AN APPRECIATION OF HILARY'S MATHEMATICAL WORK

**philosophy in an age of science
a conference in honor of Hilary Putnam's 85th birthday
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6. Evaluation of Hilary's work, and Hilary's new homework.

1. HILARY 1964-65.

I met Hilary for the first time in Fall, 1964. At that time I was an entering 16 year old Freshman at MIT.

I already "knew" Hilary from the MIT course catalog. Let me explain.

I was deeply moved in high school by Bertrand Russell's Introduction to Mathematical Philosophy, a book Russell wrote in jail in World War I.

Hilary remarked that his productivity substantially increased after his retirement from Harvard as University Professor.

Come to think of it, could it be that jail may also be a good environment for scholarly work? No teaching, no committee work, and no pressure to follow conventional academic wisdom!

In the Russell book, the status of the Axiom of Choice relative to the other axioms of set theory was raised as a central open question.

I decided then that I would work on this problem when I got to MIT. But I received the MIT course catalog, which contained the course description for an advanced course in the Spring 1965, mentioning "consistency and independence

results of Paul J. Cohen concerning the Axiom of Choice and the Continuum Hypothesis". Instructor: H. Putnam.

So I realized that "my" problem was taken away. I wanted to meet this H. Putnam guy.

I got an appointment with Hilary in the Fall of 1964 outside Walker Memorial, on the MIT campus.

I had an "easy" question for Hilary. I told Hilary that I had read Bertrand Russell's Introduction to Mathematical Philosophy, and Inquiry into Meaning and Truth, and also Hilbert and Ackermann, Principles of Mathematical logic. I wanted to know

how does logic start?

I said logic appeared to be circular because of all this careful thinking involved in setting up logic. I.e., there seemed to be logic involved in starting logic - and I was confused by this.

Hilary said I should look at the Rosenbloom and Smullyan books, and not be paralyzed by this circularity. He gave me the distinct impression that he was able to do logic even though it was circular!

Of course, I did ask Hilary something really deep, and I hope no student comes to me with that question!

I remember going home, I think, for Christmas vacation, and listening to a BBC interview with Bertrand Russell over the radio. It was a short interview, and I savored every word.

I distinctly remember thinking - Hilary Putnam is going to be the closest I will ever get to Bertrand Russell! This turned out to be essentially true - I also met Kurt Gödel in 1977.

I took Hilary's course in Spring 1965 on graduate set theory, which included the work of Gödel and Cohen as advertised in the course catalog. Dick Boyd was the TA, and George Boolos was a fellow student!

I also took Hilary's course on Hierarchy Theory the next year when he moved to Harvard. I remember traveling on the Mass Ave. bus to Harvard, on my way to Hilary's course, all the while talking to Tony Martin and Saul Kripke.

2. HILARY'S WORK ON VALIDITY AND SATISFIABILITY.

Hilary's work with Martin Davis concerns a problem in mathematical logic that is at the core of a now very large subject called automated reasoning - a branch of artificial intelligence.

Specifically, Hilary and Martin tackle the problem of determining whether a sentence in first order logic is valid - i.e., is true in all models.

It had already been established by Alonzo Church (1936) that this endeavor is impossible in the following sense: there is no algorithm for determining validity in first order logic, even with only a single binary relation symbol and no equality.

But Hilary and Martin had the good sense to try to give a practical algorithm that might work well in practical situations.

Their idea was to put the negation of the first order logic sentence in prenex form, and go through a procedure that is tantamount to constructing a model. The original sentence will be valid if and only if this procedure gets blocked at some finite stage.

This process is quite far from producing the desired practical results, but lies at the core of an enormous effort in automated reasoning that has had some substantial successes.

The most innovative of the Hilary and Martin approach concerns the way that they recognize blocks (blockades) at finite stages.

This innovation focuses entirely on propositional logic - a highly reduced form of predicate logic. In fact, "Davis Putnam algorithm" most commonly refers just to the propositional logic innovation only.

In propositional logic, we have letters $p_1, p_2, \ldots$ standing for unknown sentences. We use the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ from mathematics. A propositional formula results from combining the letters and the connectives in the obvious coherent way.

Hilary and Martin focus on the satisfiability of propositional formulas. I.e., that there be an assignment of truth values (true,false) to the letters occurring in the formula, so that the formula comes out true.

For many years, this has been referred to as SAT.

There is the obvious ancient algorithm for SAT, which is to try out all of the possible 2^n assignments, where n is the number of letters in the propositional formula. But one hopes to do much better than this, at least in practical situations. To this day, nobody knows whether you can avoid an exponential amount of steps - and instead use a polynomial number of steps. This is the famous P = NP problem in theoretical computer science.

There is now a huge industry devoted to SAT, with major software packages and major applications ranging from the design and verification of circuits, airline scheduling, industrial control, cryptography, etc.

At the core of this development is DP (Davis/Putnam), which operates on propositional formulas given, or reduced, to conjunctive normal form. This consists of a set of clauses, where each clause takes the form

$$\pm p_{i1} \vee \ldots \vee \pm p_{ir}$$

where $\pm p$ is either p or ¬p. At each stage in the process, we have a list of sets of clauses.

At any stage, The original set of clauses is satisfiable if and only if at least one of these sets of clauses is satisfiable.

Steps are performed which successively eliminate letters, but at the cost of creating more sets of clauses.

Specifically, to eliminate the letter p from one set of clauses, we can, on one hand, treating p like true; and on the other hand, treat p like false. This replaces the set of clauses by two sets of simpler clauses.

Eventually, we just get sets of clauses that become a truth value. If true appears anywhere along the process, we can quit and declare satisfiability. Otherwise, we will wind up with all falses, in which case we declare unsatisfiability.

This is the so called DP (Davis/Putnam) algorithm. It is much better than simply trying out assignments.

In the special case where a clause contains only one letter, or where a letter appears only positively or only negatively throughout the set of clauses, the letter can be eliminated without creating a new set of clauses. I.e., satisfiability/unsatisfiability is preserved. This is the basis of the DPLL algorithm (LL for Longemann and Loveland) - a very well known refinement of DP.

More sophisticated refinements involve strategies for picking the letters to be eliminated from a given set of clauses. Such refinements are being intensively investigated to this day.

3. HILARY'S WORK ON INTEGRAL POLYNOMIALS.

In 1960, Hilary published the following remarkable result:

There is no algorithm for determining whether a given polynomial with integer coefficients, in several integer variables, attains all integer values.

Around the same time, Hilary joined forces with Martin Davis and Julia Robinson to work on Hilbert's Tenth Problem. This problem is closely related to Hilary's result, but is more mathematically challenging.

This problem is part of a famous list of 23 problems published by Hilbert around 1900 covering a very large number of areas of mathematics.

H10 asks whether there is an algorithm for determining whether a polynomial with integer coefficients, in several integer variables, has a zero (i.e., attains the value 0).

This problem was eventually solved negatively through the combined efforts of four people in reverse historical order: Yuri Matiyasevich, Juiia Robinson, Martin Davis, Hilary Putnam. It is called the MRDP theorem.

The earliest of the relevant work is joint between Hilary and Martin and Julia Robinson. They tackled a related, but easier problem.

Polynomials of course only involve addition, subtraction, multiplication, and specific integers.

Exponential polynomials are allowed exponentiation as well. There is a problem with staying in the integers, and so it is customary to disallow subtraction, and use only specific positive integers.

Since we have barred subtraction, we consider so called exponential polynomial equations, where the left and right side are exponential polynomials, using specific positive integers.

By an exquisitely clever multi component argument, Hilary and Martin and Julia showed that there was no algorithm for determining the existence of a solution in integers to Exponential Polynomial Equations.

I.e., Hilary, Martin, and Julia solved H10 negatively for Exponential Polynomial Equations.

It was evident at the time that in order to solve the actual H10 negatively, one need only prove the following:

There is a polynomial P with integer coefficients such that for positive integers n,m,r,

$$n^m = r \leftrightarrow (\exists t_1,...,t_k > 0)(P(n,m,r,t_1,...,t_k) = 0).$$

Julia Robinson reduced this condition to an asymptotic one involving exponential growth rates. Finally, using Julia Robinson's work, in 1970, Yuri Matiyasevich proved the existence of the required polynomial P.

This is the way in which H10 was solved negatively. The negative solution is actually stronger than just "no algorithm" in some very important ways. The strong negative solution is generally called the MRDP theorem.

There are still some very major open problems left in this area of H10 and polynomials with integer coefficients. We provide a roadmap for the interested reader.

KNOWN ALGORITHMS.

1. Single variable Diophantine equations are easy. There are very old algorithms for determining the existence of a zero in the integers, in the rationals, in the reals, and in the complex numbers.

2. Tarski gave an algorithm in the 1950s for determining the existence of a zero in the reals, and also in the complex numbers.

3. Carl Ludwig Siegel gave an algorithm for determining the existence of a zero for quadratic P, in the integers. Grunewald and Segal did this for the positive integers. I think that the situation is the same, probably with the same authors, for rationals and positive rationals.

4. There is an algorithm for determining the existence of a zero in the integers, for two variable cubics P.

NO ALGORITHM RESULTS.

1. There is no algorithm for determining the existence of a zero, for quartic P with 58 variables, in the positive integers. James Jones. (Over the integers, somewhat more than 58 will suffice).

2. There is no algorithm for determining the existence of a zero in the integers, with 11 variables. (Zhi Wei Sun, 1992). (Over the positive integers, 9 will suffice, Matiyasevich).

CRITICAL OPEN QUESTIONS IN THIS AREA.

1. Is there is an algorithm for determining the existence of a solution in the integers, for cubic P?

2. Is there an algorithm for determining the existence of a solution in the integers, for 2 variables? For 2 variables, degree 4? For 3 variables, degree 3?

3. Is there an algorithm for determining the existence of a solution in the rationals? For 2 variables, degree 3?

The MRDP theorem has lots of applications, generally, to show that many other properties of mathematical objects can be determined by algorithms.

In my biased opinion, my favorite is the following problem formulated by yours truly.

A line condition is a triple u,v,w which asserts that u,v,w lie on some common line in the usual Euclidean plane. Here the letters are either unknown integer points in the plane, or specific integer points in the plane. E.g.,

u,v,(-1,2)

asserts that the integer point u, the integer point v, and the integer point (-1,2) lie on some common line.

PROBLEM: Given a finite set of line conditions, determine whether they are simultaneously solvable. I.e., whether there are integer points in the plane such that every one of the line conditions hold.

You can find an advanced draft on my website, where I also consider a number of closely related problems in integral Euclidean geometry.

4, HILARY'S OTHER MATHEMATICAL WORK.

I only have time to talk about a fraction of Hilary's other mathematical work.

1957. A decidable theory can have an essentially undecidable axiomatizable extension (with the same constants).

1957. There is an undecidable theory all of whose complete extensions are decidable.

1962 (1964 listed). F is the family of all sets represented in some consistent standard theory if and only if F is closed under intersection, finite set addition and finite set subtraction, and contains the null set and the "universal" set (i.e., the set of all nonnegative integers).

1962 (1964 listed). F is the family of all sets represented in some consistent axiomatizable standard theory if and only if F is a recursively enumerable family of recursively enumerable sets, F contains the null set and the "universal" set, and F is closed under intersection and finite set addition and finite set subtraction.

1963. There is an ordinal a less than constructible ω_1 such that there is no set of integers in $L_{\alpha+1}- L_\alpha$. (Gaps in the constructible hierarchy, which led ultimately to fine structure theory in set theory).

1965. P is a trial and error predicate if and only if P is Δ^0_2.

1965. Every consistent formula in predicate calculus without identity has a model in $\Sigma_1{}^* =$ the smallest class containing the recursively enumerable predicates and closed under truth-functions.

1965. Every consistent formula in predicate calculus with identity has a recursive model with a Π^0_1 domain.

1965 (with Pour-El). There is an r.e. class of r.e. sets which has no r.e. enumeration without repetition.

1969 (with Hensel and Boyd). The ramified analytic sets form the least beta model of second order arithmetic, and also corresponds exactly to "admissible degree hierarchies".

And these appearing in Hilary's forthcoming book, Philosophy in the Age of Science.

Indispensability Arguments in the Philosophy of Mathematics.

Revisiting the Liar Paradox.

Set Theory: Relaism, Replacement, and Modality.

On Axioms of Set Existence.

The Gödel Theorem and Human Nature.

After Gödel.

Nonstandard Models and Kripke's Proof of the Gödel Theorem.

A Theorem of Craig's about Ramsey Sentences.

5. TWO CONTROVERSIAL PHILOSOPHY OF MATHEMATICS PAPERS BY HILARY.

1967. Mathematics Without Foundations, Journal of Philosophy. As is inevitable, your 16 year old Freshman student has rebelled! Of course, student rebellion is nearly inevitable.

2000. (with Juliet Floyd). A Note on Wittgenstein's 'Notorious Paragraph' about the Gödel Theorem."

At the opening of the 1967 article, Hilary writes what can be read as a total rejection of my concept of the Foundational Life, at least as applied to Foundations of Mathematics:

"Philosophers and logicians have been so busy trying to provide mathematics with a "foundation" in the past half-century that only rarely have a few timid voices dared to voice the suggestion that it does not need one. I wish here to urge with some seriousness the view of the timid voices. I don't think mathematics is unclear; I don't think mathematics has a crisis in the foundations; indeed, I do not believe mathematics either has or need "foundations." The much touted problems in the philosophy of mathematics seem ro me, without exception to be problems internal to the thought of various system builders. The systems are doubtless interesting as intellectual exercises; debate between the systems and research within the systems doubtless will and should continue; but I would like to convince you (of course I won't but one can always hope) that the various system of mathematical philosophy, without exception, need not be taken seriously."

Well, this ex 16 year old student wasn't convinced, as Hilary predicted. Although Hilary touched on a large number of very interesting issues - such as determinate truth values and the continuum hypothesis - I don't see how all that discussion really justified this apparent out and out repudiation of what I mean by f.o.m. Indeed, Geoffrey Hellman has talked here about his use of ideas in Hilary's 1967 paper in connection with an approach to the foundations of set theory via modalities.

We do have a foundation for mathematics in the sense of a rather powerful model of mathematical practice, through ZFC, which does a great job on a specific task, and which allows us to derive incredibly impressive facts about mathematical practice that reveal incredibly deep features, that simply cannot be established in any other way.

For example, no less than Charles Fefferman (a famous analyst), when he was Editor of the Annals of Mathematics, said that ZFC provides an essential service in the way of codifying "the current rules of the road". He says that "in order to be accepted for publication in the Annals, it is necessary but not sufficient that the proofs must be readily formalizable in ZFC. If not, any assumptions used going beyond ZFC have to be explicitly stated."

A fact that one can reasonably conclude using theorems established about ZFC, is that a correct proof or refutation of certain problems in classical set theory will not come about - at least not come about without some completely noticeable fundamental change in the rules of the road. ZFC is a good enough model of mathematical practice to see that.

No less than David Hilbert wanted to secure the consistency of mathematics on the basis of what amounts to fragments of PA. This was shown by f.o.m. to be impossible.

Gödel proved

there exists a sentence neither provable nor refutable in ZFC.

What about the examples? Gödel proved

there exists a finitary sentence neither provable nor refutable in ZFC.

there exists a mathematically natural sentence neither provable nor refutable in ZFC. (joint with P.J. Cohen).

However, in order to secure the long term deep relevance of Incompleteness, we need to establish

there exists a mathematically natural finitary sentence neither provable nor refutable in ZFC.

I worked on this most critical of all issues in f.o.m. almost every day for the last 45 years. This is not the right place to talk about what happened...

I now come to the second of these controversial papers:

2000. (with Juliet Floyd). A Note on Wittgenstein's 'Notorious Paragraph' about the Gödel Theorem."

I recognize the historical interest of interpreting Wittgenstein. You can tell that I am a fake historian because I like to create simplified history in order to provide "context" for what I am doing.

To me, being such a fake historian, the fundamental thing that I take away from this 2000 article is

the serious challenge of presenting a philosophically coherent and unassailable account just what is accomplished by Gödel's Incompleteness Theorems.

In order to avoid certain kinds of philosophically clever criticisms, which can probably be defended against, with some difficulty, we focus on a particular restricted form of Incompleteness: there is no proof in finite set theory of the consistency of set theory. Another important version is: there is no interpretation of set theory in finite set theory. Hilbert can be interpreted as expecting and seeking such a proof.

Here is a brief description of the elements of such an account:

1. There is a model of mathematical practice, reflective of essential features of mathematical practice. Call this ZFC.

2. There is a definition of "x is a finite set" arising in 1), fully responsive to mathematical practice.

3. There is a model of finitary mathematical practice, reflective of essential features of finitary mathematical practice. It is a weak fragment of ZFC whose formalization uses 2). Call this fragment FINZFC.

4. There are sentences, with variables restricted to finite sets, as defined in 2), with various properties responsive to all sorts of considerations from mathematical practice, which are provable in ZFC but not in FINZFC.

5. There are sentences, with variables restricted to finite sets, as define in 2), which assert that mathematical practice does not lead to contradiction, through asserting that ZFC is consistent, in a way that is fully responsive to mathematical practice, and which is not provable in FINZFC.

6. EVALUATION OF HILARY'S WORK, AND HILARY'S NEW HOMEWORK.

I was rather moved by the talk of Mario de Caro yesterday when he listed one fundamental philosophical issue after another that Hilary has written about in a major way. This appears to be systematic, with seminal insights on each issue. I can only conclude that

Hilary is arguably the greatest living exponent of the Philosophical Life.

Consequently, Hilary is in dire need of being seriously challenged! So just as Hilary challenged me with some serious homework in Spring 1965 at MIT, I offer the following homework for Hilary.

1. Give a model of physical science practice that is as philosophically coherent and powerful as the model of mathematical practice through ZFC.

2. In particular, is the use of Hilbert space appropriate for use in a model of quantum mechanical practice? If so, why? If not, why not?

3. Is mathematics useful? How is it useful? Why is it useful?

4. Give a model of applied mathematical practice that is as philosophically coherent and powerful as the model of mathematical practice through ZFC.

5. Give a model of probabilistic and statistical practice that is as philosophically coherent and powerful as the model of mathematical practice through ZFC.

6. Present a model of normative reasoning that is every bit as philosophically coherent and powerful as the model of mathematical practice through ZFC.

7. Create a comprehensive system of coherent legal/political/social systems parameterized by fundamental irreducible viewpoints, that is as philosophically coherent and powerful as the model of mathematical practice through ZFC.

8. Present a model of musical judgments focusing on musical micro structure, that is as philosophically coherent and powerful as the model of mathematical practice through ZFC.

9. What is Philosophy? What is the future of Philosophy?

10. A recipient of a homework assignment always have the option to argue that (some of) the homework problems are counterproductive, improperly formulated, or otherwise wrong headed.

In keeping with Moses' life span, let us all give Hilary 35 years to complete his homework assignment!