RESTRICTIONS AND EXTENSIONS

by
Harvey M. Friedman
Ohio State University
Princeton University
friedman@math.ohio-state.edu
http://www.math.ohio-state.edu/~friedman/
February 17, 2003

Abstract. We consider a number of statements involving restrictions and extensions of algebras, and derive connections with large cardinal axioms.

1. Introduction.

By an algebra M we mean a nonempty set dom(M) together with a finite list of functions on dom(M) of various finite arities ≥ 0. The signature of M is the list of arities of the functions of M.

Let M,N be algebras. We say that M is a restriction of N if and only if M,N have the same signature, dom(M) □ dom(N), and the functions of M are restrictions of the respective functions of N. We say that N is an extension of M if and only if M is a restriction of N.

We use □,□ for cardinals throughout the paper.

We write R(□,fg) if and only if every algebra of cardinality □ has a proper restriction with the same finitely generated restrictions up to isomorphism.

We write R(□,□) if and only if every algebra of cardinality □ has a proper restriction with the same restrictions of cardinality □ up to isomorphism.

We write E(□,fg) if and only if every algebra of cardinality □ has a proper extension with the same finitely generated restrictions up to isomorphism.

We write E(□,□) if and only if every algebra of cardinality □ has a proper extension with the same restrictions of cardinality □ up to isomorphism.

In this paper, we will restrict attention to the cases □ = □. All theorems are proved in ZFC.

R(□,fg), R(□,□) are easily treated as follows.

THEOREM 1.1. The following are equivalent.
i) R(□,fg);
ii) R(□,□);
iii) □ > $2^{\square}$.

Our main results concern sufficiently large □.

THEOREM 1.2. The following are equivalent.
ii) for all sufficiently large □, E(□,fg);
ii) for all sufficiently large □, E(□,□);
iii) there exists a measurable cardinal.

There is a corresponding result concerning the “turning point”.

THEOREM 1.3. The following are equivalent.
i) □ is least such that for all □ ≥ □, E(□,fg);
ii) □ is least such that for all □ ≥ □, E(□,□);
iii) □ is the least measurable cardinal.

The Beth function is the cardinal function given by

Beth_0 = □, $\text{Beth}_{\square+1} = 2^{\text{Beth}\square}$, $\text{Beth}_{\square}$ = □{$\text{Beth}_{\square}$: □ < □}.

A Beth number is a cardinal of the form $\text{Beth}_{\square}$. Under the GCH, for all □, $\text{Beth}_{\square} = \text{Aleph}_{\square}$, and so the Beth numbers are just the infinite cardinals.

THEOREM 1.4. The following are equivalent for all Beth numbers □.
i) E(□,fg);
ii) E(□,□);
iii) □ is weakly compact or there is a measurable cardinal □ □.

We present two related results concerning arbitrary cardinals □.

THEOREM 1.5. Let □ be least such that E(□,fg). □ is a weakly Mahlo cardinal > $2^{\square}$, and □ is satisfied in L to be a weakly compact cardinal.

THEOREM 1.6. The following are mutually interpretable.
i) ZFC + there exists □ such that E(□,fg);
ii) ZFC + there exists □ such that E(□,□);

iii) ZFC + there exists a weakly compact cardinal.

The following is proved by a forcing argument.

THEOREM 1.7. The following are mutually interpretable.
i) ZFC + CH + ($\exists$□ < $2^{□1}$)(E(□,□));
ii) ZFC + there exists a weakly compact cardinal.

We now consider the use of languages (lang) for properties R and E.

Informally, we define

R(□,lang) if and only if every algebra of cardinality □ has a proper restriction satisfying the same sentences of lang.

E(□,lang) if and only if every algebra of cardinality □ has a proper extension satisfying the same sentences of lang.

We will consider the three languages WSOL, $L_{□1□}$, and SOL. Here SOL is second order logic, and WSOL is weak second order logic. WSOL is the same as second order logic except that the set quantifiers range over finite relations (rather than arbitrary relations) on the domain.

THEOREM 1.8. In Theorems 1.1 - 1.7, we can replace E(□,fg), E(□,□) by E(□,$L_{□1□}$), and R(□,fg), R(□,□) by R(□,$L_{□1□}$).

THEOREM 1.9. In Theorems 1.2 - 1.4, 1.6, we can replace E(□,fg), E(□,□) by E(□,WSOL). The following are equivalent.
i) R(□,WSOL);
ii) □ > □.

THEOREM 1.10. Let □ be least such that E(□,WSOL). □ is a weakly Mahlo cardinal > □, and □ is satisfied in L to be a weakly compact cardinal.

The following is proved by forcing.

THEOREM 1.11. The following are mutually interpretable.
i) ZFC + (□□ < $2^{□}$)(E(□,WSOL));
ii) ZFC + there exists a weakly compact cardinal.

We can also consider

R'(□,lang) if and only if every algebra of cardinality □ has a proper elementary submodel in the sense of lang.

E’(□,lang) if and only if every algebra of cardinality □ has a proper elementary extension in the sense of lang.

THEOREM 1.12. For any of lang = WSOL, $L_{□1□}$, SOL, we have E’(□,lang) iff E(□,lang), and R’(□,lang) iff R(□,lang).

A cardinal □ is extendible if and only if for all □ there exists □ and an elementary embedding from V(□) into V(□) with critical point □.

THEOREM 1.13. The following are equivalent.
i) for all sufficiently large □, E(□,SOL);
ii) for all sufficiently large □, R(□,SOL);
iii) there exists an extendible cardinal.

THEOREM 1.14. The following are equivalent.
i) □ is least such that for all □ ≥ □, E(□,SOL);
ii) □ is the least extendible cardinal.

A cardinal is called totally indescribable if and only if it is $□^n_m$ indescribable for all n,m < □.

THEOREM 1.15. Each of the following implies the next.
i) there exists a subtle cardinal;
ii) there exists a Beth number □ such that E(□,SOL);
iii) there exists a totally indescribable cardinal.

THEOREM 1.16. Let □ be the least Beth number such that E(□,SOL). □ is a totally indescribable cardinal, greater than □ totally indescribable cardinals, and less than any subtle cardinal.

THEOREEM 1.17. Let □ be least such that E(□,SOL). Then □ is a weakly Mahlo cardinal > □ and □ is satisfied in L to be a totally indescribable cardinal that is greater than □ totally indescribable cardinals.

THEOREM 1.18. Each of the following is interpretable in the next. Each proves the consistency of the previous.
i) ZFC + there exists a totally indescribable cardinal;
ii) ZFC + there exists □ such that E(□,SOL);
iii) ZFC + there exists a subtle cardinal.

We now consider R(□,SOL).

THEOREM 1.19. Each of the following implies the next.

i) there exists an extendible cardinal;
ii) for all sufficiently large □, R(□,SOL);
iii) there exists a nontrivial elementary embedding from some V(□+□) into some V(□).

THEOREM 1.20. Let □ be least such that for all □ ≥ □, R(□,SOL). Then there is a nontrivial elementary embedding from V(□+□) into some V(□), and there is no extendible cardinal □ □.

THEOREM 1.21. Each of the following implies the next.
i) there exists a third order indescribable cardinal;
ii) there exists a Beth number □ such that R(□,SOL);
iii) there exists a second order indescribable cardinal.

THEOREM 1.22. Let □ be the least Beth number such that R(□,SOL). □ is a second order indescribable cardinal, greater than □ second order indescribable cardinals, and less than any third order indescribable cardinal.

THEOREM 1.23. Let □ be least such that R(□,SOL). Then □ is a weakly Mahlo cardinal > □ and □ is satisfied in L to be a second order indescribable cardinal that is greater than □ second order indescribable cardinals.

THEOREM 1.24. Each of the following is interpretable in the next. Each proves the consistency of the previous.
i) ZFC + there exists a third order indescribable cardinal;
ii) ZFC + there exists □ such that E(□,SOL);
iii) ZFC + there exists a second order indescribable cardinal.

The following is proved by forcing.

THEOREM 1.25. Suppose ZFC + "there exists a subtle cardinal" is consistent. Then ZFC + (□□ < $2^{□}$)(R(□,SOL) □ E(□,SOL)) + (□□ < $2^{□1}$)(E(□,$L_{□1□}$)) is consistent.

Finally, we define

R(□,□,SOL) if and only if every algebra of cardinality □ has a proper restriction of cardinality □ satisfying the same sentences of SOL.

E(□,□,SOL) if and only if every algebra of cardinality □ has a proper extension of cardinality □ satisfying the same sentences of SOL.

THEOREM 1.26. The following are equivalent for all Beth numbers □.

i) R(□,□,SOL);

ii) R(□,□,SOL);

iii) there is a nontrivial elementary embedding from V(□+1) into V(□+1).