A THEORY OF STRONG INDISCERNIBLES

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The Complete Theory of Everything (CTE) is based on certain axioms of indiscernibility. Such axioms of indiscernibility have been given a philosophical justification by Kit Fine. I want to report on an attempt to give strong indiscernibility axioms which might also be subject to such philosophical analysis, and which prove the consistency of set theory; i.e., ZFC or more. In this way, we might obtain a (new kind of) philosophical consistency proof for mathematics.

We start with the usual impredicative theory of types with infinity, but not extensionality, and without equality. We call this ITT. We use infinitely many types (sorts). Objects of type 1 are the individuals. Objects of type k+1 are the unary predicates on objects of type k. We use the binary relation between objects of type k+1 and ob-jects of type k; namely, the former holds of the latter.

The idea is that sort 1 - the individuals - is to encompass absolutely every-thing, inclu-ding objects of higher types. But that idea is in no way, shape, or form, to be reflec-ted in the axioms considered about the typed objects. In fact, any statement such as

"every object of type 2 is an object of type 1"

cannot even be expressed in our language.

Specifically, for each $k \geq 1$, we have infinitely many vari-ables x^k ranging over objects of type k. The atomic formulas are of the form $x^{k+1}(x^k)$, where x^{k+1} is a variable of type k+1 and y^k is a variable of type k, and $k \geq 1$.

Formulas are built up as usual using logical connectives, and quanti-fiers $(\forall x^k)$, $(\exists x^k)$.

We use the usual axioms and rules of first order predicate calculus for this language, including the appropriate axioms of identity (in each sort). The nonlogsical axioms are:

1) full comprehension. $(\exists x^{k+1})(\forall x^k)(x^{k+1}(x^k) \leftrightarrow \varphi)$, where φ is a formula in the language of ITT in which x^{k+1} is not free;
2) axiom of infinity. There exists x^3 which is nonvacuous, and if x^3 holds at x^2 then x^3 holds at some y^2 whose exten-sion is more inclusive than the extension of x^2.

NOTE: There are many equiv-alent forms of the axiom of infinity. This one seems to be the simplest. It cannot be formulated using only types 1 and 2.

We define an equality relation as x ≡ y. Using ≡, we can give the usual robust formulation and development of basic concepts such as functions, car-dinal comparisons, finite sets, etc., in ITT.

We are now ready to introduce the axioms of strong indiscernibility.

We say that x^{k+1} is a part of y^{k+1} if and only if (□z^k)(x^{k+1}(z^k) □ y^{k+1}(z^k)).

Let □ be a formula with at most the free variable x^k.

We say that x^{k+1} is a part of □ if and only if (□z^k)(x^{k+1}(z^k) □ □(z^k)).

We use "infinite part" to indicate that the extension of the part is infinite.

We let "□ is large" be the sentence asserting that there is no one-one correspondence between the vari-ous x^k satisfying □ and some objects of sort 1. (This expresses the idea that x^k does not have car-dinality ≤ the class of individuals). Thus if k = 1 then "□ is large" is refutable.

AXIOMS OF STRONG INDISCERNIBILITY (SI). Let □,□ have at most the free variable x^k. If □ is large then there exists an infinite part x^{k+1} of □, where for all finite and equinumer-ous parts x^{k+1},y^{k+1} of x^{k+1}, we have □(x^{k+1}) □ □(y^{k+1}).

We consider the system ITT+SI.

THEOREM. ZFC + “there exists a cardinal that is n-ineffable for all finite n” is interpre-table in ITT + SI. On the other hand, ITT + SI is interpretable in ZC + “there exists a cardinal □ which arrows □.” ZC_0 + “there exists a cardinal □ which arrows □” is equicon-sistent with ITT + SI over PRA (primitive recursive arithmetic). ITT + SI states and proves the consistency of ZFC + “there is a cardinal that is n-ineffable for all n < □.”

[Here ZC_0 refers to ZC with bounded separation only.]