INTERPRETING SET THEORY IN DISCRETE MATHEMATICS: BOOLEAN RELATION THEORY

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1. WQO THEORY.

Wqo theory = well quasi ordering theory, is a branch of combinatorics which has proved to be a fertile source of deep metamathematical pheneomena.

In wqo theory, one freely uses some overtly set theoretic arguments. They are definitely noticeable, although they come fairly early in the interpretation hierarchy. Way before, say, ZFC.

A qo (quasi order) is a reflexive transitive relation $(A,\leq)$. A wqo (well quasi order) is a qo $(A,\leq)$ such that for all $x_1,x_2,\ldots$ from A, there exists $i < j$ such that $x_i \leq x_j$.

Highlights of wqo theory: that certain qo's are wqo's.

There are many equivalent definitions of wqo.

THEROEM 1.1. Let $(A,\leq)$ be a qo. The following are equivalent.
i. $(A,\leq)$ is a wqo.
ii. Every infinite sequence from A has an infinite subsequence which is increasing $(\leq)$.
iii. For all $x_1,x_2,\ldots \in A$ there exists n such that every term is $\geq$ at least one of $x_1,\ldots,x_n$.
iv. Every infinite subset of A has a two element chain.

v. Every infinite subset of A has an infinite chain of type □.
vi. Every subset of A has a finite subset such that every element of A is ≥ some element of that finite subset.

Proof: i □ ii. Color (i,j), i < j, according to whether xi □ xj or not. By Ramsey's theorem for pairs, there is an infinite subsequence which is either increasing (□), or with y_i □□ y_j, i < j. Only the former is possible.

ii □ iii. Suppose not. Define an infinite subsequence y1,y2,... so that y1 = x1, and each yi+1 is not ≥ y1,...,yi. This contradicts wqo.

iii □ iv. Enumerate the set without repetition.

iv □ v. Enumerate the set without repetition, and argue as in i □ ii.

v □ vi. Enumerate the set without repetition, and argue as in ii □ iii.

vi □ i. Let $x_1,x_2,...$ □ A. Let $\{x_1,x_2,...,x_n\}$ include the finite set. QED

Mainly application of Ramsey's theorem for pairs. Also simple inductive constructions. QED

J.B. Kruskal treats finite trees as finite posets, where there is a root, and the set of strict predecessors of any vertex is linearly ordered, and studies the qo

there exists an inf preserving embedding from T_1 into T_2.

I.e., h:T_1 □ T_2, where h is one-one, preserves □, and h(x inf y) = h(x) inf h(y). (Every finite tree has an obvious inf operation on the vertices).

THEOREM. (J.B. Kruskal). The above qo of finite trees as posets is a wqo.

Let's see what is set theoretic about the proof. Nash Williams's introduction of the minimal bad sequence technique seriously simplified the original Kruskal proof, and is also used extensively in wqo theory. See [NW65].

Suppose we want to prove that a given qo $(A,\le)$ is a wqo. Usually A is countably infinite, with a reasonable norm $| \, |$ mapping A into positive integers, where for all n, $\{x \in A: |x| \le n\}$ is finite. (Sometimes there is just a norm from A into positive integers).

We argue as follows. Suppose $(A,\le)$ is not a wqo. We call an sequence ***bad*** if and only if no term is $\le$ any later term.

Since $(A,\le)$ is not a wqo, there is an infinite bad sequence. We construct what is called a minimal infinite bad sequence. Take y_1 to be of minimal norm such that y1 starts an infinite bad sequence. Take y_2 to be of minimal norm such that y_1,y_2 starts an infinite bad sequence. Take y_3 to be of minimal norm such that y_1,y_2,y_3 starts an infinite bad sequence. Etcetera.

Note that we are building this infinite minimal bad sequence in stages, by considering all possible infinite bad sequences - including the infinite bad sequence being constructed!

The only way to make sense of this is to accept the idea that infinite sequences, say, of natural numbers, exist prior to any construction of them.

This is normally viewed as a serious, but limited, commitment to the set theoretic point of view. It is called impredicativity, and was strongly objected to by Poincare and Weyl.

It can be shown that the infinite minimal bad sequence statement sits at Π^1_1-CA_0 in the interpretation hierarchy of lecture #1.

The infinite minimal bad sequence argument gives the simplest proof of Kruskal's theorem and of a key lemma due to Graham Higman. Let $(A,\le)$ be a qo. We write $(A^*,\le^*)$ for the qo of all finite sequences from A (< > allowed) under

$(x_1,...,x_n) \le^* (y_1,...,y_m) \leftrightarrow \exists$ strictly increasing $f:\{1,...,n\} \to \{1,...,m\}$ such that each $x_i \le y_{f(i)}$.

THEOREM 1.2. ([Hi52]). If $(A,\le)$ is a wqo then $(A^*,\le^*)$ is a wqo.

Proof: ([NW65]). Suppose not. Let □[1],□[2],... be an infinite minimal bad sequence in A* (minimal with respect to length). No □[i] is empty. Look at the sequence x[1],x[2],... □ A of last terms. Let x[i1],x[i2],... be an infinite subsequence which is increasing (□). Look at □[1],...,
□[i_1-1],□[i_1]′,□[i_2]′,..., where □[i_j]′ is the result of chopping off the last term of □[i_j]. By the minimality, this cannot be bad. Let □[i_j]′ □* □[i_k]′, j < k. Then □[i_j] □* □[i_k]. This contradicts that □[1],□[2],... is bad. QED

It is known that one can avoid the infinite minimal bad sequence argument for Theorem 1.2 in favor of an argument in the much weaker ACA0. However, so such avoidance is possible with the following.

THEROEM 1.3. (J.B. Kruskal). The finite trees as posets, under the qo, inf preserving embedding, is a wqo.

Proof: ([NW65]). Let T[1],T[2],... be a minimal bad sequence (using number of vertices). Knock off the roots and let K be the set of all resulting trees (immediate subtrees of the T's). First suppose (K,□) is a wqo. By Theorem 1.2, (K*,□*) is a wqo. Knock off the roots of T[1],T[2],... . This results in T[1]*,T[2]*,... □ K*. Let T[i]* □* T[j]*, i < j. Then T[i] □ T[j], which contradicts that T[1],T[2],... is bad.

Suppose (K,□) is not a wqo. Let K[1],K[2],... be an infinite bad sequence of trees in K. Let i be least such that at least one of these trees, K[j], is an immediate subtree of T[i]. Then T[1],...,T[i-1],K[j],K[j+1],... is bad. Since K[j] has fewer vertices than T[i], this violates the minimality of T[1],T[2],... . QED

We know that Kruskal's theorem in the form of Theorem 1.2 lies between $□^1_2$-TI_0, $□^1_2$-TI on our interpretation list (see [RW93]). This tells us that Kruskal's theorem cannot be proved without using some significant set theoretic methods.

We now extract finite information from KT.

COROLALRY 1.4. (Linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded sequence of finite trees. There exist i < j such that T_i is inf preserving embeddable into T_j.

COROLLARY 1.5. (Computational KT). Let $T_1,T_2,\ldots$ be a polynomial time sequence of finite trees. There exist $i < j$ such that T_i is inf preserving embeddable into T_j.

Note Corollary 1.5 is purely arithmetical. In fact, Theorem 1.3, Corollary 1.4, and Corollary 1.5 all sit between Π^1_2-TI_0, Π^1_2-TI on our interpretation list. This is true even for nonconstant coefficient 1, and for linear time/log space.

By an obvious application of weak Konig's lemma, Corollary 1.4 has very strong uniformities.

THEOREM 1.6. (Uniform linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded sequence of finite trees. There exists $i < j \leq n$ such that T_i is inf preserving embeddable into T_j, where n depends only on the given linear bound, and not on $T_1,T_2,\ldots$

With this kind of strong uniformity, we can obviously strip the statement clear of infinite sequences of trees.

For nonconstant coefficient 1, we have:

THEOREM 1.7. (finite KT). Let n >> k. For all finite trees $T_1,\ldots,T_n$ with each $|T_i| \leq i+k$, there exists $i < j$ such that T_i is inf preserving embeddable into T_j.

All of these variants of KT lie between Π^1_2-TI_0 and Π^1_2-TI.

The growth rate for the least n as a function of k in Theorem 1.7 is truly enormous. It is a provably recursive function of Π^1_2-TI, but grows faster than all provably recursive functions of Π^1_2-TI_0. See [Fr02].

Other Π^0_2 forms of KT involving only the internal structure of a single finite tree can be found in [Fr02].

THEOREM 1.8. (Robertson/Seymour Graph Minor Theorem). Let $G_1,G_2,\ldots$ be finite graphs. There exists $i < j$ such that G_i is minor included in G_j.

GMT is significantly stronger than KT, and lives above $\Pi^1$1-CA_0. It seems to go somewhat higher than Π^1_1-CA, but not too much higher. See [FRS87].

An extremely interesting consequence of GMT is the subcubic graph theorem. A subcubic graph is a graph where every vertex has valence $\leq$ 3. (Loops and multiple edges are allowed).

THEOREM 1.8. Let $G_1,G_2,...$ be subcubic graphs. There exists $i < j$ such that G_i is embeddable into G_j as topological spaces (with vertices going to vertices).

This is also stronger than KT and lives above Π^1_1-CA_0. Growth conditions and finite forms apply just as for KT. See [FRS87] and [Fr02].

2. COUNTABLE POINTSETS.

I now very briefly mention a perhaps surprising result that can be told to Freshman(?), that puts us at impredicativity, but not quite as high as KT. The set theoretic methods involved are quite close to Cantor's thinking as he started to develop set theory.

THEOREM 2.1. Let A,B be countable sets of reals. There is a pointwise continuous one-one function from A into B, or there is a pointwise continuous one-one function from B into A.

The proof involves constructing the Cantor Bendixson decomposition of A,B (normally this is done only for closed sets). This can have any countable ordinal length. Then one carefully builds the function from the "shortest" one to the "longest" one, sequentially. See [Fr05].

There is no way to avoid set theory. Theorem 2.1 appears exactly at ATR_0 in the interpretation list.

We can state in purely mathematical terms what is so exotic about Theorem 2.1.

THEROEM 2.2. There is no Borel function which, when given, two countable sets A,B of rationals(!), indicates a correct direction A into B or B into A, for a pointwise continuous one-one function. There is no Borel function which, when given two countable sets A,B of rationals with a pointwise continuous one-one function from A into B, produces a pointwise continuous one-one function from A into B.

3. BOREL SELECTION.

Let S □ ¬^{2} and E □ ¬. A selection for A on E is a function f:E □ ¬ whose graph is contained in S.

A selection for S is a selection for S on ¬.

We say that S is symmetric if and only if S(x,y) □ S(y,x).

THEOREM 3.1. Let S □ ¬^{2} be a symmetric Borel set. Then S or ¬^{2}\S has a Borel selection.

My proof of Theorem 3.1 relies heavily on Borel determinacy, due to D.A. Martin. See [Fr71], [Ma85], and [Fr02].

Theorem 3.1 sits far higher in the interpretation list than anything we have discussed yet. It is provable in ZC + (□□ < □$_1$)(V(□)□), and sits close to it, way above ZC. This is still quite shy of ZFC.

Theorem 3.1 can be proved using uncountably many iterations of the power set operation, but not with only countable many iterations of the power set operation. It cannot be proved in ZFC without the replacement axiom.

There is another kind of Borel selection theorem from the work of Debs and Saint Raymond of Paris VII. They take the general form: if there is a nice selection for S on compact subsets of E, then there is a nice selection for S on E. See [DS02], [DS04], [DS].

THEOREM 3.2. Let S □ ¬^{2} be Borel and E □ ¬ be Borel with empty interior. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

THEOREM 3.3. Let S □ ¬^{2} be Borel and E □ ¬ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

THEROEM 3.4. Theorem 3.2 is provable in ZC + (□□ < □$_1$)(V(□)□) but not in ZC. Theorem 3.3 is neither provable nor refutable in ZFC.

However, the interpretation levels of Theorem 3.2 and 3.3 are not so clear. They are certainly bounded above by ZC + (□□ < □$_1$)(V(□)□).

4. BOOLEAN RELATION THEORY.

We begin with two examples of statements in BRT of special importance for the theory.

THIN SET THEOREM. Let $k \geq 1$ and $f:N^k \to N$. There exists an infinite set $A \subseteq N$ such that $f[A^k] \neq N$.

COMPLEMENTATION THEOREM. Let $k \geq 1$ and $f:N^k \to N$. Suppose that for all $x \in N^k$, $f(x) > \max(x)$. There exists an infinite set $A \subseteq N$ such that $f[A^k] = N\backslash A$.

These two theorems are official statements in BRT. In the complementation theorem, A is unique.

We now write them in BRT form.

THIN SET THEOREM. For all $f \in MF$ there exists $A \in INF$ such that $fA \neq N$.

COMPLEMENTATION THEOREM. For all $f \in SD$ there exists $A \in INF$ such that $fA = N\backslash A$.

The thin set theorem lives in IBRT in A,fA. There are only $2^{2^\wedge 2} = 16$ statements in IBRT in A,fA. These are easily handled. IBRT = inequational Boolean relation theory.

The complementation theorem lives in EBRT in A,fA. There are only $2^{2^\wedge 2} = 16$ statements in IBRT in A,fA. These are easily handled. EBRT = equational Boolean relation theory.

THIN SET THEOREM. Let $k \geq 1$ and $f:N^k \to N$. There exists an infinite set $A \subseteq N$ such that $f[A^k] \neq N$.

Proof: (J. Remmel's version). Let $f:N^k \to N$. Let ot(k) be the number of order types of k tuples from N. Let $\alpha_1,\ldots,\alpha_{ot(k)}$ be an enumeration of these order types. Apply the usual infinite Ramsey theorem for k tuples in the usual way to obtain $A \in INF$ such that for all $m \in [0,ot(k)]$,

$(\forall x,y \in A^k)$(if x,y have the same order type then $f(x) = m \leftrightarrow f(y) = m$).

It is clear that the $x \in A^k$ of any given order type can only map to at most one element of $[0,ot(k)]$. Hence $rng(f|A^k) \subseteq$

[0,ot(k)] has at most ot(k) elements. Therefore rng($f|A^k$) omits at least one element of [0,ot(k)].

There are conjectures to the effect that the infinite Ramsey theorem is required for the Thin Set Theorem. It is known that some level of nonconstructivity is needed to prove the Thin Set Theorem even for k = 2.

COMPLEMENTATION THEOREM (with uniqueness). For all $f \in SD$ there exists a unique $A \subseteq N$ with fA = N\A. Moreover, $A \in$ INF.

Proof: Let $f \in SD$. We inductively define a set $A \subseteq N$ as follows. Suppose $n \geq 0$ and we have defined membership in A for all $0 \leq i < n$. We then define $n \in A$ if and only if $n \notin f(A \cap [0,n))$. Since $f \in SD$, we have for all n, $n \in A \leftrightarrow n \notin fA$ as required.

Now suppose fB = N\B. Let m be least such that A,B differ. Then $m \in B \leftrightarrow m \notin fB \leftrightarrow m \notin f(B \cap [0,m))$, and $m \in A \leftrightarrow m \notin f(A \cap [0,m))$. Since $A \cap [0,m) = B \cap [0,m)$, we have $m \in A \leftrightarrow m \in B$. This is a contradiction. Hence A = B.

If A is finite then fA is finite and N\A is infinite. Hence A is infinite. QED

There is a variant of the Complementation Theorem that applies to all $f \in MF$. Let $f:N^k \to Z$. Define the "upper image"

$f_<A = \{f(x_1,...,x_k): f(x_1,...,x_k) > \max(x_1,...,x_k)\}$

where f has arity k. Obviously, if $f \in SD$ then $f_<A = fA$.

UPPER COMPLEMENTATION THEOREM. Every $f:N^k \to Z$ has a unique upper complementation. ($f_<A$ = N\A). This unique upper complement is infinite.

Let an affine map be given from N^k into Z. We can try to determine its unique upper complementation, and understand its structure.

We have looked at the Thin Set Theorem and the Complementation Theorem in continuous settings, a little.

CONTINUOUS COMPLEMENTATION THEOREM (with uniqueness). Every strictly dominating continuous $f:E^k$ □ E, where E □ ¬ is closed, has a unique complementation.

CONTINUOUS THIN SET THEOREM. For all continuous f:¬ □ ¬, there exists an open A □ ¬ of full measure such that fA ≠ ¬. fails for $¬^3$.

There are 16 statements in EBRT in A,fA on (SD,INF), and 16 statements in IBRT in A,fA on (MF,INF). It is not hard to determine all 32 truth values. The only nontrivial ones are the Complementation and Thin Set theorems, with a footnote. There is a slightly sharper Thin Set Theorem: (□f □ MF)(□A □ INF)(fA □ A ≠ N).

We analyzed EBRT in A,B,fA,fB,□ on (SD,INF). Here □ is required in the conclusion, A □ B □ N. Without □, 2^{16} = 65,536. With □, 2^9 = 512.

There are lots of tricky ones. For all f □ SD there exist infinite A □ B □ N such that

1. B □. fA = N, A = B □ fB.
2. A □. fB = N, fA □ B, B □ fB □ fA.
3. A □ fB = ∅, fB □ B.

1,2 correct. 3 incorrect.

ELG (expansive linear growth) is also important for the theory. f:Nk □ N □ ELG if and only if

there exist constants c,d > 1 such that

$$c|x| \; □ \; f(x) \; □ \; d|x|$$

holds for all but finitely many x □ N^k.

We analyzed EBRT in A,B,fA,fB,□ on (ELG,INF). Here □ is required in the conclusion, A □ B □ N.

Again there are lots of tricky ones. For all f □ ELG there exist infinite A □ B □ N such that

1. B □. fA = A □ fB = N.
2. A □, fB = N, B □ fA = ∅.
3. A □ fB = ∅, B □ fB = N, fB □ B □ fA.
4. B □. fA = N, A □ fB = ∅.

5. B □. fA = N, A □ fB.
6. A □ fA = ∅, B □ fB = N, B □ fB □ A □ fA.
7. A □. fB = N, fA □ B.
8. fB □ B □. fA, A = B □ fB.

1,2,5,7,8 true. 3,4,6 false. We also did some classifications for IBRT.

What about EBRT in A,B,fA,fB on (ELG,INF)? 65,536. Could tie up the entire set of logic students at Berkeley for a year.

EBRT in A,B,fA,fB,gA,gB on (ELG,INF) has 2^{32} = 4,204,067,296 cases. Could tie them up for life.

Jump to EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF) (smile). There are 2^{512} cases.

Here we know that we run out of axioms in ZFC for this. The case that I know is independent of ZFC can be done using Mahlo cardinals of finite order, but not without. We now say much more.

TEMPLATE. For all f,g □ ELG there exist A,B,C □ INF such that

X □. fY □ V □. gW
P □. fR □ S □. gT.

Here X,Y,V,W,P,R,S,T are among the three letters A,B,C.

Note that there are 6561 such statements. We have been able to analyze all of these 6561. There is a surprise.

1. All 6561 are provable or refutable using no set theoretic methods at all (RCA_0) with exactly 12 exceptions.

2. These 12 exceptions are really exactly one exception up to the obvious symmetry: permuting A,B,C, and switching the two clauses.

3. The single exception is the **exotic case**

PROPOSITION A. For all f,g □ ELG there exist A,B,C □ INF such that

A □. fA □ C □. gB

A □. fB □ C □. gC.

4. This statement is provably equivalent to the 1-consistency of MAH, over ACA’.

5. If we replace “infinite” by “arbitrarily large finite” then we don’t change the classification, and the 12 exceptions are also handled (positively) without any set theoretic methods.

THEOREM 4.1. The following is provable in MAH+ but not in MAH, even with the axiom of constructibility. An instance of the Template holds if and only if in that instance, “infinite” is replaced by “arbitrarily large finite”.

How can we use large cardinals to prove Proposition A? Start with f,g □ ELG. There is a way of massaging f,g using the infinite Ramsey theorem, introducing a lot of symmetries.

We then have a structure

$M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,...)$,

where the c’s are indexed by N, and form indiscernibles for atomic formulas, which generate N*.

We then canonically lift M* to

$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},c_0^{**},...)$

where this time the c’s are indexed by a large cardinal □, and also form indiscernibles for atomic formulas, which generate N**.

Since f**,g** are still in ELG (in an appropriate sense), we obtain a weak form of well foundedness for M**. It is enough to support a useful form of the Complementation theorem for M**. We get a complementation, W, for g**.

We then make a Skolem hull type argument, getting W1 □ W2 □ ... . We start with W_1 = a highly indiscernible sequence of type □ of generators, so that each W_1 □ f**Wn = ∅. Also, W_1 is so good that all W_n’s are of order type □. This level of indiscernibility can only be obtained through large cardinals.

So we obtain the required

A □. fA □ C □. gB
A □. fB □ C □. gC.

except with A = W_1, B = W_2, C = W_3, f = f**, g = g**, sitting in never never land. But order type □ and the canonical nature of M*,M** tells us that we can stick this back into the natural numbers with f,g. QED

5. FINITE GRAPHS.

Here we present an explicitly $□^0_1$ sentence that is independent of ZFC involving finite graphs. This is intensively ongoing research, and Proposition 11.2 represents the current state of the art.

A simple graph G is a pair (V,E), where V = V(G) is a nonempty set (the vertices), and E = E(G) is a set of subsets of V of cardinality 2 (the edges).

We say that A □ V(G) is a G independent set if and only if there is no {x,y} □ E(G) with x,y □ A.

We will consider graphs on any set $[t]^k$, where k,n ≥ 1. I.e., where V(G) = $[t]^k$. Here [t] = {1,...,t}.

For A □ $[t]^k$, the neighborhood of A consists of the y such that {x,y} □ E(G). The upper neighborhood of A consists of the y >lex x such that {x,y} □ E(G).

THEOREM 11.1. Every simple graph on any $[t]^k$ has an independent set, where every vertex outside the set lies in its upper neighborhood. The independent set is unique.

Let x,y □ $[t]^k$. We say that x,y are order equivalent if and only if for all 1 □ i,j □ k, $x_i < x_j$ iff $y_i < y_j$.

We say that G on $[t]^k$ is order invariant if and only if for all x,y,x',y' □ V(G), if (x,x'),(y,y') are order equivalent then {x,x'} □ E(G) □ (y,y') □ E(G). Thus connections are made in G only according to the relative size of the coordinates involved.

For x □ $[t]^k$, we write $2^x = (2^{x1},...,2^{xk})$, and x-1 = $(x_1-1,...,x_k-1)$.

PROPOSITION 11.2. Every simple order invariant graph on any $[t]^k$ has an independent set, where any $2^{8k!x}$ lying on a 4 clique outside the set, also lies on a 4 clique in its upper neighborhood, without $2^{8k!x}-1$.

THEOREM 11.3. Proposition 11.2 is provably equivalent to Con(SMAH) over ACA. Proposition 11.2 follows immediately from Theorem 11.1, if we remove "without $2^{8k!x}-1$".

Here ACA is the arithmetic comprehension axiom scheme with full induction. SMAH = ZFC + {there exists a strongly n-Mahlo cardinal}$_n$. ACA can be weakened somewhat.

Note that Proposition 11.2 is explicitly Π^0_1.

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