RELATIONAL SYSTEM THEORY

ABSTRACT

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1. Introduction.

Here we present some formal systems concerning ternary relations which relate to informal conceptual ideas that are arguably more fundamental than those that drive modern set theory.

We establish mutual translations between these formal systems and various systems of set theory, ranging from countable set theory, ZFC, to large cardinal hypotheses such as the existence of a Ramsey cardinal, and the existence of an elementary embeddings from V into V.

The primitives are identity, and x[y,z,w]. The latter is read "x is a ternary relation which holds at the objects y,z,w."

The use of x[y,z,w] rather than the more usual $y \in x$ has many advantages for this work. One of them is that we have found a convenient way to eliminate any need for axiom schemes. All axioms considered are single sentences with clear meaning. (In one case only, the axiom is a conjunction of a manageable finite number of sentences).

The theories are single sorted, and are based on the idea that some objects appear as arguments of other objects (ternary relations), whereas some objects do not so appear. The former objects are called arguments or argumental, whereas the remaining objects are called nonarguments or nonargumental. This is totally analogous to one sorted formalizations of the theory of classes where sets are defined to be classes that are an element of a class.

The basic axioms form a system called Ternary, which are, with one minor exception, all easy consequences of the axiom scheme asserting the following. We can form an object x which holds of any argumental objects y,z,w if and only if a given first order formula holds of y,z,w, where x does not appear and where parameters for objects are allowed, and all quantifiers range over argumental objects only. The minor exception is the axiom that asserts that there is an argument without any arguments. As indicated above, here we avoid any use of axiom schemes.

Although Ternary is not strong enough to derive the above mentioned axiom scheme, it does derive instances sufficient for our purposes when combined with the main axioms.

The main axioms are stated in terms of what we call 3-systems. A 3-system consists of three objects x,y,z, where x and y and z all have at least one argument (nonemptiness), and where every two distinct arguments of any of x or y or z combined, are "related" by x or y or z, using some third argument. We use the most appropriate notion of subsystem of 3-systems.

We use two different notions of reduction of 3-systems. One is a-reduction, where argumental components remain the same and nonargumental components become argumental.

The other is na-reduction, where argumental components remain the same and nonargumental components remain nonargumental, but are cut back.

All of the axioms considered take the following two general forms:

Every 3-system (of a specific kind) has an a-reduction (of the same specific kind).

Every 3-system (of a specific kind) has an na-reduction (of the same specific kind).

The axioms of the first form correspond to roughly ZF and fragments thereof. The axioms of the second form correspond to large cardinal hypotheses ranging from roughly Ramsey cardinals (just below measurable cardinals) and the existence of an elementary embedding from V into V (over NBG without choice).

Now let me now go somewhat beyond what has been established at this point.

I believe that set theory is the canonical mathematical limit of informal common sense thinking. Let me explain with an example you are all familiar with.

People, using common sense, think about, say, a full head of hair. They think that if you remove one strand of hair from a full head of hair, then it remains a full head of hair.

Scientific thinking has a problem with this. After all, one can perform a thought experiment whereby the number of strands of hair is counted, and pulled out one by one, until there are no more. Clearly complete baldness is not a full head of hair.

At this point, set theory enters the picture. The idea of a full head of hair is associated with the precise set theoretic notion of: infinite set. It is provable in set theory that if you take an element out of an infinite set, then it remains infinite. It is provable in set theory that infinite sets are not empty. It is provable in set theory that infinite sets cannot be numerically counted - the count never terminates.

There is a more sophisticated idea of this sort. There is the common sense idea of a large system. Not just an inert clump like a head of hair, but rather a large system with a number of interlocking components with complicated internal connections. Like the physical universe, or like the human body, or like the world of living organisms, or like the world economic system.

There is the idea that in any large system, we can take something away without the system falling apart.

In the case of the head of hair, it doesn’t make any difference which strand of hair you take away. Here, we are merely asserting that *something* can be taken away - not that *anything* can be taken away. And I am not asserting that you can get away with taking only one item away. It may be a significant amount of stuff.

Furthermore, in any large system, we can take something away without the system falling apart, and where the system remains “similar”.

So what are the missing parts of this analogy?

1. Any large clump stays large after some (any) point removal
□ infinite sets.
2. Any large system remains a large system after some portion removal □ XXX.
3. Any large system remains a large system, unaffected, after some portion removal □ YYY.
4. Any large system remains unaffected after some expansion □ ZZZ.

XXX ~ Jonsson cardinals ~ Ramsey cardinals.
YYY ~ ZZZ ~ elementary embedding axioms roughly around a rank into itself.

We have already backed up these statements with precise theorems in set theory involving set theoretic structures of sufficiently large cardinality. We call this subject, *the theory of large algebras.* Initial developments along these lines have been published in [Fr04]. Some further developments are implicit in the unpublished notes [Fr03].

The break point (how large is large?) depends on the notion of “unaffected” one uses. In [Fr04}, the break point is the first measurable cardinal. The break points discussed in [Fr03] are much higher. Some relevant notions are language oriented, whereas others are more directly mathematical.

But the bold new idea here goes well beyond any theory of large algebras within set theory. Here is what we anticipate:

A. There is a nonmathematical common sense oriented theory of systems and components which corresponds to various well studied levels of set theory with/without large cardinals.

B. More generally, one can formulate transparent principles of a plausible nature about ANY common sense notions, which correspond to various well studied levels of set theory with/without large cardinals.

A word of caution here. It may well be the case that in any region of common sense thinking, if one goes far enough, one reaches outright contradictions. That is to be expected. What we are anticipating is the designation of formal systems closely associated with commonsense thinking, that are in some ways extrapolations of commonsense thinking, and in other ways are restrictions of commonsense thinking, which correspond exactly to various levels of abstract set theory.

As we shall see, A is arguably already implicit in the development below, where we focus entirely on the notion of a ternary relation. However, we look forward to reworking the development here based on richer notions that are an integral part of everyday thinking.

In fact, it can be argued that common sense thinking is incredibly richer, logically, than mathematics or science. Of course, it is not subject to the same kind of deep and subtle constraints that mathematics and science operate under. In a separate longstanding development, I struggle, with increasing success, to find normal mathematics that requires large cardinals. But I speculate that large cardinals are everywhere implicit in common sense thinking.

We will hopefully get to the point where set theory with large cardinals emerges as the one mathematical area which applies to just about everything outside of science - across the board.

In fact, set theory with large cardinals may be to common sense thinking as the Newton/Leibniz calculus is to science.

The “calculus” aspect of set theory with large cardinals is as follows. There will be a proliferation of natural formal systems involving various groups of common sense notions. One will want to know how these systems compare under interpretability. One will see that, in fact, there is a quasi linear ordering under interpretability. One will want to “calculate”, for any pair of such systems, how they compare in this quasi linear ordering.

The only way to make such comparisons will be to have a manageable set of representatives for each level that arises, and first identify where each of the two systems to be compared fits in.

The manageable set of representatives is, of course, just various well studied levels of set theory with large cardinals.

So set theory with large cardinals may be the appropriate measuring tool for the comparison of systems based on common sense notions. Perhaps it can then emerge as the most generally and transparently useful area of modern mathematics.

2. Ternary.

The base theory for our investigation is the system Ternary. Ternary is a one sorted theory, with only equality and x[y,z,w].

We say that x is an argument (arg) if and only if (□y,z,w)(y[x,z,w]   y[z,x,w]   y[z,w,x]). We say that x is argumental if and only if x is an argument. We say that x is nonargumental if and only if x is not an argument.

Here are the axioms of Ternary.

ATOM. (□arg x)(□y,z,w)(□x[y,z,w]).

COMPLEMENTATION. (□x)(□args y,z,w)(x[y,z,w] □ □u[y,z,w]).

UNION. (□x)(□y,z,w)(x[y,z,w] □ (u[y,z,w]   v[y,z,w])).

ATOMIC COMPREHENSION. (□x)(□args y,z,w)(x[y,z,w] □ □), where □ is an atomic formula not mentioning x.

PROJECTION. (□x)(□args y,z,w)(x[y,z,w] □ (□u)(v[u,z,w])).

Obviously, every one of these axioms is a single sentence, except Atomic Comprehension. But here there are only finitely many instances up to change of variables. E.g., we can insist that □ be an atomic formula whose variables are among y,z,w,t,u,v.

3. Systems, Subsystems, Reductions, Full Systems, Complete Systems.

We say that x is argumental if and only if x is an argument. We say that x is nonargumental if and only if x is not an argument. We use the term "object" for any x. Hence objects can be argumental or nonargumental.

We say that x is a subobject of y if and only if (□t,u,v)(x[t,u,v] □ y[t,u,v]). We write this as x □ y. We write x ≡ y if and only if x □ y □ y □ x.

Note that x □ y does not really say that x is similar to y - at least not very strongly. We would also like the inner workings of x to be the same as the inner workings of y, with regard to objects that fall within the purview of x. This motivates the following definition.

We say that x′ is a restriction of x if and only if for all arguments t,u,v of x′, x[t,u,v] □ x′[t,u,v].

We will not be using the above definition. Instead, we work with triples of objects x,y,z. The components of the triple x,y,z are x and y and z.

We say that t is an argument of the triple x,y,z if and only if t is an argument of at least one of its three components.

We say that two distinct object t,u are related by x if and only if x holds of some three objects that include both t,u.

A 3-system is a triple x,y,z, where

i. x,y,z each have at least one argument.
ii. Any two distinct arguments of the triple x,y,z are related by at least one of its three components.

We say that x′,y′,z′ is a subsystem of the 3-system x,y,z if and only if x′,y′,z′ is a 3-system such that for all arguments t,u,v of the 3-system x′,y′,z′,

x[t,u,v] □ x′[t,u,v];
y[t,u,v] □ y′[t,u,v];
z[t,u,v] □ z′[t,u,v].

We introduce two related notions of reduction.

An a-reduction of a 3-system S is a subsystem of S, where the argumental components of S remain the same, and the nonargumental components of S become argumental.

An na-reduction of a 3-system S is a subsystem of S, where the argumental components of S remain the same, and the nonargumental components of S remain nonargumental, but not ≡.

More formally, let x,y,z be a 3-system. We say that x′,y′,z′ is an a-reduction of the 3-system x,y,z if and only if

i. x′,y′,z′ is a subsystem of x,y,z.
ii. x argumental □ x = x′.
iii. y argumental □ y = y′.
iv. z argumental □ z = z′.
v. x nonargumental □ x′ argumental.
vi. y nonargumental □ y′ argumental.
vii. z nonargumental □ z′ argumental.

We say that x′,y′,z′ is an na-reduction of the 3-system x,y,z if and only if

i. x′,y′,z′ is a subsystem of x,y,z.
ii. x argumental □ x = x′.
iii. y argumental □ y = y′.
iv. z argumental □ z = z′.
v. x nonargumental □ (x′ nonargumental □ □x′ ≡ x).
vi. y nonargumental □ (y′ nonargumental □ □y′ ≡ y).
vii. z nonargumental □ (z′ nonargumental □ □z′ ≡ z).

We say that a 3-system S is full if and only if every subobject of an argument of S is an argument of S.

We say that a 3-system S is complete if and only if every argumental object agrees with some argument of S at all triples of arguments of S.

More formally, we say that a 3-system S is complete if and only if (□args x)(□y)(y is an argument of S □ (□t,u,v)(t,u,v are arguments of S □ (x[t,u,v] □ y[t,u,v]))).

We are now prepared to state the following axioms.

A-REDUCTION. Every 3-system has an a-reduction.

NA-REDUCTION. Every 3-system has an na-reduction.

FULL A-REDUCTION. Every full 3-system has a full a-reduction.

FULL NA-REDUCTION. Every full 3-system has a full na-reduction.

COMPLETE A-REDUCTION. Every complete 3-system has a complete a-reduction.

COMPLETE NA-REDUCTION. Every complete 3-system has a complete na-reduction.

There is a weak form of A-Reduction that we also work with. We say that a 3-system is argumental if and only if each of its three components is argumental.

ARGUMENTAL SUBSYSTEM. Every 3-system has an argumental subsystem.

4. Results.

THEOREM 4.1. Ternary + A-Reduction is mutually interpretable with Z_2 = second order arithmetic. The same is true of Ternary + Argumental Subsystem.

THEOREM 4.2. Ternary + NA-Reduction is mutually interpretable with NBG + “On is a Ramsey cardinal”. In particular, it interprets ZFC + “there exists an almost Ramsey cardinal”, and is interpretable in ZF\P + “there exists a Ramsey cardinal”. The same is true of Ternary + A-Reduction + Full A-Reduction + NA-Reduction.

THEOREM 4.3. Ternary + Full A-Reduction is mutually interpretable with ZFC. The same is true of Ternary + A-Reduction + Full A-Reduction.

THEROEM 4.4. Ternary + Full NA-Reduction, Ternary + Complete A-Reduction, are both inconsistent.

We say that V(□) is strongly inaccessible if and only if every function from an element of V(□) into V(□) is itself an element of V(□).

THEOREM 4.5. Ternary + Complete NA-reduction interprets NBG + {there exists a nontrivial $\square_n$ elementary embedding from V into V}$_n$, and is interpretable in ZF + “there exists a cardinal □ and a nontrivial elementary embedding from V(□+1) into V(□+1), where V(□) is strongly inaccessible”. The same is true of Ternary + A-Reduction + Full A-Reduction + NA-Reduction + Complete NA-Reduction.

COROLLARY 4.6. Ternary + Complete NA-reduction interprets ZFC + “there exists a nontrivial elementary embedding from a rank into itself”.

Here a Ramsey cardinal is a cardinal □ such that for all partitions of the finite subsets of □ into two pieces, there is a subset of □ of cardinality □ such that any two finite subsets of the subset of the same finite cardinality lie in the same piece.

An almost Ramsey cardinal is an uncountable cardinal □ such that for all partitions of the finite subsets of □ into two pieces, there is a subset of □, of any given cardinality < □, such that any two finite subsets of the subset of the same finite cardinality lie in the same piece.

Almost Ramsey cardinals are incompatible with the axiom of constructability.

In [Mi79], the Dodd Jensen core model is used in order to establish the mutual interpretability of

ZFC + "there exists a Jonsson cardinal".
ZFC + "there exists a Ramsey cardinal".

For our results (Theorems 4.2, 5.3), we use the mutual interpretabilty of the following triple and the following pair:

NBG + “On is a Jonsson cardinal”.
NBG\P + “On is a Jonsson cardinal”.
NBG + “On is a Ramsey cardinal”.

ZF\P + "there exists a Jonsson cardinal".
ZF\P + "there exists a Ramsey cardinal".

Mitchell has confirmed that his published proof will establish this with “tentative strong belief”.

It is well known that NBG + {there exists a nontrivial $\square_n$ elementary embedding from V into V$\}_n$ is stronger than ZFC + many measurable cardinals. Using known inner model theory, it is well known that it is stronger than ZFC + projective determinacy, or ZFC + Woodin cardinals. Woodin, using forcing arguments, has shown that NBG + {there exists a nontrivial Σ_n elementary embedding from V into V}$_n$ interprets ZFC + "there exists a nontrivial elementary embedding from a rank into itself". Hence Corollary 4.6.

5. Schematic Versions.

We now freely use schemes. While we believe that the avoidance of schemes is an important development, we also believe that there is still some importance to be attached to the systems based on schemes.

The most mild use of schemes is to use the following.

ARGUMENTAL COMPREHENSION. (∃x)(∀args y,z,w)(x[y,z,w] ↔ φ), where φ is a formula not mentioning x, whose quantifiers range over argumental objects only.

We can also use this stronger form.

COMPREHENSION. (∃x)(∀args y,z,w)(x[y,z,w] ↔ φ), where φ is a formula not mentioning x.

THEOREM 5.1. All of the systems discussed In 4.1 -4.6 are equivalent to the systems obtained by replacing Ternary with Atom + Argumental Comprehension.

If we use Atom + Comprehension instead of Ternary, then the systems are somewhat stronger. We restate the results using Atom + Comprehension as follows.

THEOREM 5.2. Atom + Comprehension + A-Reduction is mutually interpretable with Z_3 = third order arithmetic. The same is true of Atom + Comprehension + Argumental Subsystem.

THEOREM 5.3. Atom + Comprehension + NA-Reduction is mutually interpretable with ZF\P + "there exists a Ramsey cardinal". In particular, it interprets ZFC + "there exists an almost Ramsey cardinal" and is interpretable in ZFC + "there exists a Ramsey cardinal". The same is true of Atom + Comprehension + A-Reduction + Full A-Reduction + NA-Reduction.

THEOREM 5.4. Atom + Comprehension + Full A-Reduction is mutually interpretable with MKGC = Morse Kelley with Global

Choice. The same is true of Atom + Comprehension + A-Reduction + Full A-Reduction.

THEROEM 5.5. Atom + Comprehension + Full NA-Reduction, Atom + Comprehension + Complete A-Reduction, are both inconsistent.

THEOREM 5.6. Atom + Comprehension + Complete NA-reduction interprets MK + {there exists a nontrivial $\square_n$ elementary embedding from V into $V\}_n$, and is interpretable in ZF + "there exists a cardinal □ and a nontrivial elementary embedding from V(□+1) into V(□+1), where V(□) is strongly inaccessible". The same is true of Atom + Comprehension + A-Reduction + Full A-Reduction + NA-Reduction + Complete NA-Reduction.

COROLLARY 5.7. Atom + Comprehension + Complete NA-reduction interprets ZFC + "there exists a nontrivial elementary embedding from a rank into itself".

There are other ways in which schemes can be used for alternative formulations. We will not go into this here. Earlier versions of this work employed schemes in many ways that we have avoided here.

All of our results (sections 4 and 5) remain the same if we use extensionality. Of course, if we use extensionality, then there is no need to use ≡.

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