where the reducing in Definition 7.1 is not strict.

PROPOSITION 7.3.6. For all k, n, r, every order invariant $R \subseteq Q^{2k}$ has a free $\{x_1, \ldots, x_r, ush(x_1), \ldots, ush(x_r)\} \subseteq Q^k[(8knr)!]$, each $\{x_1, \ldots, x_{(8kni)!}\}$ reducing $fld(\{x_1, \ldots, x_i\} \cup \{0, \ldots, n\})^k$.

DEFINITION 7.2. $[Q]^{\leq k} = \{x \subseteq Q: |x| \leq k\}$. Let $x, y, z, w \in [Q]^{\leq k}$. (x, y) and (z, w) are order equivalent if and only if $|x| = |z| \land |y| = |w| \land (\forall i, j)(x_i < y_j \leftrightarrow z_i < w_j)$, where x_i is the i-th element of x, counting from 1 to $|x|$ (and analogously for y, z, w). An order invariant graph on $[Q]^{\leq k}$ is a pair $G = ([Q]^{\leq k}, E)$, where $E \subseteq [Q]^{\leq k} \times [Q]^{\leq k}$ irreflexive and symmetric, and invariant under order equivalence. For $x, y \in [Q]^{\leq k}$, $x <_{lex} y \leftrightarrow A$ is reducing B if and only if $(\forall x \in B)(\exists y \in A)(x = y \lor (x \, E \, y \land x >_{lex} y))$, where $x <_{lex} y$ if and only if the least i with $x_i \neq y_i$ exists and has $x_i < y_i$. $[Q]^{\leq k}[t] = \{x \in [Q]^{\leq k}: $ all elements of x can be written with numerator and denominator of magnitude $\leq t\}$. The upper shift $ush(x)$ of $x \in [Q]^{\leq k}$ is obtained from x by adding 1 to its nonnegative elements.

There are several variants of the lexicographic ordering that can be used here without changing the results.

PROPOSITION 7.3.7. For all $k, n, r \geq 0$, every order invariant graph on $[Q]^{\leq k}$ has a free $\{x_1, \ldots, x_r, ush(x_1), \ldots, ush(x_r)\} \subseteq [Q]^{\leq k}[(8knr)!]$, each $\{x_1, \ldots, x_{(8kni)!}\}$ reducing $[x_1 \cup \ldots \cup x_i \cup \{0, \ldots, n\}]^{\leq k}$.

THEOREM 7.3.8. Propositions 7.3.6 and 7.3.7 are provably equivalent to Con(SRP) over WKL_0.

7.4. Ω BASES

DEFINITION 7.4.1. $\Omega:\wp(Q^k) \rightarrow \wp(Q^k)$ is defined as follows. $\Omega(S)$ is the set of all lexicographically least values over S of the various order theoretic $F:Q^k \rightarrow Q^k$ in $\{0,...,k\}$.

DEFINITION 7.4.2. Let $R \subseteq Q^{2k}$. S is an Ω basis if and only if for all $x \in \Omega(S)^k$, $x \in S$ if and only if x does not R reduce to any $y \in S$.

THEOREM 7.4.1. If $S \subseteq Q^k$ is finite then $\Omega(S)$ is finite. Every $R \subseteq Q^{2k}$ has a nonempty finite Ω basis.

Proof: The first claim follows from the fact that there are only finitely many order theoretic $F:Q^k \rightarrow Q^k$ in $0,...,k$. For the second claim, let R be a relation on Q^k. Define $S \subseteq \{0,...,k\}^k$ by lexicographic induction on $\{0,...,k\}^k$ by $x \in S$ if and only if x does not R-reduce to any element of S. Then $(0,...,0) \in S$. Clearly S is an Ω basis since $\Omega(S) \subseteq \{0,...,k\}^k$. QED

PROPOSITION 7.4.2. Every order invariant $R \subseteq Q^{2k}$ has a finite Ω basis containing its upper shift below k.

Proposition 7.4.2 is explicitly arithmetic. It is easy to give an a priori upper bound on the cardinality of S relative to k. We can then use the well known decision procedure for $(Q,<,0,+1)$ to put Proposition 7.4.2 in Π^0_1 form. Alternatively, we can use an explicit upper bound.

PROPOSITION 7.4.3. Every order invariant $R \subseteq Q^{2k}$ has an Ω basis in $Q[(8k)!]^k$ containing its upper shift below k.

We prove Proposition 7.4.2 from the following variant of Proposition 5.3.3. We use $\#'(S) = (fld(S) \cup N)^k$.

PROPOSITION 5.3.3'. Every order invariant relation on every Q^k has a $\#'$ basis containing its upper shift.

The proof of Theorem 5.3.4 goes through with virtually no

modification for Proposition 5.3.3'.

THEOREM 7.4.4. Propositions 7.4.2 and 7.4.3 are provably equivalent to Con(SRP) over EFA.

Proof: We prove Proposition 7.4.2 from WKL_0 + Con(SRP). The remaining claims will be proved elsewhere (including reducing WKL_0 to EFA). By Theorem 5.3.4, it suffices to derive Proposition 7.4.2 from Proposition 5.3.3'. Let R be an order invariant relation on Q^k. Let S be a #' basis for R containing its upper shift. We now construct a finite S* $\subseteq$ S with the appropriate witnesses. Specifically, first make sure that Ω(S*) $\subseteq$ #'(S) by putting the arguments from S^k that lex minimize the order theoretic functions in 0,...,k, into S*. Then put these lexmin's or their lexicographically least R-reductions into S*. Finally, close under the upper shift below k. Note that S* is finite, and is an Ω basis for R containing its upper shift below k. QED

8. ORDINALS

DEFINITION 8.1. S $\subseteq$ Q^k is normal if and only if U = {$-1-n^{-1}$: n ≥ 1} is an initial segment of fld(S).

DEFINITION 8.2. Let R be a relation on Q^k. REL(R) is the relation on U given by p REL(R) q if and only if any normal # basis S of R containing its upper shift and a given (p,...,p,p'),(q,...,q,q') has p' < q'. REL*(R) is the relation on U given by p REL*(R) q if and only if any normal ≥# basis S of R containing its upper shift and a given (p,...,p,p'),(q,...,q,q'), whose N^k translated projections are projections, has p' < q'.

Some REL(R)'s are well orderings on U, whereas "most" are not.

DEFINITION 8.3. The provable ordinals of an extension K of ZFC are the ordinals with a recursive representation which can be proved to be well ordered in K.

THEOREM 8.1. (SRP^+) The provable ordinals of SRP are the ordinals of the REL(R) that are well orderings on U, for order invariant relations R on some Q^k.

THEOREM 8.2. ($HUGE^+$) The provable ordinals of HUGE are the ordinals of the REL*(R), where R is an order invariant

relation on some Q^k.

The proofs of Theorems 8.1 and 8.2 will appear elsewhere.

9. TEMPLATES

The first four Templates have already been introduced in section 1.

MASTER TEMPLATE. R HAS A T INVARIANT MAXIMAL ROOT. $R, T \subseteq Q^{2k}$ are order theoretic.

MASTER TEMPLATE/≥0. R HAS A T INVARIANT MAXIMAL NONNEGATIVE ROOT. $R, T \subseteq {}^{Q2}k$ are order theoretic.

MAX TEMPLATE. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T INVARIANT MAXIMAL ROOT. $T \subseteq Q^{2k}$ is order theoretic.

MAX TEMPLATE/≥0. EVERY ORDER INVARIANT $R \subseteq Q^{2k}$ HAS A T INVARIANT MAXIMAL NONNEGATIVE ROOT. $T \subseteq Q^{2k}$ is order theoretic.

CONJECTURE 9.1. Every instance of these four Templates is provable or refutable in SRP. In fact, provable in SRP or refutable in RCA_0. It follows from Theorem 4.1.6 that this conjecture for the Master Template implies this conjecture for the Max Template, and also this conjecture for the Master Template/≥0 implies this conjecture for the Max Template/≥0.

CONJECTURE 9.2. There are instances of the Master Template/≥0 with k = 4 and 4 parameters, which cannot be proved or refuted in ZFC.

THEOREM 9.3. (RCA_0) If Con(SRP) then Conjecture 9.1 for the Master Template/≥0 and Max Template/≥0 fails for fixed SRP[n].

Proof: Assume Con(SRP). Let SRP[n] be given. By the last claim of Theorem 4.3.4 (proved elsewhere), let $R \subseteq Q^{2k}$ be order invariant, where Proposition 4.3.1 for R implies Con(SRP[n]) over RCA_0, and where Proposition 4.3.1 for R is provable in SRP. Hence Proposition 4.3.1 for R is neither provable nor refutable in SRP[n], and Proposition 4.3.1 for R is an instance of Template A. Also Proposition 4.3.1 for

k implies Con(SRP[n]), and is also provable in SRP. Hence Proposition 4.3.1 for k is neither provable nor refutable in SRP, and Proposition 4.3.1 for k is an instance of Template B. QED

We make the following conjecture concerning Proposition 4.3.1.

CONJECTURE 9.4. The statement "every order invariant $R \subseteq Q^{2k}$ has a maximal nonnegative root, where projections at two equal length subsequences of (0,1,2,3,4} agree above their combined terms" is not provable or refutable in ZFC.

Conjecture 9.1 might be false because algorithmic unsolvability enters into Master Template or Master Template/$\geq$0. This seems much more likely for the Master Templates than for the other two. In any case, Conjectures 9.2, 9.4 point the way toward an appropriate modification of Conjecture 9.1 in case it fails. We can simply restate Conjecture 9.1 in k = 4, or other small k, with or without restricting the number of parameters. If we don't restrict the number of parameters, there will be infinitely problem instances. If we do restrict the number of parameters, there will be only finitely many problem instances up to obvious equivalence.

It is our general view that incompleteness - particularly ZFC incompleteness - arises at lower complexity statements than algorithmic unsolvability. Any substantial discussion of this viewpoint is beyond the scope of this paper.

We now come to # bases.

CONJECTURE 9.5. The statement "every order invariant $R \subseteq Q^8$ has a # basis containing its upper shift" is provably equivalent to Con(SRP) over WKL_0.

According to Theorem 5.3.4, Conjecture 9.5 holds with 4 replaced by some integer.

The upper shift on Q is an example of a rational piecewise linear function from Q into Q. In Proposition 5.3.3, we use its diagonal image ush(S) on any $S \subseteq Q^k$.

Below "ot" abbreviates "order theoretic, and "oi" abbreviates "order invariant".

TEMPLATE (ot). Let f:Q → Q be rational piecewise linear and R ⊆ Q^{2k} be order theoretic. R has a # basis containing its diagonal f image.

TEMPLATE (oi). Let f:Q → Q be rational piecewise linear. Every order invariant R ⊆ Q^{2k} has a # basis containing its diagonal f image.

CONJECTURE 9.6. Every instance of # Template (ot) is provable or refutable in SRP^+. In fact, provable in SRP^+ or refutable in RCA_0. This follows for # Template (oi).

Once again, we believe that if algorithmic unsolvability occurs in the # Templates then it is reflected only at higher levels of complexity than where ZFC or SRP incompleteness occurs.

10. FORMAL SYSTEMS USED

EFA   Exponential function arithmetic. Based on exponentiation and bounded induction. Same as $I\Sigma_0(\exp)$, [HP93], p. 37, 405.
RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si09].
WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si09].
ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si09].
ACA ACA_0 with induction for all formulas.
Z(C)   Zermelo set theory (with the axiom of choice).
ZF(C)   Zermelo Frankel set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Je14].
SRP[k]   ZFC + (∃λ)(λ has the k-SRP), for fixed k. Definition 2.9.
SRP   ZFC + (∃λ)(λ has the k-SRP), as a scheme in k. Definition 2.9.
SRP^+ ZFC + (∀k)(∃λ)(λ has the k-SRP). Definition 2.9.
HUGE[n]   ZFC + "there exists an elementary embedding j:V(λ) → V(μ) with critical point κ such that $j^n(\kappa) < \lambda$", for fixed n.
HUGE   ZFC + "there exists an elementary embedding j:V(λ) → V(μ) with critical point κ such that $\alpha = j^n(\kappa) < \lambda$", as a scheme in n.

HUGE$^+$ ZFC + ($\forall$n)(there exists an elementary embedding j:V(λ) → V(μ) with critical point κ such that $j^n(\kappa) < \lambda$).

The above HUGE hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings). For more about huge cardinals, see [Ka94], p. 331.

For background concerning the SRP hierarchy, see [Fr14a], section 10.1, and [Fr01].

11. CURRENT ONGOING DEVELOPMENTS

We are currently working with the notion of continuation which appears to support a major simplification of Proposition 4.3.1 and the other Propositions in section 4.

It is not yet clear which related notions will also support major simplifications in Proposition 5.3.3 and the other Propositions in section 5. Many ways of elaborating on the continuation idea are being currently explored.

DEFINITION 11.1. For finite sequences x,y, x*y is the concatenation of x,y. S' is a nonnegative continuation of S ⊆ Q^k if and only if S ⊆ S' ⊆ Q^k|≥0 ∧ ($\forall$x,y ∈ S')($\exists$z,w ∈ S)(x*y and z*w are order equivalent).

PROPOSITION 11.1. Every finite E ⊆ Q^k|>n has a maximal nonnegative continuation, where $S_{1...n}$|>n = $S_{0...n-1}$|>n.

DEFINITION 11.2. S' is a nonnegative r-continuation of S ⊆ Q^k if and only if S ⊆ S' ⊆ Q^k|≥0 ∧ ($\forall$$x_1$,...,$x_r$ ∈ S')($\exists$$y_1$,...,$y_r$ ∈ S)(x_1*...*x_r and y_1*...*y_r are order equivalent).

PROPOSITION 11.2. Every finite E ⊆ Q^k|>n has a maximal nonnegative r-continuation, where $S_{1...n}$|>n = $S_{0...n-1}$|>n.

We have been able to show that Proposition 11.1 and the more general Proposition 11.2 are provably equivalent to Con(SRP) over WKL_0, and provable in SRP for fixed n.

The advantage in bringing in a new parameter, r, in Proposition 11.2 is that this will likely help considerably in getting very small choices of parameters so that the statement is independent of ZFC.

REFERENCES

[Av03] J. Avigad, Number theory and elementary arithmetic, *Philosophia Mathematica,* 11:257-284, 2003.

[Co63,64] P.J. Cohen, The independence of the continuum hypothesis. Proceedings of the National Academy of Sciences of the United States of America 50(6):1143-1148. P.J. Cohen, The independence of the continuum hypothesis. Proceedings of the National Academy of Sciences of the United States of America 51(1):105-110.

[Fr01] H. Friedman, Subtle cardinals and linear orderings, Annals of pure and applied logic, 107(1-3): 1-34.

[Fr11] H. Friedman, Invariant Maximal Cliques and Incompleteness, Downloadable Manuscripts, #71, October 7, 2011, 132 pages.

[Fr14a] H. Friedman, Invariant Maximality and Incompleteness, in: Foundations and Methods from Mathematics to Neuroscience: Essays Inspired by Patrick Suppes. (2014) Edited by Colleen E. Crangle, Adolfo Garcia de la Sienra, and Helen E. Longino, Stanford, California: CSLI Publications, p. 25-51. Downloadable Manuscripts, #77, https://u.osu.edu/friedman.8/foundationaladventures/downloadable-manuscripts.

[Fr14b] H. Friedman, Testing the Consistency of Mathematics, 7 pages, July 23, 2014. Extended abstract. https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ #83.

[Fr15] H. Friedman, Boolean Relation Theory and Incompleteness. Lecture Notes in Logic, ASL Publications. https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/.

[Go29,30] K. Gödel, On the completeness of the calculus of logic. On the completeness of the axioms of the functional calculus of logic. In S. Feferman, J. W. Dawson, S. C. Kleene, G. H. Moore, R. M. Solovay, and J. V. Heijenoort, eds., Kurt Gödel: Collected Works, vol. I, p. 61-101. p. 103-123. Oxford University Press, 1931.

[Go31] K. Gödel, On formally undecidable propositions of principia mathematica and related systems I. In S. Feferman, J. W. Dawson, S. C. Kleene, G. H. Moore, R. M. Solovay, and J. V. Heijenoort, eds., Kurt Gödel: Collected Works, vol. I, p. 145-195. Oxford University Press, 1931.

[Go40] K. Gödel. The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory. In S. Feferman, J. W. Dawson, S. C. Kleene, G. H. Moore, R. M. Solovay, and J. V. Heijenoort, eds., Kurt Gödel: Collected Works, vol. II, p. 33-101. Oxford University Press, 1940.

[HP93] Hájek, P. and P. Pudlák. Metamathematics of First-Order Arithmetic. Perspectives in Mathematical Logic. Berlin: Springer-Verlag.

[Je14]. T.J. Jech, Set Theory: The Third Millennium Edition, revised and expanded. New York: Springer-Verlag Berlin Heidelberg.

[Ka94] A. Kanamori, The Higher Infinite. Large Cardinals in Set Theory from their Beginnings, Perspectives in Mathematial Logic. Berlin: Springer-Verlag.

[Mc10] C. McLarty, What does it take to prove Fermat's Last Theorem? Grothendieck and the logic of number theory, Bulletin of Symbolic Logic Volume 16, Issue 3 (2010), 359-377.

[Mc11] C. McLarty, A finite order arithmetic foundation for cohomology, 2011, http://arxiv.org/abs/1102.1773v3

[Ro49] J. Robinson, Definability and Decision Problems in Arithmetic, Journal of Symbolic Logic, Volume 14, Number 2, June 1949, p. 98-114.

[Si09] Simpson, S. G. 2009. Subsystems of Second Order Arithmetic. Perspectives in Logic. New York: ASL and Cambridge University Press, 2nd edition.

[Wi95] Wiles, Andrew (1995). "Modular elliptic curves and Fermat's Last Theorem" (PDF). *Annals of Mathematics* (Annals of Mathematics) **141** (3): 443–551. doi:10.2307/2118559. ISSN 0003-486X. JSTOR 2118559. OCLC 37032255. Zbl 0823.11029

[WIKIa] http://en.wikipedia.org/wiki/Reverse_mathematics

[WIKIb]
http://en.wikipedia.org/wiki/Elementary_function_arithmetic