SIMILAR SUBCLASSES

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Abstract. Reflection, in the sense of [Fr03a] and [Fr03b], is based on the idea that a category of classes has a subclass that is "similar" to the category. Here we present axiomatizations based on the idea that a category of classes that does not form a class has extensionally different subclasses that are "similar". We present two such similarity principles, which are shown to interpret and be interpretable in certain set theories with large cardinal axioms.

1. Introduction.

As in [Fr03a], [Fr03b], we use "class" as a neutral term, without commitment to the developed notions of "set" and "class" that have become standard in set theory and mathematical logic. We use $\in$ for membership.

This framework supports interpretations of sentential reflection that may differ from conventional set theory or class theory. However, we do not pursue this direction here.

As in [Fr03a], [Fr03b], this framework is intended to accommodate objects that are not classes. Such nonclasses are treated as classes with no elements. Thus we are careful not to assume extensionality. In fact, we will not assume any form of extensionality.

As in [Fr03a], [Fr03b], all of our formal theories of classes are in the language $L(\in)$, which is the usual classical first order predicate calculus with only the binary relation symbol $\in$ (no equality).

As in [Fr03a], [Fr03b], we use "category of classes" or just "category" as a neutral term, not specifically related to category theory. They are given by a formula of $L(\in)$ with a distinguished free variable, with parameters allowed.

The Similar Subclass Principle (SSP) asserts the following:

any category not forming a class has two extensionally different nonempty subclasses which are similar with respect to any two elements of the first.

We formalize SSP in the expected way by approximation. SSP consists of the axioms

□(□y)(□x)(x □ y □ □) □ (□y,z)((□x □ y)(□) □ (□x □ z)(□) □
(□x)(x □ y) □ (□x)(x □ z) □ (□x)(x □ y □ x □ z) □
(□u,v □ y)(□$_1$ □ □$_1$[y/z] □ ... □ □$_k$ □ □$_k$[y/z]))

where x,y,z,u,v are distinct variables, □,□$_1$,...,□$_k$ are formulas of L(□), y,z not in □, and all free variables in □$_1$,...,□$_k$ are among y,u,v.

The following definition is used in [Ba75] and [Fr01]. We say that an ordinal □ is subtle if and only if

i) □ is a limit ordinal;
ii) Let C □ □ be closed and unbounded, and for each □ < □ let A$_□$ □ □ be given. There exists □,□ □ C, □ < □, such that A$_□$ = A$_□$ □ □.

It is well known that every subtle ordinal is a subtle cardinal (see [Fr01], p. 3).

We will use the following schematic form of subtlety. SSUB is the following scheme in the language of ZFC with □,=. Let □,□ be formulas, where we view □ as carving out a class on the variable x, and □ as carving out a binary relation on the variables x,y. Parameters are allowed in □,□.

if □ defines a closed and unbounded class of ordinals and □ defines a system A$_□$ □ □, for all ordinals □, then there exists □,□ □ C, □ < □, where A$_□$ = A$_□$ □ □.

THEOREM 1. SSP is mutually interpretable with SSUB. SSP is provable in ZFC + there exists arbitrarily large subtle cardinals. SSP is provable in SSUB + V = L.

The Strong Similar Subclass Principle (SSP*) asserts the following:

for any category K not forming a class and any class x, there is a nonempty y K such that for all z y, y and y\z are similar with respect to any element of x.

We formalize SSP* in the expected way by approximation, just as we formalized SSP.

THEOREM 2. ZFC + "there exists a measurable cardinal" is interpretable in SSP*. SSP* is provable in NBG + "there are elementary embeddings from V into V with arbitrarily large critical points".

REFERENCES

[Fr03a] Sentential reflection, abstract. http://www.math.ohio-state.edu/~friedman/

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[Ba75] J. Baumgartner, Ineffability properties of cardinals I, in: A. Hajnal, R. Rado, and V. Sos (eds.), Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolylai, vol. 10 (Amsterdam: North-Holland, 1975), 109-130.

[Fr01] H. Friedman, Subtle Cardinals and Linear Orderings, Annals of Pure and Applied Logic 107 (2001) 1-34.